**Thai Journal of Mathematics**
Volume 19 Number 2 (2021)
Pages 725–738

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

# Chromatic Numbers of Suborbital Graphs for Some Hecke Groups

**Woratham Khangtragool**[1] **and Khuanchanok Chaichana**[2,3,4,*]

[1] *Graduate Master's Degree Program in Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
*e-mail: woratham_k@cmu.ac.th*

[2] *Advanced Research Center for Computational Simulation, Chiang Mai University, Chiang Mai 50200, Thailand*

[3] *Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand*

[4] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
*e-mail: khuanchanok.c@cmu.ac.th*

**Abstract** In this paper, we study the chromatic numbers of suborbital graphs for Hecke groups $H(\sqrt{2})$ and $H(\sqrt{3})$. We decompose each suborbital graph into disjoint isomorphic subgraphs and show that each subgraph is homomorphic to the Farey graph. Finally, we show that each suborbital graph has chromatic number 2 or 3.

**MSC:** 05C15; 20H10; 05C25

**Keywords:** Hecke group; suborbital graph; chromatic number

Submission date: 15.03.2020 / Acceptance date: 25.06.2020

## 1. Introduction

For each integer $q \geq 3$, the Hecke group $H(\lambda_q)$ is the group of transformations on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ generated by $S(z) = -\frac{1}{z}$ and $T(z) = z + \lambda_q$ where $\lambda_q = 2\cos(\frac{\pi}{q})$ [1]. In the cases $q = 3, 4,$ or $6$, the elements of $H(1), H(\sqrt{2})$, and $H(\sqrt{3})$ are completely describable. For $m \in \{1, 2, 3\}$, the Hecke group $H(\sqrt{m})$ consists of the transformations

(1) $T_1(z) = \dfrac{a\sqrt{m}z + b}{cz + d\sqrt{m}}$ where $a, b, c, d \in \mathbb{Z}$ and $adm - bc = 1$, and

(2) $T_2(z) = \dfrac{az + b\sqrt{m}}{c\sqrt{m}z + d}$ where $a, b, c, d \in \mathbb{Z}$ and $ad - bcm = 1$.

Each $H(\sqrt{m})$ acts transitively on the set $\sqrt{m}\widehat{\mathbb{Q}} = \{\frac{r}{s}\sqrt{m} \mid \frac{r}{s} \in \mathbb{Q}\} \cup \{\infty\}$ by the action $T \cdot z = T(z)$.

---

*Corresponding author.

A *graph* $G$ consists of a vertex set $V(G) \neq \emptyset$ and an edge set $E(G) \subseteq V(G) \times V(G)$. If $V(G)$ is finite, we say that $G$ is finite. Otherwise, we say that $G$ is infinite. We denote a directed edge $(v, w) \in E(G)$ by $v \to w$. Given $m \in \{1, 2, 3\}$ and $\frac{u}{N} \in \mathbb{Q}$, the *suborbital graph* $G(\infty, \frac{u}{N}\sqrt{m})$ is the infinite graph consisting of the vertex set $\sqrt{m}\widehat{\mathbb{Q}}$ and the edge set $\{T(\infty) \to T(\frac{u}{N}\sqrt{m}) \mid T \in H(\sqrt{m})\}$. In other words, the edges in the graph $G(\infty, \frac{u}{N}\sqrt{m})$ are the images of the edge $\infty \to \frac{u}{N}\sqrt{m}$ under the action of $H(\sqrt{m})$. We use the letters $u$ and $N$ in order to be consistent with [2–4] and we reserve the lower case $n$ for the divisors of $N$. For more introductory and comprehensive treatments of suborbital graphs, we refer the reader to [5, Chapter 5], [6, Chapter 4], and [7, Chapter 1.11]; there they are called *orbital graphs*.

We will visualize the graph $G(\infty, \frac{u}{N}\sqrt{m})$ by drawing it on the closed upper half-plane $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\} \cup \{\infty\}$. For an edge $T(\infty) \to T(\frac{u}{N}\sqrt{m})$, if one endpoint is $\infty$, the edge is represented by the vertical straight line in $\overline{\mathbb{H}}$ meeting the other endpoint on $\mathbb{R}$. Otherwise, both endpoints are in $\mathbb{R}$ and we represent this edge by the half-circle in $\overline{\mathbb{H}}$ centered on $\mathbb{R}$ meeting both endpoints. For example, consider the suborbital graph $G(\infty, 1)$ where the edges are the images of $\infty \to 1$ under all transformations in $H(1)$. This graph, also known as the *Farey* graph, is visualized as shown in Figure 1.

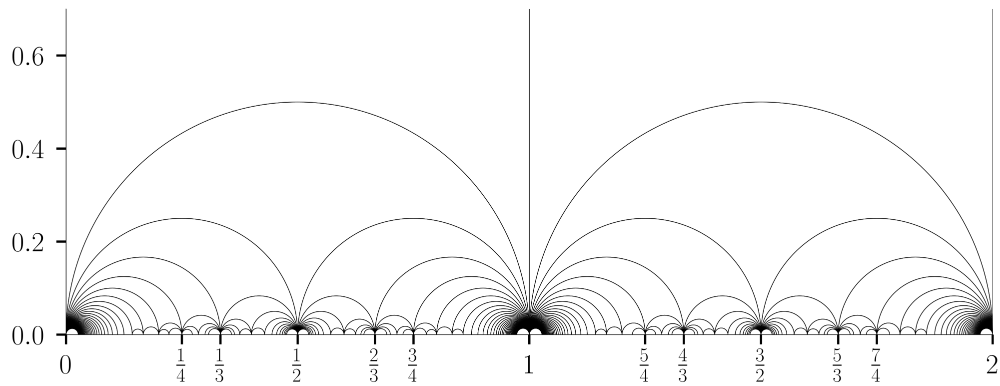


FIGURE 1. $G(\infty, 1)$.

The suborbital graphs for the modular group $H(1)$ were first studied by Jones, Singerman, and Wicks [2]. Many properties of the suborbital graphs $G(\infty, \frac{u}{N})$ were investigated by Jones et al. They gave sufficient and necessary conditions for the existence of an edge in $G(\infty, \frac{u}{N})$ and conjectured conditions for each suborbital graph to be a forest. The conjecture was later proven by Akbas [4]. Keskin extended their works to the suborbital graphs for $H(\sqrt{2})$ and $H(\sqrt{3})$ [3]. Some of their results that are relevant to our work are restated or rephrased in this section. Throughout this paper, we always assume that a fraction $\frac{x}{y}$ in $\widehat{\mathbb{Q}}$ is reduced, i.e., $\gcd(x, y) = 1$. The element $\infty$ is represented by the fractions $\frac{1}{0}$ and $\frac{-1}{0}$.

