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Related Gauss-Winkler Type Inequality for Fuzzy and Pseudo-Integrals

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Abstract In this paper, we prove generalizations of the related Gauss-Winkler type inequality for fuzzy and pseudo-integrals. Indeed, we prove fuzzy version provided by D. H. Hong, and also state and prove these inequalities for pseudo-integrals.

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1. Introduction

Fuzzy meaures and fuzzy integrals, which were originally introduced by Sugeno in 1974 [1], are important analytical methods of measuring uncertain information [2]. The study of inequalities for Sugeno integrals was initiated by Román-Flores et al. [3–5] and then followed by the authors [6–9].

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subseteq [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ [10–14]. Based on this structure, there were developed the concepts of $\oplus$ measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The advantage of the pseudo-analysis is that there are covers with one theory, and so with unified methods, problem (usually nonlinear and under uncertainty) from many different fields (system theory, optimization, decision making, control theory, differential equation, etc.). Pseudo-analysis uses many mathematical tools from different fields as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc.

The classical Gauss-Winkler inequality provides the following inequality [15]:

$$\left(\int_0^\infty x^2 f(x)d\mu\right)^2 \leq \frac{5}{9}\left(\int_0^\infty f(x)d\mu\right)\left(\int_0^\infty x^4 f(x)d\mu\right), \tag{1.1}$$

where $f : [0, \infty) \to [0, \infty)$ is decreasing. We suppose that all involved integrals exist.

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Example 1.1. Let $f(x) = \dfrac{1}{1+x^6}$. We have

$$\int_0^\infty x^2 f(x)dx = \int_0^\infty \frac{x^2}{1+x^6}dx = \frac{\pi}{6},$$
$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{1+x^6}dx = \frac{\pi}{3},$$
$$\int_0^\infty x^4 f(x)dx = \int_0^\infty \frac{x^4}{1+x^6}dx = \frac{\pi}{3}.$$

Finally

$$\left(\frac{\pi}{6}\right)^2 \leq \frac{5}{9}\left(\frac{\pi}{3}\right)\left(\frac{\pi}{3}\right) \Rightarrow 0.0277 \leq 0.0617.$$

In [16], Dug Hun Hong replaced the condition of decreasing with non-decreasing and reduced the domain of the integrals into the interval $[0,1]$. Then he showed that the classical Gauss-Winkler inequality is not valid for the Sugeno integrals. In the continue, he only obtained the lowest optimal value for the related Gauss-Winkler type inequality for fuzzy version and showed, by an example that, the obtained optimum value in the following Theorems are valid. D. H. Hong showed with an example that the bound is obtained, is optimal. Let's take a look at this example.

Example 1.2 ([16])**.** Let $f(x) = \begin{cases} 0 & x \in [0, 0.4255] \\ 3.1731 & x \in (0.4255, 1] \end{cases}$. A simple calculation shows that

$$\left(⨍_0^1 x^2 f(x)d\mu\right)^2 = \left(⨍_0^1 3.1731x^2 d\mu\right)^2 = (0.5745)^2 = 0.3300,$$

$$⨍_0^1 f(x)d\mu = ⨍_0^{0.4255} 0d\mu \vee ⨍_{0.4255}^1 3.1731d\mu = 0.5745,$$

and

$$⨍_0^1 x^4 f(x)d\mu = ⨍_0^1 3.1731x^4 d\mu = 0.4030.$$

Consequently,

$$0.3300 = \left(⨍_0^1 x^2 f(x)d\mu\right)^2 \approx 1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) = 0.3300.$$

Therfore, the constant 1.4255 is optimal.

In the present paper, we prove related Gauss-Winkler type inequality for fuzzy and pseudo-integrals. This paper is organized as follows: Section 2 contains some of the preliminaries. Section 3 provides generalizations of the related Gauss-Winkler type inequality for fuzzy and pseudo-integrals. Finally, we closed this paper by a conclusion.