**Theorem 1.1** ([3, Theorems 1, 2]). *Let $m = 1$, 2, or 3.*

*If $\gcd(m, N) = 1$, then the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $G(\infty, \frac{u}{N}\sqrt{m})$ if and only if either*

*(G1.1)* $ry - sx = \pm N, \quad x \equiv \pm ur \pmod N, \quad y \equiv \pm us \pmod N, \quad m \mid s;$ *or*

*(G1.2)* $ry - sx = \pm N, \quad x \equiv \pm mur \pmod N, \quad y \equiv \pm mus \pmod N, \quad m \mid y.$

*If $\gcd(m, N) = m$, then the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $G(\infty, \frac{u}{N}\sqrt{m})$ if and only if either*

*(G2.1)* $ry - sx = \pm N, \quad x \equiv \pm ur \pmod N, \quad y \equiv \pm us \pmod N, \quad m \mid s;$ *or*

*(G2.2)* $ry - sx = \pm \frac{N}{m}, \quad x \equiv \pm ur \pmod{\frac{N}{m}}, \quad y \equiv \pm us \pmod N.$

*The signs are the same at every occurence of $\pm$ in the same line.*

For example, if Theorem 1.1(G1.1) holds, then there are two possibilities:

(1) $ry - sx = N, \quad x \equiv ur \pmod N, \quad y \equiv us \pmod N, \quad m \mid s;$ or

(2) $ry - sx = -N, \quad x \equiv -ur \pmod N, \quad y \equiv -us \pmod N, \quad m \mid s.$

Note that if $N < 0$, then $\frac{-u}{-N} = \frac{u}{N}$ with $-N > 0$. So we may assume that $N > 0$. Furthermore, observe that if $v \equiv u \pmod N$, then the conditions in Theorem 1.1 for $G(\infty, \frac{v}{N}\sqrt{m})$ are equivalent to those for $G(\infty, \frac{u}{N}\sqrt{m})$. Consequently, we may assume that $0 \le u < N$.

*Graph coloring* is the process of assigning color to each vertex in a graph such that no adjacent vertices have the same color. If we can color a graph $G$ with $k$ colors, we say that $G$ is *$k$-colorable*. We define the *chromatic number* of $G$ to be the smallest positive integer $k$ such that $G$ is $k$-colorable. If $G$ has chromatic number $k$, we say that $G$ is $k$-chromatic. The chromatic number of a graph can be obtained through certain properties of the graph itself, or by relating it to another graph under certain mappings.

For any two vertices $v$ and $w$ in a graph, other than referring to the directed edge from $v$ to $w$ itself, we also write $v \to w$ to indicate the existence of the edge when no confusion can arise. Moreover, we use the expression $v \rightleftharpoons w$ to indicate that $v \to w$ or $w \to v$. A *circuit* of length $l$ in a graph is a sequence of distinct vertices $v_1, v_2, \dots, v_l$ such that $v_1 \rightleftharpoons v_2 \rightleftharpoons \cdots \rightleftharpoons v_l \rightleftharpoons v_1$ where $l \ge 3$. A circuit of length 3 is called a *triangle*. A graph that contains no circuits is called a *forest* and has chromatic number at most 2. Under the axiom of choice, we can weaken the assumption by requiring only that the graph contains no circuits of odd lengths. On the other hand, a graph that contains a triangle is at least 3-chromatic. Sufficient and necessary conditions regarding these properties were given in [3] and we rephrase them as follows.

**Theorem 1.2** ([3, Theorems 4-6, Corollary 1]). *Let $N > 1$. Then*

*(1) $G(\infty, \frac{u}{N})$ has at most circuits of length 3, and $G(\infty, \frac{u}{N})$ contains a triangle if and only if $u^2 \pm u + 1 \equiv 0 \pmod N$;*

*(2) $G(\infty, \frac{u}{N}\sqrt{2})$ has at most circuits of length 4;*

*(3) $G(\infty, \frac{u}{N}\sqrt{3})$ has at most circuits of lengths 3 and 6, and $G(\infty, \frac{u}{N}\sqrt{3})$ contains a triangle if and only if $u^2 \pm u + 1 \equiv 0 \pmod N$ and $3 \mid N$.*

A *homomorphism* from graph $G$ to graph $H$ is a function $\varphi : V(G) \to V(H)$ such that if $(u, v) \in E(G)$, then $(\varphi(u), \varphi(v)) \in E(H)$. If there exists a homomorphism from $G$ to $H$, we say that $G$ is homomorphic to $H$. In this case, the chromatic number of $G$ is at most that of $H$. An *isomorphism* between $G$ and $H$ is a bijective homomorphism from $G$ to $H$ such that its inverse is a homomorphism. Clearly, isomorphisms preserve chromatic numbers. Let $G$ be a graph and $\{G_i\}_i$ be a family of graphs. If $V(G)$ and $E(G)$ are the unions of pairwise disjoint sets $\{V(G_i)\}_i$ and $\{E(G_i)\}_i$, respectively, we say that $G$ is the disjoint union of $\{G_i\}_i$. Moreover, if all graphs $G_i$ are isomorphic, then the chromatic numbers of $G_i$ and $G$ are equal. We shall see that the suborbital graphs admit such decompositions.

Next, we discuss how we can decompose suborbital graphs into disjoint isomorphic subgraphs using certain equivalence relations on the vertex sets. Keskin [3] gave a class of $H(\sqrt{m})$-invariant equivalence relations and we briefly discuss their characterizations here. For each $N \in \mathbb{N}$ and $m = 1,\ 2,$ or $3$, define $H_0^m(N)$ to be the set consisting of $T \in H(\sqrt{m})$ where

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d} \quad \text{or} \quad T(z) = \frac{a\sqrt{m}z + b}{cz + d\sqrt{m}} \quad \text{such that } c \equiv 0 \pmod{N}.$$

Then there is an $H(\sqrt{m})$-invariant equivalence relation $\approx_N$ on $\sqrt{m}\widehat{\mathbb{Q}}$ given by

$$\tfrac{r}{s}\sqrt{m} \approx_N \tfrac{x}{y}\sqrt{m} \iff T^{-1}S \in H_0^m(N)$$

where

$$T(z) = \frac{rz + *}{(s/m)\sqrt{m} + *} \quad \text{if } m \mid s, \quad T(z) = \frac{r\sqrt{m} + *}{sz + *} \quad \text{otherwise,}$$

and

$$S(z) = \frac{xz + *}{(y/m)\sqrt{m} + *} \quad \text{if } m \mid y, \quad S(z) = \frac{x\sqrt{m} + *}{yz + *} \quad \text{otherwise.}$$

We can characterize this relation by

$$\tfrac{r}{s}\sqrt{m} \approx_N \tfrac{x}{y}\sqrt{m} \iff \begin{cases} \frac{ry-sx}{m} \equiv 0 \pmod{N}, & \text{if } m \mid s \text{ and } m \mid y; \\ ry - sx \equiv 0 \pmod{N}, & \text{otherwise.} \end{cases}$$

If $\gcd(m, N) = 1$, then the statements $\frac{ry-sx}{m} \equiv 0 \pmod{N}$ and $ry - sx \equiv 0 \pmod{N}$ are equivalent and the relation can be reduced to

$$\tfrac{r}{s}\sqrt{m} \approx_N \tfrac{x}{y}\sqrt{m} \iff ry - sx \equiv 0 \pmod{N}.$$

All vertex-induced subgraphs of $G(\infty, \frac{u}{N}\sqrt{m})$ by each equivalence class of $\approx_N$ are isomorphic. In the case that $\gcd(m, N) = 1$, if there is an edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ in $G(\infty, \frac{u}{N}\sqrt{m})$, then $ry - sx = \pm N$. So, $\frac{r}{s}\sqrt{m} \approx_N \frac{x}{y}\sqrt{m}$. This means that each connected component of $G(\infty, \frac{u}{N}\sqrt{m})$ lies in a single equivalence class of $\approx_N$. It follows that $G(\infty, \frac{u}{N}\sqrt{m})$ is the disjoint union of vertex-induced subgraphs by each equivalence class. Let $F(\infty, \frac{u}{N}\sqrt{m})$ denote the vertex-induced subgraph of $G(\infty, \frac{u}{N}\sqrt{m})$ by the equivalence class $[\infty]_{\approx_N} = \{\frac{x}{y}\sqrt{m} : y \equiv 0 \pmod{N}\}$. We obtain that $G(\infty, \frac{u}{N}\sqrt{m})$ is a disjoint union of subgraphs each isomorphic to $F(\infty, \frac{u}{N}\sqrt{m})$. From Theorem 1.1, we can characterize the existence of an edge in $F(\infty, \frac{u}{N}\sqrt{m})$ as follows.

**Theorem 1.3.** *Let $m = 1,\ 2,$ or $3$. If $\gcd(m, N) = 1$, then the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $F(\infty, \frac{u}{N}\sqrt{m})$ if and only if either*

*(F1.1)* $ry - sx = \pm N, \quad x \equiv \pm ur \pmod{N}, \quad mN \mid s, \quad N \mid y;$ *or*
*(F1.2)* $ry - sx = \pm N, \quad x \equiv \pm mur \pmod{N}, \quad N \mid s, \quad mN \mid y.$

*The signs are the same at every occurence of $\pm$ in the same line.*

We emphasize here that the graphs $F(\infty, \frac{u}{N}\sqrt{m})$ have yet to be defined for the case $\gcd(m, N) = m > 1$. In that case, the relation $\approx_N$ may not guarantee that the edge sets of the subgraphs are disjoint. Instead, we will use $\approx_{N/m}$ and similarly define $F(\infty, \frac{u}{N}\sqrt{m})$ for each $G(\infty, \frac{u}{N}\sqrt{m})$.

Recall that any suborbital graph $G(\infty, \frac{u}{N}\sqrt{m})$ is identical to $G(\infty, \frac{v}{|N|})$ where $0 \leq v < |N|$ and $v \equiv u \pmod{N}$. So we may assume that $0 \leq u < N$. Keskin showed that the graphs $F(\infty\frac{u}{N}\sqrt{m})$ and $F(\infty, \frac{N-u}{N}\sqrt{m})$ where $\gcd(m, N) = 1$ are reflections of each other across the vertical line meeting $\frac{1}{2}\sqrt{m}$.

**Lemma 1.1** ([3, Lemma 6])**.** *If $\gcd(m, N) = 1$, then $F(\infty, \frac{u}{N}\sqrt{m})$ is isomorphic to $F(\infty, \frac{N-u}{N}\sqrt{m})$ by the mapping $v \mapsto \sqrt{m} - v$.*

Jones et al. gave sufficient conditions for any two graphs $F(\infty, \frac{u}{N})$ and $F(\infty, \frac{u}{n})$ to be homomorphic.

**Lemma 1.2** ([2, Lemma 5.3(ii)])**.** *If $n \mid N$, then $F(\infty, \frac{u}{N})$ is homomorphic to $F(\infty, \frac{u}{n})$ by the mapping $v \mapsto \frac{N}{n}v$.*

In the case that $m = 1$, the chromatic numbers for $G(\infty, \frac{u}{N})$ were obtained by Tapanyo and Jaipong. They first proved that the Farey graph $F(\infty, 1)$ has chromatic number 3 then applied the previous lemma to give an upper bound for the chromatic number for every graph $F(\infty, \frac{u}{N})$. Finally, they extended their results to the graphs $G(\infty, \frac{u}{N})$.

**Theorem 1.4** ([8, Theorem 8])**.** *The Farey graph is 3-chromatic.*

**Corollary 1.3** ([8, Corollary 11])**.** *Let $\chi$ denote the chromatic number of $G(\infty, \frac{u}{N})$. Then*

*(1) $\chi = 2$ if $N \nmid (u^2 \pm u + 1)$, and*
*(2) $\chi = 3$ if $N \mid (u^2 \pm u + 1)$.*

Our work concerns the remaining cases $m = 2$ or $3$. Here, we state our main theorem and describe our approach to obtain the chromatic number of the suborbital graphs $G(\infty, \frac{u}{N}\sqrt{2})$ and $G(\infty, \frac{u}{N}\sqrt{3})$. Note that the statement $\gcd(m, N) = m$ is equivalent to $m \mid N$ and we use them interchangeably.

**Main Theorem.**

*(1) $G(\infty, \frac{u}{N}\sqrt{2})$ has chromatic number 2.*
*(2) Let $\chi$ be the chromatic number of $G(\infty, \frac{u}{N}\sqrt{3})$.*
 *(a) If $3 \nmid N$, then $\chi = 2$.*
 *(b) If $9 \nmid N$ and $3 \mid N$, then*
  *(i) $\chi = 2$ if $N \nmid (u^2 \pm u + 1)$, and*
  *(ii) $\chi = 3$ if $N \mid (u^2 \pm u + 1)$.*
 *(c) If $9 \mid N$, then $\chi = 2$.*

*Outline of the proof.* We begin by decomposing $G(\infty, \frac{u}{N}\sqrt{m})$ into disjoint isomorphic subgraphs using an $H(\sqrt{m})$-invariant equivalence relation on the vertex set. As the chromatic numbers of each decomposed subgraph are equal to the chromatic number of $G(\infty, \frac{u}{N}\sqrt{m})$, we focus on one special subgraph, which we denote by $F(\infty, \frac{u}{N}\sqrt{m})$. We show that each $F(\infty, \frac{u}{N}\sqrt{m})$ is homomorphic to the Farey graph $F(\infty, 1)$, thereby making the graph at most 3-chromatic. Then, we investigate the exact chromatic numbers for each case. ∎

## 2. Main Results

This section is divided into three parts. Subsection 2.1 begins with the decomposition of suborbital graphs $G(\infty, \frac{u}{N}\sqrt{m})$ where $m \mid N$ into disjoint subgraphs. Subsection 2.2 then discusses homomorphisms and shows that all graphs $F(\infty, \frac{u}{N}\sqrt{m})$ are homomorphic to the Farey graph $F(\infty, 1)$. Finally, Subsection 2.3 studies the chromatic numbers of the graphs $G(\infty, \frac{u}{N}\sqrt{m})$.

### 2.1. Graph Decompositions

Our goal is to decompose the suborbital graphs $G(\infty, \frac{u}{N}\sqrt{m})$ where $m \mid N$ into disjoint subgraphs induced by the equivalence classes of $\approx_{N/m}$. Recall that

$$\frac{r}{s}\sqrt{m} \approx_{N/m} \frac{x}{y}\sqrt{m} \iff \begin{cases} \frac{ry-sx}{m} \equiv 0 \pmod{\frac{N}{m}}, & \text{if } m \mid s \text{ and } m \mid y; \\ ry - sx \equiv 0 \pmod{\frac{N}{m}}, & \text{otherwise.} \end{cases}$$

We first verify that every edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ in $G(\infty, \frac{u}{N}\sqrt{m})$ has endpoints in the same class. If $N = m$, then the relation is trivial and there is only one equivalence class. Next, suppose $N > m$. In order to verify the relation $\approx_{N/m}$, we have to know whether or not $m \mid s$ and $m \mid y$ hold simultaneously. We show that $m \mid s$ and $m \mid y$ if Theorem 1.1(G2.1) holds, and $m \nmid s$ or $m \nmid y$ if Theorem 1.1(G2.2) holds.