2. Preliminary

We denote by $\mathbb{R}$ the set of all real numbers. Let X be a non-empty set and Σ be a $\sigma-$algebra of subsets of X. Throughout this paper, all considered subsets are supposed to be in Σ.

Definition 2.1 (Ralescu and Adams [17]). A set function $\mu : \Sigma \to [0, +\infty)$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$,
(FM2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$,
(FM3) $A_1 \subseteq A_2 \subseteq \ldots \Rightarrow \lim \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$,
(FM4) $A_1 \supseteq A_2 \supseteq \ldots$ and $\mu(A_1) < \infty \Rightarrow \lim \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$.

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space. If f is a non-negative real-valued function on X, we will denote

$$F_\alpha = L_\alpha f = \{x \in X \mid \ f(x) \geq \alpha\} = \{f \geq \alpha\},$$

the α-level of f, for $\alpha > 0$. We know that $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$. If μ is a fuzzy measure on X, we define $\mathfrak{F}^\sigma(X) = \{f : X \to [0, \infty) | \ f \text{ is } \mu - \text{measurable}\}$.

Definition 2.2 (Wang and Klir [18]). Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^\mu(X)$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as

$$⨍_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$ and $\wedge$ denotes the operations *sup* and *inf* on $[0, \infty]$, respectively. In particular, if $A = X$ then

$$⨍_X f d\mu = ⨍ f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following properties of the Sugeno integral can be found in [18].

Proposition 2.3 (Wang and Klir [18]). *Let (X, Σ, μ) be a fuzzy measure space, with $A, B \in \sum$ and $f, g \in \mathfrak{F}^\mu(X)$. We have*

(1) $⨍_A f d\mu \leq \mu(A)$.
(2) $⨍_A k d\mu = k \wedge \mu(A)$, for k non-negative constant.
(3) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Leftrightarrow ⨍_A f d\mu \leq \alpha$.
(4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Leftrightarrow ⨍_A f d\mu \geq \alpha$.
(5) $⨍_{A\cup B} f d\mu \geq ⨍_A f d\mu \vee ⨍_B f d\mu$.

In the following, we are going to review some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals in details, we refer to [18–20].

Let $[a, b]$ be a closed (in some cases can be considered semi-closed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.4 (Wang and Klir [18]). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b], \mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.5 (Wang and Klir [18])**.** The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b]\times[a,b]\to[a,b]$ which is commutative, positively non-decreasing, i.e., $x\preceq y$ implies $x\odot z\preceq y\odot z$ for all $z\in[a,b]_+$, associative and for which there exists a unit element $\mathbf{1}\in[a,b]$, i.e., for each $x\in[a,b]$, $\mathbf{1}\odot x=x$.

We assume also $\mathbf{0}\odot x=\mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x\odot(y\oplus z)=(x\odot y)\oplus(x\odot z)$.

We shall consider the semiring $([a,b],\oplus,\odot)$ for two important (with completely different behavior) cases. The first case is when pseudo-operations are generated by a monotone and continuous function $g:[a,b]\to[0,\infty)$, i.e., pseudo-operations are given with:

$$x\oplus y=g^{-1}(g(x)+g(x))\qquad\text{and}\qquad x\odot y=g^{-1}(g(x)g(x)).\tag{2.1}$$

Then, the pseudo-integral for a function $f:[c,d]\to[a,b]$ reduces on the $g-$integral

$$\int_{[c,d]}^{\oplus}f(x)dx=g^{-1}\left(\int_c^d g(f(x))dx\right).\tag{2.2}$$

More details on this structure as well as corresponding measures and integrals can be found in [11]. The second class is when $x\oplus y=\max(x,y)$ and $x\odot y=g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f:\mathbb{R}\to[a,b]$ is given by

$$\int_{\mathbb{R}}^{\oplus}f\odot dm=\sup_{x\in\mathbb{R}}\left(f(x)\odot\psi(x)\right),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-additine. Then any continuous function $f:[0,\infty]\to[0,\infty]$ the integral $\int^{\oplus}f\odot dm$ can be obtained as a limit of g-integrals.