Suppose Theorem 1.1(G2.1) holds. Then

$$ry - sx = \pm N, \quad x \equiv \pm ur \pmod{N}, \quad y \equiv \pm us \pmod{N}, \text{ and } m \mid s.$$

From $y \equiv \pm us \pmod{N}$, we can write $y = \pm us + Nk$ for some $k \in \mathbb{Z}$. Since $m$ divides both $s$ and $N$, we have $m$ also divides $y$. From $ry - sx = \pm N$, it follows that $\frac{ry-sx}{m} = \pm\frac{N}{m}$. So $\frac{r}{s}\sqrt{m} \approx_{N/m} \frac{x}{y}\sqrt{m}$ and both endpoints belong to the same equivalence class of $\approx_{N/m}$.

On the other hand, suppose Theorem 1.1(G2.2) holds. Then

$$ry - sx = \pm\frac{N}{m}, \quad x \equiv \pm ur \pmod{\frac{N}{m}}, \text{ and } y \equiv \pm us \pmod{N}.$$

Suppose for sake of contradiction that $m \mid s$ and $m \mid y$. Then equality $ry - sx = \pm\frac{N}{m}$ implies that $\frac{ry-sx}{m} = \pm\frac{N}{m^2}$. From the relation $y \equiv \pm us \pmod{N}$, we get $\frac{y}{m} \equiv \pm u\frac{s}{m} \pmod{\frac{N}{m}}$. Then

$$\frac{ry-sx}{m} = r\frac{y}{m} - \frac{s}{m}x \equiv r(\pm u\tfrac{s}{m}) - \tfrac{s}{m}(\pm ur) \equiv 0 \pmod{\frac{N}{m}}.$$

However, this contradicts $\frac{ry-sx}{m} = \pm\frac{N}{m^2}$. Therefore, $m \nmid s$ or $m \nmid y$. Since $ry - sx = \pm\frac{N}{m}$, we have $\frac{r}{s}\sqrt{m} \approx_{N/m} \frac{x}{y}$. So both endpoints are in the same equivalence class of $\approx_{N/m}$.

In either case, any edge in $G(\infty, \frac{u}{N}\sqrt{m})$ where $m \mid N$ has both endpoints contained in the same equivalence class of $\approx_{N/m}$. This means that all subgraphs induced by the equivalence classes of $\approx_{N/m}$ are disjoint. We consider the subgraph of $G(\infty, \frac{u}{N}\sqrt{m})$ induced by the equivalence class $[\infty]_{\approx_{N/m}}$ and denote this subgraph by $F(\infty, \frac{u}{N}\sqrt{m})$.

Next, we will find sufficient and necessary conditions for the existence of an edge in $F(\infty, \frac{u}{N}\sqrt{m})$ where $m \mid N$. Observe that

$$\frac{1}{0}\sqrt{m} \approx_{N/m} \frac{x}{y}\sqrt{m} \iff \begin{cases} \frac{y}{m} \equiv 0 \pmod{\frac{N}{m}}, & \text{if } m \mid y; \\ y \equiv 0 \pmod{\frac{N}{m}}, & \text{otherwise.} \end{cases}$$

If $m^2 \nmid N$, then $\gcd(m, \frac{N}{m}) = 1$ and the relation can be reduced to

$$\frac{1}{0}\sqrt{m} \approx_{N/m} \frac{x}{y}\sqrt{m} \iff y \equiv 0 \pmod{\frac{N}{m}}.$$

On the other hand, if $m^2 \mid N$, then the second case is impossible ($\frac{N}{m} \mid y$ but $m \nmid y$) while the first case is equivalent to $y \equiv 0 \pmod{N}$. In this case, we have

$$\tfrac{1}{0}\sqrt{m} \approx_{N/m} \tfrac{x}{y}\sqrt{m} \iff y \equiv 0 \pmod{N}.$$

As the relation $\approx_{N/m}$ behaves differently depending on whether $m^2$ divides $N$ or not, we consider the subgraphs $F(\infty, \frac{u}{N}\sqrt{m})$ with $m^2 \nmid y$ and with $m^2 \mid y$ separately. From Theorem 1.1(G2.1) and (G2.2), we obtain the following characterization of $F(\infty, \frac{u}{N}\sqrt{m})$ where $m^2 \nmid N$.

**Lemma 2.1.** *If $m \mid N$ and $m^2 \nmid N$, then the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $F(\infty, \frac{u}{N}\sqrt{m})$ if and only if either*

*(F2.1) $ry - sx = \pm N$, $x \equiv \pm ur \pmod{N}$, $N \mid s$, $N \mid y$; or*
*(F2.2) $ry - sx = \pm \frac{N}{m}$, $x \equiv \pm ur \pmod{\frac{N}{m}}$, $\frac{N}{m} \mid s$, $\frac{N}{m} \mid y$, $y \equiv \pm us \pmod{N}$.*

*The signs are the same at every occurence of $\pm$ in the same line.*

*Proof.* From Theorem 1.1, the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $F(\infty, \frac{u}{N}\sqrt{m})$ if and only if $\frac{N}{m} \mid s$ and $\frac{N}{m} \mid y$ and either

(G2.1) $ry - sx = \pm N$, $x \equiv \pm ur \pmod{N}$, $y \equiv \pm us \pmod{N}$, $m \mid s$; or
(G2.2) $ry - sx = \pm \frac{N}{m}$, $x \equiv \pm ur \pmod{\frac{N}{m}}$, $y \equiv \pm us \pmod{N}$.

Since $\gcd(m, \frac{N}{m}) = 1$, the condition that $\frac{N}{m} \mid s$ and $m \mid s$ is equivalent to $N \mid s$. Recall that (G2.1) implies $m \mid y$. Likewise, we can replace the condition that $\frac{N}{m} \mid y$ and $m \mid y$ by $N \mid y$. Consequently, the requirement that $y \equiv \pm us \pmod{N}$ is redundant. Therefore, (G2.1) together with $\frac{N}{m} \mid s$ and $\frac{N}{m} \mid y$ is equivalent to (F2.1). On the other hand, (F2.2) is a restatement of (G2.2) with $\frac{N}{m} \mid s$ and $\frac{N}{m} \mid y$. ∎

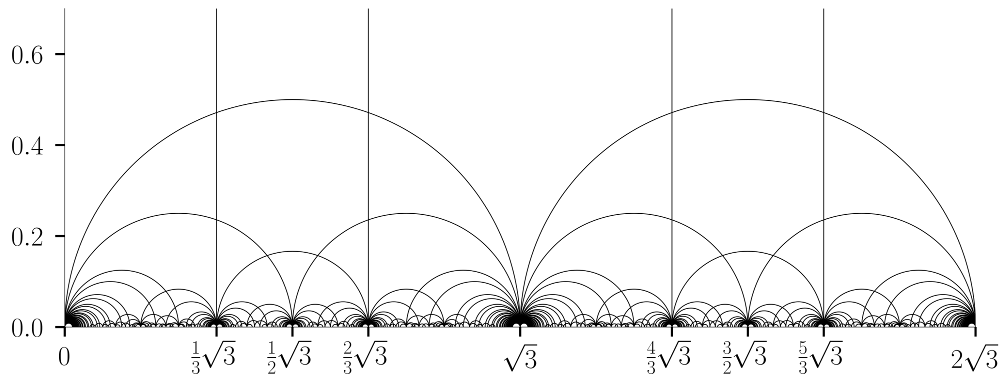


FIGURE 2. $F(\infty, \frac{1}{3}\sqrt{3})$.