We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have $m(A)=\text{ess}\sup_{\mu}(x|x\in A)=\sup\{a|\mu(x|x\in A,x>a)>0\}$.

Theorem 2.6 (Mesiar and Pap [20])**.** *Let m be a* sup*-measure on $([0,\infty],\mathbb{B}[0,\infty])$, where $\mathbb{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A)=\text{ess}\sup_{\mu}(\psi(x)|x\in A)$, and $\psi:[0,\infty]\to[0,\infty]$ is a continuous. Then for any pseudo-addition $\oplus$ with a generator g there exists a family m_λ of $\oplus_\lambda$-measure on $([0,\infty],\mathbb{B})$, where $\oplus_\lambda$ is a generated by g^λ (the function g of the power $\lambda,\lambda\in(0,\infty)$) such that $\lim_{\lambda\to\infty}m_\lambda=m$.*

Theorem 2.7 (Mesiar and Pap [20])**.** *Let $([0,\infty],\sup,\odot)$ be a semiring , when $\odot$ is a generated with g, i.e., we have $x\odot y=g^{-1}(g(x)g(y))$ for every $x,y\in(0,\infty)$. Let m be the same as in Theorem 2.6, Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$ -measures, where $\oplus_\lambda$ is a generated by $g^\lambda,\lambda\in(0,\infty)$ such that for every continuous function $f:[0,\infty]\to[0,\infty]$,*

$$\int^{\sup}f\odot dm=\lim_{\lambda\to\infty}\int^{\oplus_\lambda}f\odot dm_\lambda=\lim_{\lambda\to\infty}(g^\lambda)^{-1}\left(\int g^\lambda(f(x))dx\right).\tag{2.3}$$

Recall that, functions $f,g:X\to\mathbb{R}$ are said to be comonotone if for all $x,y\in X$,

$$\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\geq 0,$$

and f and g said to be countermonotone if for all $x,y\in X$,

$$\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\leq 0.$$

The comonotonicity of functions f and g is equivalent to the nonexistence of points $x, y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$. Similarly, if f and g are countermonotone then $f(x) < f(y)$ and $g(x) < g(y)$ cannot happen.

Now, we recall some integral inequalities at the following which are used in the next section.

Theorem 2.8 (Fuzzy Chebyshev's inequality [21]). *Suppose that f, g are two real-valued functions from $[0,1]$ to $[0,\infty)$ and that μ is the Lebesgue measure. If f and g are non-decreasing functions, then the inequality*

$$-\!\!\!\!\!\!\int_0^1 f \cdot g d\mu \leq \left(-\!\!\!\!\!\!\int_0^1 f d\mu\right)\left(-\!\!\!\!\!\!\int_0^1 g d\mu\right), \tag{2.4}$$

holds.

In [22], Xu and Ouyang proved the following lemma.

Lemma 2.9 (Xu and Ouyang [22]). *Let (X, Σ, μ) be a fuzzy measure space, let $A \in \Sigma$ and let $f : X \to \mathbb{R}$ be a measurable function such that $-\!\!\!\!\!\int_A f d\mu \leq 1$. Then for any $s \geq 1$, we have*

$$\left(-\!\!\!\!\!\!\int_A f d\mu\right)^s \leq -\!\!\!\!\!\!\int_A f^s d\mu. \tag{2.5}$$

Theorem 2.10 (Agahi, Mesiar and Ouyang [23]). *Let $u, v : [0,1] \to [a,b]$ be two measurable functions and a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\int_{[0,1]}^{\oplus} (u \odot v) dx \geq \left(\int_{[0,1]}^{\oplus} u dx\right) \odot \left(\int_{[0,1]}^{\oplus} v dx\right),$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

3. Main Results

In this section, as main results, we prove related Gauss-Winkler type inequalities for fuzzy and pseudo-integrals.

3.1. Related Gauss-Winkler Type Inequalities for Fuzzy Integrals

In this part, we are going to prove Gauss-Winkler inequality for Sugeno integral.