An example of a graph $F(\infty, \frac{u}{N}\sqrt{m})$ with $m \mid N$ and $m^2 \nmid N$ is $F(\infty, \frac{1}{3}\sqrt{3})$ which is pictured in Figure 2. Looking at this graph, one may wonder of the possibility of decomposing it further. Indeed, this is the case for graphs $F(\infty, \frac{u}{N}\sqrt{m})$ where $m \mid N$ and $m^2 \nmid N$. Denote the vertex set and the edge set of $F(\infty, \frac{u}{N}\sqrt{m})$ by $V$ and $E$, respectively.

The set $E$ can be partitioned into disjoint sets $E_1$ and $E_2$ where

$$E_1 = \{v \to w \in E : \text{(F2.1) holds for } v \to w\} \text{ and}$$
$$E_2 = \{v \to w \in E : \text{(F2.2) holds for } v \to w\}.$$

Let $F_1(\infty, \frac{u}{N}\sqrt{m})$ and $F_2(\infty, \frac{u}{N}\sqrt{m})$ denote the edge-induced subgraphs of $F(\infty, \frac{u}{N}\sqrt{m})$ by $E_1$ and $E_2$, respectively. The vertex sets of $F_1(\infty, \frac{u}{N}\sqrt{m})$ and $F_2(\infty, \frac{u}{N}\sqrt{m})$, denoted by $V_1$ and $V_2$, respectively, are given by

$$V_1 = \{v \in V : v \to w \text{ is in } E_1 \text{ or } w \to v \text{ is in } E_1 \text{ for some } w \in V\} \text{ and}$$
$$V_2 = \{v \in V : v \to w \text{ is in } E_2 \text{ or } w \to v \text{ is in } E_2 \text{ for some } w \in V\}.$$

Then we have the following result.

**Lemma 2.2.** *Suppose $m \mid N$ and $m^2 \nmid N$. Then $F(\infty, \frac{u}{N}\sqrt{m})$ is the disjoint union of $F_1(\infty, \frac{u}{N}\sqrt{m})$ and $F_2(\infty, \frac{u}{N}\sqrt{m})$ with an edgeless graph.*

*Proof.* The graph $F(\infty, \frac{u}{N}\sqrt{m})$ is the union of $F_1(\infty, \frac{u}{N}\sqrt{m})$ and $F_2(\infty, \frac{u}{N}\sqrt{m})$ with the edgeless graph having vertex set $V - (V_1 \cup V_2)$. We already know that $E_1 \cap E_2 = \emptyset$. It suffices to show that $V_1 \subseteq V - V_2$ which would imply $V_1 \cap V_2 = \emptyset$. Let $\frac{r}{s}\sqrt{m}$ be a vertex in $V_1$. Then it is the case that $N \mid s$ (Lemma 2.1(F2.1)).

If $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is an edge in $E_2$, then $y \equiv \pm us \equiv 0 \pmod N$ and $ry - sx = \pm\frac{N}{m}$ (Lemma 2.1(F2.2)). However, the relations $s \equiv 0 \pmod N$ and $y \equiv 0 \pmod N$ imply that $ry - sx \equiv 0 \pmod N$ contradicting $ry - sx = \pm\frac{N}{m}$.

If $\frac{x}{y}\sqrt{m} \to \frac{r}{s}\sqrt{m}$ is an edge in $E_2$, then $\pm uy \equiv s \equiv 0 \pmod N$ and $ry - sx = \pm\frac{N}{m}$ (Lemma 2.1(F2.2)). Since $\gcd(u, N) = 1$, we have $y \equiv 0 \pmod N$. Similarly, we obtain that $ry - sx \equiv 0 \pmod N$ which contradicts $ry - sx = \pm\frac{N}{m}$.

Therefore, $\frac{r}{s}\sqrt{m}$ is not a vertex in $V_2$. ∎

The decomposition of $F(\infty, \frac{1}{3}\sqrt{3})$ into $F_1(\infty, \frac{1}{3}\sqrt{3})$ and $F_2(\infty, \frac{1}{3}\sqrt{3})$ is shown in Figures 3 and 4. This lemma allows us to study $F_1(\infty, \frac{u}{N}\sqrt{m})$ and $F_2(\infty, \frac{u}{N}\sqrt{m})$ separately because the chromatic number of $F(\infty, \frac{u}{N}\sqrt{m})$ is the maximum of the two disjoint subgraphs.

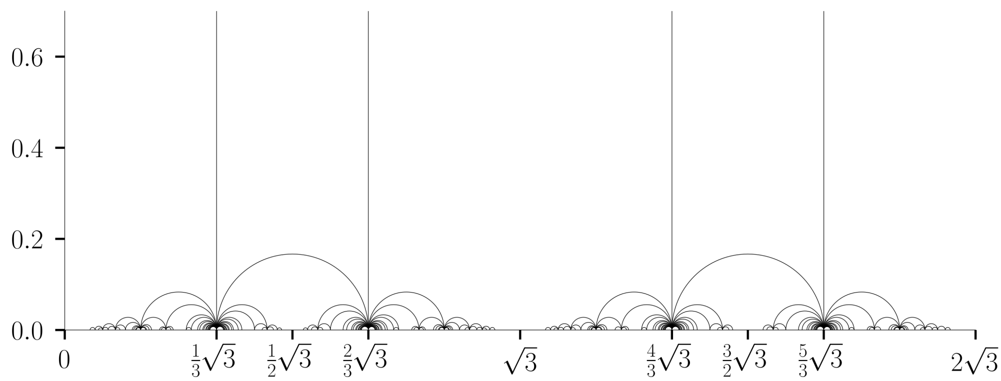


FIGURE 3. $F_1(\infty, \frac{1}{3}\sqrt{3})$.

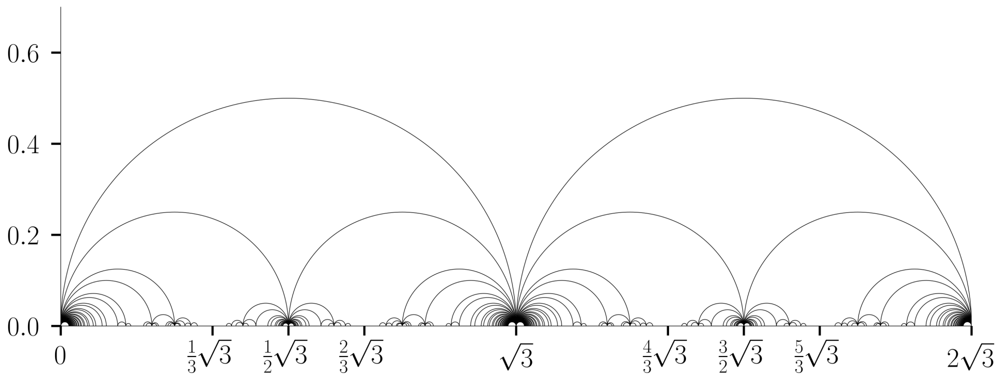


FIGURE 4. $F_2(\infty, \frac{1}{3}\sqrt{3})$.

Next, we find sufficient and necessary conditions for the existence of an edge in the suborbital graphs $F(\infty, \frac{u}{N}\sqrt{m})$ where $m^2 \mid N$. Recall that the equivalence class $[\infty]_{\approx_{N/m}}$ consists of vertices $\frac{x}{y}\sqrt{m}$ where $N \mid y$. One property regarding $F(\infty, \frac{u}{N}\sqrt{m})$ is that they do not contain edges of the second condition. In particular, we have the following lemma.

**Lemma 2.3.** *If $m^2 \mid N$, then the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $F(\infty, \frac{u}{N}\sqrt{m})$ if and only if*

*(F3.1)* $ry - sx = \pm N, \quad x \equiv \pm ur \pmod N, \quad N \mid s, \quad N \mid y.$

*The signs are the same at every occurence of $\pm$ in the same line.*

*Proof.* From Theorem 1.1, the edge $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ is in $F(\infty, \frac{u}{N}\sqrt{m})$ if and only if $N \mid s$ and $N \mid y$ and either

(G2.1) $ry - sx = \pm N, \quad x \equiv \pm ur \pmod N, \quad y \equiv \pm us \pmod N, \quad m \mid s;$ or
(G2.2) $ry - sx = \pm \frac{N}{m}, \quad x \equiv \pm ur \pmod{\frac{N}{m}}, \quad y \equiv \pm us \pmod N.$

As $N \mid s$ implies $m \mid s$, the latter is redundant. Similarly, the condition that $N$ divides both $s$ and $y$ implies the relation $y \equiv \pm us \pmod N$, thereby making it redundant. So can rewrite the conditions as

(F3.1) $ry - sx = \pm N, \quad x \equiv \pm ur \pmod N, \quad N \mid s, \quad N \mid y;$ or
(F3.2) $ry - sx = \pm \frac{N}{m}, \quad x \equiv \pm ur \pmod{\frac{N}{m}}, \quad N \mid s, \quad N \mid y.$

However, (F3.2) never holds because $N$ dividing both $s$ and $y$ implies $ry - sx \equiv 0 \pmod N$ which contradicts $ry - sx = \pm \frac{N}{m}$. ■

Observe that the edge conditions for $F_1(\infty, \frac{u}{N}\sqrt{m})$ (Lemma 2.1(F2.1)) are identical to those for $F(\infty, \frac{u}{N})$ (Theorem 1.3(F1.1)). On the other hand, the conditions for $F_2(\infty, \frac{u}{N}\sqrt{m})$ (Lemma 2.1(F2.2)) are exactly those for $F(\infty, \frac{u}{N/m})$ with an extra condition (Theorem 1.3(F1.2)). As we can identify $\widehat{\mathbb{Q}}$ with $\sqrt{m}\widehat{\mathbb{Q}}$ by mapping $\frac{x}{y} \leftrightarrow \frac{x}{y}\sqrt{m}$, we obtain the following lemma.

**Lemma 2.4.** *If $m \mid N$ and $m^2 \nmid N$, then*

*(1) $F_1(\infty, \frac{u}{N}\sqrt{m})$ is isomorphic to $F(\infty, \frac{u}{N})$, and*
*(2) $F_2(\infty, \frac{u}{N}\sqrt{m})$ is homomorphic to $F(\infty, \frac{u}{N/m})$.*

*If $m^2 \mid N$, then $F(\infty, \frac{u}{N}\sqrt{m})$ is isomorphic to $F(\infty, \frac{u}{N})$.*

In other words, $F_1(\infty, \frac{u}{N}\sqrt{m})$ is the graph $F(\infty, \frac{u}{N})$ scaled by a factor of $\sqrt{m}$ while $F_2(\infty, \frac{u}{N}\sqrt{m})$ is a scaled copy of $F(\infty, \frac{u}{N/m})$ where some edges are removed.

## 2.2. Graph Homomorphisms

In this subsection, we show how some suborbital graphs might be homomorphic to other suborbital graphs. We then extend this result to show that each $F(\infty, \frac{u}{N}\sqrt{m})$ is homomorphic to the Farey graph $F(\infty, 1)$. This tells us that the chromatic number of $F(\infty, \frac{u}{N}\sqrt{m})$ is at most 3. Throughout this subsection, we assume that $m = 2$ or 3. So the statement $\gcd(m, N) = 1$ is equivalent to $m \nmid N$ and we use them interchangeably.

**Lemma 2.5.** *Suppose $m \nmid N$. If $n \mid N$, then $F(\infty, \frac{u}{N}\sqrt{m})$ is homomorphic to $F(\infty, \frac{u}{n}\sqrt{m})$ by the mapping $v \mapsto \frac{N}{n}v$.*

*Proof.* We first verify that $v \mapsto \frac{N}{n}v$ maps $[\infty]_{\approx_N}$ into $[\infty]_{\approx_n}$. Let $\frac{r}{s}\sqrt{m} \in [\infty]_{\approx_N}$ from which it follows that $N \mid s$. We must find the reduced form of $\frac{Nr}{ns}$. Observe that $\frac{Nr}{ns} = \frac{r}{n(s/N)}$. Since $1 \le \gcd(r, n\frac{s}{N}) \le \gcd(r, n\frac{s}{N}\frac{N}{n}) = \gcd(r, s) = 1$, the fraction $\frac{r}{n(s/N)}$ is reduced. Clearly, $n\frac{s}{N} \equiv 0 \pmod{n}$. So, $\frac{Nr}{ns} \in [\infty]_{\approx_n}$.

Next, we verify that the mapping is a homomorphism. Let $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ be an edge in $F(\infty, \frac{u}{N}\sqrt{m})$. Then, by Theorem 1.3(1), either

(F1.1) $ry - sx = \pm N, \quad x \equiv \pm ur \pmod{N}, \quad mN \mid s, \quad N \mid y;$ or
(F1.2) $ry - sx = \pm N, \quad x \equiv \pm mur \pmod{N}, \quad N \mid s, \quad mN \mid y.$

In a way similar to the previous paragraph, we can show that $\frac{r}{n(s/N)}$ and $\frac{x}{n(y/N)}$ are the reduced forms of $\frac{Nr}{ns}$ and $\frac{Nx}{ny}$, respectively. Suppose (F1.1) holds. We can rewrite $mN \mid s$ as $(mn\frac{N}{n}) \mid (n\frac{s}{N}\frac{N}{n})$. It follows that $mn \mid (n\frac{s}{N})$. Next,

$$r(n\tfrac{y}{N}) - (n\tfrac{s}{N})x = \frac{ry - sx}{N/n} = \pm n.$$

Finally, $n \mid N$ and $N \mid (x \mp ur)$ imply $n \mid (x \mp ur)$. So, $x \equiv \pm ur \pmod{n}$. By Theorem 1.3(F1.1), $\frac{Nr}{ns}\sqrt{m} \to \frac{Nx}{ny}\sqrt{m}$ is an edge in $F(\infty, \frac{u}{n}\sqrt{m})$. If (F1.2) holds, a similar proof applies. ■

Observe that the homomorphisms are, in fact, scalings. So, if $\gcd(m, N) = 1$ and $n \mid N$, then $F(\infty, \frac{u}{N}\sqrt{m})$ is a scaled copy of $F(\infty, \frac{u}{n}\sqrt{m})$ where some edges are removed. For instance, the graph $F(\infty, \frac{1}{3}\sqrt{2})$ (Figure 5) is the graph $F(\infty, \sqrt{2})$ (Figure 6) scaled by a factor of $\frac{1}{3}$ with some edges removed.