Theorem 3.1. (Fuzzy Gauss-Winkler inequality: non-decreasing case). *Let $f : [0,1] \to [0,\infty)$ be a non-decreasing function and μ be the Lebesgue measure on $\mathbb{R}$. Then the inequality*

$$\left(-\!\!\!\!\!\!\int_0^1 x^2 f(x) d\mu\right)^2 \leq 1.4255 \left(-\!\!\!\!\!\!\int_0^1 f(x) d\mu\right)\left(-\!\!\!\!\!\!\int_0^1 x^4 f(x) d\mu\right), \tag{3.1}$$

holds.

Proof. Since 1.4255 is greater than 1, it can be written:

$$1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) \geq \left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right).$$

Now, from (2.4), we have:

$$\begin{aligned} 1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) &\geq \left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 d\mu\right)\left(⨍_0^1 f(x)d\mu\right) \\ &= \left(⨍_0^1 f(x)d\mu\right)^2\left(⨍_0^1 (x^2)^2 d\mu\right). \end{aligned}$$

Now by using (2.5), we get:

$$\begin{aligned} 1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) &\geq \left(⨍_0^1 f(x)d\mu\right)^2\left(⨍_0^1 x^2 d\mu\right)^2 \\ &= \left(\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^2 d\mu\right)\right)^2. \end{aligned}$$

Applying (2.4), it follows that

$$1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) \geq \left(⨍_0^1 f(x)\cdot x^2 d\mu\right)^2.$$

Thereby, the theorem is proved. ∎

Theorem 3.2. (Fuzzy Gauss-Winkler inequality: non-increasing case). *Let $f:[0,1]\to[0,\infty)$ be a non-increasing function and μ be the Lebesgue measure on $\mathbb{R}$. Then the inequality*

$$\left(⨍_0^1 (1-x)^2 f(x)d\mu\right)^2 \leq 1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 (1-x)^4 f(x)d\mu\right), \tag{3.2}$$

holds.

Proof. The proof is similar to the proof of the Theorem 3.1. ∎

In the following, we present an example to illustrate the validity of Theorem 3.1.

Example 3.3. Let $f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} < x \leq 1 \end{cases}$ and μ be the Lebesgue measure on $\mathbb{R}$. A straightforward calculus shows that $⨍_0^1 f(x)d\mu = 0.5$, $⨍_0^1 x^2 f(x)d\mu = 0.267$ and $⨍_0^1 x^4 f(x)d\mu = 0.202$. Consequently

$$\begin{aligned} 0.071 = \left(⨍_0^1 x^2 f(x)d\mu\right)^2 &\leq 1.4255\left(⨍_0^1 f(x)d\mu\right)\left(⨍_0^1 x^4 f(x)d\mu\right) \\ &= 1.4255(0.5)(0.202) = 0.144. \end{aligned}$$

3.2. Related Gauss-Winkler Type Inequalities for Pseudo-Integrals

Now, we state and prove generalizations of the Theorems 3.1 and 3.2 for pseudo integrals.

Theorem 3.4. (Gauss-Winkler type inequality for pseudo-integrals: non-decreasing case). *Let $f : [0,1] \to [a,b]$ be a non-decreasing and measurable function and a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function and let μ be a Lebesgue measure. Then the inequality*

$$\left(\int_{[0,1]}^{\oplus} (x^2 \odot f(x))d\mu\right)^2 \leq \left(\int_{[0,1]}^{\oplus} f(x)d\mu\right) \odot \left(\int_{[0,1]}^{\oplus} (x^4 \odot f(x))d\mu\right), \tag{3.3}$$

holds.

Proof. Set $\alpha = \left(\int_{[0,1]}^{\oplus} f(x)d\mu\right) \odot \left(\int_{[0,1]}^{\oplus} (x^4 \odot f(x))d\mu\right)$. From (2.1), we have

$$\begin{aligned}
\alpha &= \left(\int_{[0,1]}^{\oplus} f(x)d\mu\right) \odot \left(\int_{[0,1]}^{\oplus} (x^4 \odot f(x))d\mu\right) \\
&= \left(g^{-1}\int_0^1 g(f(x))dx\right) \odot \left(g^{-1}\int_0^1 g(x^4 \odot f(x))dx\right) \\
&= \left(g^{-1}\int_0^1 g(f(x))dx\right) \odot \left(g^{-1}\int_0^1 g\left(g^{-1}(g(x^4)g(f(x))\right)dx\right) \\
&= \left(g^{-1}\int_0^1 g(f(x))dx\right) \odot \left(g^{-1}\int_0^1 \left(g(x^4)g(f(x))\right)dx\right) \\
&= g^{-1}\left(g\left(g^{-1}\int_0^1 g(f(x))dx\right) g\left(g^{-1}\int_0^1 \left(g(x^4)g(f(x))\right)dx\right)\right) \\
&= g^{-1}\left(\int_0^1 g(f(x))dx \int_0^1 \left(g(x^4)g(f(x))\right)dx\right).
\end{aligned}$$

Since $g(x^4)$ and $g(f(x))$ are comonotone, so by using Chebyshev's inequality, we have

$$\begin{aligned}
\alpha &\geq g^{-1}\left[\int_0^1 g(f(x))dx \int_0^1 g(x^4)dx \int_0^1 g(f(x))dx\right) \\
&= g^{-1}\left[gg^{-1}\left(\int_0^1 g(f(x))dx\right) gg^{-1}\left(\int_0^1 g(x^4)dx\right) gg^{-1}\left(\int_0^1 g(f(x))dx\right)\right) \\
&= g^{-1}\left[g\left(\int_{[0,1]}^{\oplus} f(x)dx\right) g\left(\int_{[0,1]}^{\oplus} x^4dx\right) g\left(\int_{[0,1]}^{\oplus} f(x)dx\right)\right) \\
&= \int_{[0,1]}^{\oplus} f(x)dx \odot \int_{[0,1]}^{\oplus} x^4dx \odot \int_{[0,1]}^{\oplus} f(x)dx.
\end{aligned}$$

Using Theorem 2.10, we obtain that

$$\alpha \geq \int_{[0,1]}^{\oplus} \left(f(x) \odot x^4 \odot f(x)\right)dx. \tag{3.4}$$

Now by using continuity and commutativity $\odot$, we get

$$\begin{aligned}\alpha &\geq \int_{[0,1]}^{\oplus} \left((x^2)^2 \odot (f(x))^2\right) d\mu \\ &= \int_{[0,1]}^{\oplus} \left(x^2 \odot f(x)\right)^2 d\mu. \end{aligned} \tag{3.5}$$

Applying Theorem 2.9, we get

$$\alpha \geq \left(\int_{[0,1]}^{\oplus} (x^2 \odot f(x)) d\mu\right)^2.$$

Hence we have

$$\left(\int_{[0,1]}^{\oplus} (x^2 \odot f(x)) d\mu\right)^2 \leq \left(\int_{[0,1]}^{\oplus} f(x) d\mu\right) \odot \left(\int_{[0,1]}^{\oplus} (x^4 \odot f(x)) d\mu\right).$$

Therefore we get the desired result. ∎

In the following, we present an example to illustrate the validity of Theorem 3.4.

Example 3.5. Let $f : [0,1] \to [a,b]$ be a non-decreasing and measurable function and let $g(x) = x^\alpha$ for some $\alpha \in (0,\infty)$. The corresponding pseudo-operations are

$$x \oplus y = g^{-1}\left(g(x) + g(y)\right) = \sqrt[\alpha]{x^\alpha + y^\alpha},$$
$$x \odot y = g^{-1}\left(g(x)g(y)\right) = \sqrt[\alpha]{x^\alpha y^\alpha} = xy.$$

Now, we have

$$\left(\int_{[0,1]}^{\oplus} (x^2 f(x))^\alpha dx\right)^{\frac{1}{\alpha}} \leq \left(\int_{[0,1]}^{\oplus} (f(x))^\alpha d\mu\right)^{\frac{1}{\alpha}} \left(\int_{[0,1]}^{\oplus} (x^4 f(x))^\alpha d\mu\right)^{\frac{1}{\alpha}}.$$