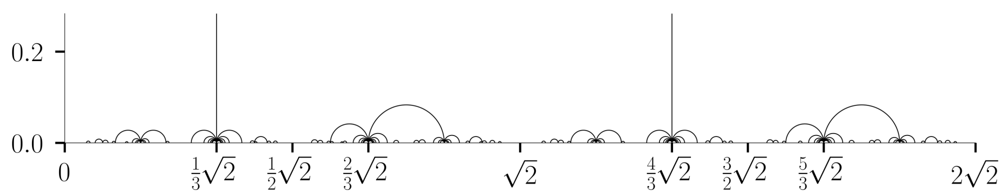


Figure 5. $F(\infty, \frac{1}{3}\sqrt{2})$.

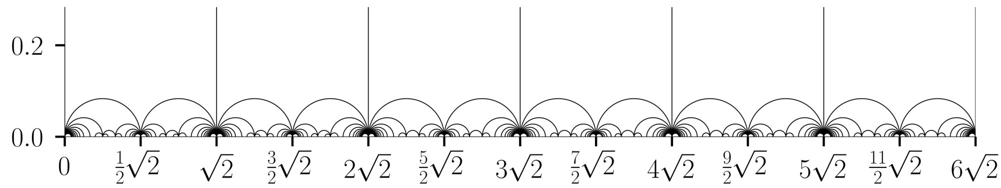


FIGURE 6. $F(\infty, \sqrt{2})$.

**Lemma 2.6.** *$F(\infty, \sqrt{m})$ is homomorphic to $F(\infty, 1)$.*

*Proof.* One can verify directly from Theorem 1.3 that the conditions for $F(\infty, \sqrt{m})$ are those for $F(\infty, 1)$ with extra conditions. ∎

**Theorem 2.1.** *$F(\infty, \frac{u}{N}\sqrt{m})$ is homomorphic to $F(\infty, 1)$.*

*Proof.* If $m \nmid N$, then there are homomorphisms $\varphi_1$ (Lemma 2.5) and $\varphi_2$ (Lemma 2.6) as follows:

$$F(\infty, \tfrac{u}{N}\sqrt{m}) \xrightarrow{\varphi_1} F(\infty, \sqrt{m}) \xrightarrow{\varphi_2} F(\infty, 1).$$

The composition $\varphi_2 \circ \varphi_1$ then is a homomorphism from $F(\infty, \frac{u}{N}\sqrt{m})$ to $F(\infty, 1)$.

If $m \mid N$ and $m^2 \nmid N$, then there are homomorphisms $\varphi_1, \psi_1$ (Lemma 2.4) and $\varphi_2, \psi_2$ (Lemma 1.2) as follows:

$$F_1(\infty, \tfrac{u}{N}\sqrt{m}) \xrightarrow{\varphi_1} F(\infty, \tfrac{u}{N}) \xrightarrow{\varphi_2} F(\infty, 1),$$

$$F_2(\infty, \tfrac{u}{N}\sqrt{m}) \xrightarrow{\psi_1} F(\infty, \tfrac{u}{N/m}) \xrightarrow{\psi_2} F(\infty, 1).$$

Then $\varphi_2 \circ \varphi_1$ is a homomorphism from $F_1(\infty, \frac{u}{N}\sqrt{m})$ to $F(\infty, 1)$, and $\psi_2 \circ \psi_1$ from $F_2(\infty, \frac{u}{N}\sqrt{m})$ to $F(\infty, 1)$. Since $F_1(\infty, \frac{u}{N}\sqrt{m})$ and $F_2(\infty, \frac{u}{N}\sqrt{m})$ share no vertices and contain all edges of $F(\infty, \frac{u}{N}\sqrt{m})$, we can naturally extend $\varphi_2 \circ \varphi_1$ and $\psi_2 \circ \psi_1$ to a homomorphism from $F(\infty, \frac{u}{N}\sqrt{m})$ to $F(\infty, 1)$.

If $m^2 \mid N$, then there are homomorphisms $\varphi_1$ (Lemma 2.4) and $\varphi_2$ (Lemma 1.2) as follows:

$$F(\infty, \tfrac{u}{N}\sqrt{m}) \xrightarrow{\varphi_1} F(\infty, \tfrac{u}{N}) \xrightarrow{\varphi_2} F(\infty, 1).$$

It follows that $\varphi_2 \circ \varphi_1$ is a homomorphism from $F(\infty, \frac{u}{N}\sqrt{m})$ to $F(\infty, 1)$. ∎

**Corollary 2.7.** *$G(\infty, \frac{u}{N}\sqrt{m})$ is at most 3-chromatic.*

Next, we extend Lemma 1.1 to the case $m \mid N$. In particular, we show that $F(\infty, \frac{u}{N}\sqrt{m})$ and $F(\infty, \frac{N-u}{N}\sqrt{m})$ where $m \mid N$ are reflections of each other across the vertical line meeting $\frac{1}{2}\sqrt{m}$.

**Lemma 2.8.** *If $m \mid N$, then $F(\infty, \frac{u}{N}\sqrt{m})$ is isomorphic to $F(\infty, \frac{N-u}{N}\sqrt{m})$ by the mapping $v \mapsto \sqrt{m} - v$.*

*Proof.* There are two cases: $m^2 \nmid N$ or $m^2 \mid N$, both of which can be proved similarly. We give a proof for the former case. If $m^2 \nmid N$, then $F(\infty, \frac{u}{N}\sqrt{m})$ and $F(\infty, \frac{N-u}{N}\sqrt{m})$ have the same vertex set $[\infty]_{\approx_{N/m}}$. Given $\frac{r}{s}\sqrt{m} \in [\infty]_{\approx_{N/m}}$, the fraction $\sqrt{m} - \frac{r}{s}\sqrt{m}$ has the reduced form $\frac{s-r}{s}\sqrt{m}$ which belongs to $[\infty]_{\approx_{N/m}}$. So, $v \mapsto \sqrt{m} - v$ indeed maps $[\infty]_{\approx_{N/m}}$ to itself. Let $\frac{r}{s}\sqrt{m} \to \frac{x}{y}\sqrt{m}$ be an edge in $F(\infty, \frac{u}{N}\sqrt{m})$. Then either

(F2.1) $ry - sx = \pm N$, $x \equiv \pm ur \pmod{N}$, $N \mid s$, $N \mid y$; or
(F2.2) $ry - sx = \pm \frac{N}{m}$, $x \equiv \pm ur \pmod{\frac{N}{m}}$, $\frac{N}{m} \mid s$, $\frac{N}{m} \mid y$, $y \equiv \pm us \pmod{N}$.

We show that $\frac{s-r}{s}\sqrt{m} \to \frac{y-x}{y}\sqrt{m}$ is an edge in $F(\infty, \frac{N-u}{N}\sqrt{m})$. Suppose that (F2.2) holds. Then

$$(s-r)y - s(y-x) = -(ry - sx) = \mp \tfrac{N}{m}$$

and

$$y - x \equiv \mp ur \equiv \mp(N-u)(s-r) \pmod{\tfrac{N}{m}}$$

and

$$y \equiv \pm us \equiv \mp(N-u)s \pmod{N}.$$

The statements $\frac{N}{m} \mid s$ and $\frac{N}{m} \mid y$ are vacuously true. By Lemma 2.1(F2.2), the edge $\frac{s-r}{s}\sqrt{m} \to \frac{y-x}{y}\sqrt{m}$ is in $F(\infty, \frac{N-u}{N}\sqrt{m})$. If (F2.1) holds, a similar proof applies. ∎

As a consequence of Lemmas 1.1 and 2.8, the we obtain the following result. For a real number $x$, we denote the greatest integer not exceeding $x$ by $\lfloor x \rfloor$.

**Corollary 2.9.**

*(1) If $N = 1$, then $F(\infty, \frac{u}{N}\sqrt{m})$ is isomorphic to $F(\infty, \sqrt{m})$.*
*(2) Otherwise, $F(\infty, \frac{u}{N}\sqrt{m})$ is isomorphic to $F(\infty, \frac{v}{N}\sqrt{m})$ for some $1 \leq v \leq \lfloor \frac{N}{2} \rfloor$.*

*Proof.* Recall that we may assume $N \geq 1$. It can be easily seen from Theorem 1.3 and Lemma 2.1 that $F(\infty, \frac{u}{N}\sqrt{m}) = F(\infty, \frac{v}{N}\sqrt{m})$ for $u \equiv v \pmod{N}$. If $N = 1$, choose $v = 1$. Otherwise, choose $v$ such that $0 \leq v < N$. Since $N > 1$, the condition $(v, N) = 1$ forces $v \geq 1$. The corollary then follows from Lemma 1.1 and Lemma 2.8. ∎

## 2.3. Chromatic Numbers

In this subsection, we apply our preceeding results to prove our main theorem. The suborbital graphs $G(\infty, \frac{u}{N}\sqrt{2})$ are all 2-chromatic which follows readily from Theorem 1.2. As for the graphs $G(\infty, \frac{u}{N}\sqrt{3})$, we separate them into three classes: $3 \nmid N$, $3 \mid N$ and $9 \nmid N$, and $9 \mid N$. First, we consider the case $3 \nmid N$.

**Lemma 2.10.** *If $3 \nmid N$, then $G(\infty, \frac{u}{N}\sqrt{3})$ is 2-chromatic.*

*Proof.* If $G(\infty, \frac{u}{N}\sqrt{3})$ is a forest, then the graph is 2-chromatic. Otherwise, $G(\infty, \frac{u}{N}\sqrt{3})$ contains triangles or circuits of length 6 (Theorem 1.2). We show that the graph cannot contain triangles which would imply that it has chromatic number 2.

Suppose there exists a triangle $\frac{r}{s}\sqrt{3} \rightleftharpoons \frac{x}{y}\sqrt{3} \rightleftharpoons \frac{u}{v}\sqrt{3} \rightleftharpoons \frac{r}{s}\sqrt{3}$ in $G(\infty, \frac{u}{N}\sqrt{3})$. This triangle must be in one of the disjoint subgraphs induced by equivalence classes of $[\infty]_{\approx_N}$ Since all subgraphs are isomorphic, we may assume that the triangle is in $F(\infty, \frac{u}{N}\sqrt{3})$. From Theorem 1.3, there must be a pair of adjacent vertices in the triangle with denominators divisible by $mN$. We may assume the pair to be $\frac{r}{s}\sqrt{3}$ and $\frac{x}{y}\sqrt{3}$ since we can always rearrange the circuit. But $mN \mid s$ and $mN \mid y$ implies that $ry - sx \neq N$, contradicting the adjacency of both vertices (Theorem 1.3). ∎

Next, we consider the case $9 \mid N$.

**Lemma 2.11.** *If $9 \mid N$, then $G(\infty, \frac{u}{N}\sqrt{3})$ is 2-chromatic.*

*Proof.* We will consider the subgraph $F(\infty, \frac{u}{N}\sqrt{3})$. Since $9 \mid N$, $F(\infty, \frac{u}{N}\sqrt{3})$ is homomorphic to $F(\infty, \frac{u}{9}\sqrt{3})$. We will show that $F(\infty, \frac{u}{9}\sqrt{3})$ is 2-chromatic.

By Corollary 2.9, we may assume that $1 \leq u \leq 4$. Since $\gcd(u, 9) = 1$, the possible values for $u$ are 1, 2, and 4. By computation, all values of $u$ do not satisfy $N \mid (u^2 \pm u + 1)$. By Theorem 1.2(3), $F(\infty, \frac{u}{9}\sqrt{m})$ is a forest. ∎

Finally, we prove our main theorem. In the remaining case that $3 \mid N$ and $9 \nmid N$, if the graph $G(\infty, \frac{u}{N}\sqrt{3})$ contains a triangle, then it is 3-chromatic. Otherwise, the graph is 2-chromatic.

**Theorem 2.2.**

*(1)* $G(\infty, \frac{u}{N}\sqrt{2})$ *has chromatic number* 2.
*(2) Let* $\chi$ *be the chromatic number of* $G(\infty, \frac{u}{N}\sqrt{3})$.
  *(a) If* $3 \nmid N$, *then* $\chi = 2$.
  *(b) If* $9 \nmid N$ *and* $3 \mid N$, *then*
    *(i)* $\chi = 2$ *if* $N \nmid (u^2 \pm u + 1)$, *and*
    *(ii)* $\chi = 3$ *if* $N \mid (u^2 \pm u + 1)$.
  *(c) If* $9 \mid N$, *then* $\chi = 2$.

*Proof.* The graph $G(\infty, \frac{u}{N}\sqrt{2})$ either is a forest or contains no circuits of odd lengths (Theorem 1.2). So $G(\infty, \frac{u}{N}\sqrt{2})$ is 2-chromatic. Next, we consider the graph $G(\infty, \frac{u}{N}\sqrt{3})$. The cases $3 \nmid N$ and $9 \mid N$ follow from Lemmas 2.10 and 2.11, respectively.

Suppose $3 \mid N$ and $9 \nmid N$. If $N \mid (u^2 \pm u + 1)$, then $G(\infty, \frac{u}{N}\sqrt{3})$ contains a triangle (Theorem 1.2). This and Corollary 2.7 imply that the graph is 3-chromatic. On the other hand, if $N \nmid (u^2 \pm u + 1)$, then $G(\infty, \frac{u}{N}\sqrt{3})$ either is a forest or contains only circuits of length 6 (Theorem 1.2). In either case, the graph is 2-chromatic. ∎

## Acknowledgement

This research was supported by Chiang Mai University.

## References

[1] M. Conder, P. Dobscanyi, Normal subgroups of the modular group and other Hecke groups, In G. Rosenberger, G. Fine, A.M. Gaglione, D. Spellman, Editors, Combinatorial Group Theory, Discrete Groups, and Number Theory, American Mathematical Society (2006), 65–86.

[2] G.A. Jones, D. Singerman, K. Wicks, The modular group and generalized Farey graphs, In C.M Campbell, E.F. Robertson, Editors, Groups St Andrews 1989, London Mathematical Society Lecture Note Series, Vol. 2, Cambridge University Press; Cambridge (1991), 316–338.

[3] R. Keskin, On suborbital graphs for some Hecke groups, Discrete Mathematics 234 (2001) 53–64.

[4] M. Akbas, On suborbital graphs for the modular group, Bulletin of the London Mathematical Society 33 (2001) 647–652.

[5] M. Bhattacharjee, D. Macpherson, R.G. Moller, P.M Neumann, Notes on Infinite Permutation Groups, Texts and Readings in Mathematics, Hindustan Book Agency, 1997.

[6] J. Lauri, R. Scapellato, Topics in Graph Automorphisms and Reconstruction, London Mathematical Society Lecture Note Series, Cambridge University Press; Cambridge, 2016.

[7] P.J. Cameron, Permutation Groups, London Mathematical Society Student Texts, Cambridge University Press; Cambridge, 1999.

[8] W. Tapanyo, P. Jaipong, Chromatic numbers of suborbital graphs for the modular group and the extended modular group, Journal of Mathematics 2017 (2017) Article ID 7458318.