Theorem 3.6. (Gauss-Winkler type inequality for pseudo-integrals: non-increasing case). *Let $f : [0,1] \to [a,b]$ be a non-increasing and measurable function and let a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an decreasing function and let μ be a Lebesgue measure. Then the inequality*

$$\left(\int_{[0,1]}^{\oplus} ((1-x)^2 \odot f(x)) d\mu\right)^2 \leq \left(\int_{[0,1]}^{\oplus} f(x) d\mu\right) \odot \left(\int_{[0,1]}^{\oplus} ((1-x)^4 \odot f(x)) d\mu\right),$$

holds.

Proof. The proof is similar to the proof of the Theorem 3.4. ∎

In the sequal, we generalize the Gauss-Winkler type inequality by the semiring $([a,b], \max, \odot)$, where $\odot$ is generated.

Theorem 3.7. *Let $f : [0,1] \to [a,b]$ be a non-decreasing, measurable and continuous function and $\odot$ be represented by an increasing multiplicative generator g. Let m be the same as in Theorem 2.6. Then the inequality*

$$\left(\int_{[0,1]}^{\sup} (x^2 \odot f(x)) \odot dm\right)^2 \leq \left(\int_{[0,1]}^{\sup} f(x) \odot dm\right) \odot \left(\int_{[0,1]}^{\sup} (x^4 \odot f(x)) \odot dm\right), \tag{3.6}$$

holds.

Proof. Set $\alpha = \left(\int_{[0,1]}^{\sup} f(x)\odot dm\right)\odot\left(\int_{[0,1]}^{\sup}(x^4\odot f(x))\odot dm\right)$. From (2.3), we have

$$\begin{aligned}
\alpha &= \left(\int_{[0,1]}^{\sup} f(x)\odot dm\right)\odot\left(\int_{[0,1]}^{\sup}(x^4\odot f(x))\odot dm\right)\\
&= \left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} f(x)\odot dm_\lambda\right)\odot\left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda}(x^4\odot f(x))\odot dm_\lambda\right)\\
&= \left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)\odot\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(x^4\odot f(x)\right)dx\right)\right)\\
&= \left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)\odot\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(\left(g^\lambda\right)^{-1}\left(g^\lambda(x^4)g^\lambda(f(x))\right)\right)dx\right)\\
&= \left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)\odot\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1\left(g^\lambda(x^4)g^\lambda(f(x))\right)dx\right)\right)\\
&= \left(g^\lambda\right)^{-1}\left\{g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1\left(g^\lambda(x^4)g^\lambda(f(x))\right)dx\right)\right)\right\}.
\end{aligned}$$

Set $\beta=\left(g^\lambda\right)^{-1}\left\{g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1\left(g^\lambda(x^4)g^\lambda(f(x))\right)dx\right)\right)\right\}$.

From Theorem 2.10, comonotonicity $g^\lambda(x^4)$ and $g^\lambda(f(x))$, and above relations, we obtain that

$$\begin{aligned}
\beta &\ge \left(g^\lambda\right)^{-1}\left\{g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1\left(g^\lambda(x^4)g^\lambda(f(x))\right)dx\right)\right)\right\}\\
&= \left(g^\lambda\right)^{-1}\left\{g^\lambda\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f(x))\right)dx\right)\right.\\
&\qquad \left. g^\lambda\left(\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\int_0^1 g^\lambda(x^4)dx\right)\left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\int_0^1 g^\lambda(f(x))dx\right)\right)\right\}\\
&= \left(g^\lambda\right)^{-1}\left\{g^\lambda\left(\int_{[0,1]}^{\sup} f(x)dx\right)g^\lambda\left(\int_{[0,1]}^{\sup} x^4dx\int_{[0,1]}^{\sup} f(x)dx\right)\right\}\\
&= \left(g^\lambda\right)^{-1}\left\{g^\lambda\left(\int_{[0,1]}^{\sup} f(x)dx\right)g^\lambda\left(\int_{[0,1]}^{\sup} x^4dx\right)g^\lambda\left(\int_{[0,1]}^{\sup} f(x)dx\right)\right\}\\
&= \int_{[0,1]}^{\sup}\left(f(x)\odot x^4\odot f(x)\right)dx.
\end{aligned}$$

Using continuity and commutativity of $\odot$, we have

$$\begin{aligned}
\left(\int_{[0,1]}^{\sup} f(x)\odot dm\right)\odot\left(\int_{[0,1]}^{\sup}(x^4\odot f(x))\odot dm\right) &\ge \int_{[0,1]}^{\sup}(x^4\odot f^2(x))\odot dm\\
&= \int_{[0,1]}^{\sup}\left(x^2\odot f(x)\right)^2\odot dm\\
&\ge \left(\int_{[0,1]}^{\sup}(x^2\odot f(x))\odot dm\right)^2.
\end{aligned}$$

The proof is now complete. ■

Example 3.8. Let $f : [0,1] \to [a,b]$ be a non-decreasing and measurable function, $g^{\lambda}(x) = e^{\lambda x}$ and $\psi(x)$ be the same as in Theorem 2.6. Then

$$x \oplus y = \lim_{\lambda\to\infty} \left(\frac{1}{\lambda} \ln \left(e^{\lambda x} + e^{\lambda y} \right) \right) = \max(x,y),$$
$$x \odot y = \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln \left(e^{\lambda x} e^{\lambda y} \right) = x + y.$$

Therefore (3.6) reduces on the following inequality

$$\sup \left(x^2 f(x) + \psi(x) \right)^2 \le \Big(\sup \left(f(x) + \psi(x) \right) \Big) \Big(\sup \left(x^4 f(x) + \psi(x) \right) \Big).$$

Theorem 3.9. *Let $f : [0,1] \to [a,b]$ be a non-increasing, measurable and continuous function and $\odot$ be represented by a decreasing multiplicative generator g. Let m be the same as in Theorem 2.6. Then the inequality*

$$\left(\int_{[0,1]}^{\sup} ((1-x)^2 \odot f(x)) \odot dm \right)^2 \le \left(\int_{[0,1]}^{\sup} f(x) \odot dm \right) \odot \left(\int_{[0,1]}^{\sup} ((1-x)^4 \odot f(x)) \odot dm \right),$$

holds.

Proof. Using the same arguments in Theorem 3.7, the proof is obvious. ■

Example 3.10. Let $g^{\lambda}(x) = x^{-\lambda}$. We have

$$x \oplus y = (x^{-\lambda} + y^{-\lambda})^{-1/\lambda} \qquad \text{and} \qquad x \odot y = xy.$$

Therefore (3.7) reduces on the following inequality:

$$\sup((1-x)^2 f(x) + \psi(x))^2 \le (\sup(f(x) + \psi(x)))(\sup((1-x)^4 f(x) + \psi(x))),$$

where ψ is the same as in Theorem 2.6.

4. Conclusion

We have proved the related Gauss-Winkler type inequality for fuzzy and pseudo-integrals. More precisely, we show that

$$\left(⨍_0^1 x^2 f(x) d\mu \right)^2 \le 1.4255 \left(⨍_0^1 f(x) d\mu \right) \left(⨍_0^1 x^4 f(x) d\mu \right),$$

holds where $f : [0,1] \to [0,\infty)$ is a non-decreasing function and μ is the Lebesgue measure on $\mathbb{R}$ and also

$$\left(\int_{[0,1]}^{\oplus} (x^2 \odot f(x)) d\mu \right)^2 \le \left(\int_{[0,1]}^{\oplus} f(x) d\mu \right) \odot \left(\int_{[0,1]}^{\oplus} (x^4 \odot f(x)) d\mu \right),$$

holds where $f : [0,1] \to [a,b]$ is a non-decreasing function and μ is the Lebesgue measure on $\mathbb{R}$.

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