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Interpolative Ćirić-Reich-Rus Type Contractions in b-Metric Spaces

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Abstract In this paper, we analyze the notion of interpolative Ćirić-Reich-Rus type contractions in b-metric spaces to identify the fixed point and attempts to prove certain fixed point results under such mappings. In addition to this the manuscript also has an illustration to substantiate the results achieved.

MSC: 47H10; 54H25

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1. Introduction

In 1906, Fréchet introduced the concept of metric spaces. Later, many authors strive to generalize the concept of metric like quasimetric, semimetric and ultrametric, G-metric etc. Non Hausdroff spaces are widely used in domain theory, optimization theory and advanced computer sciences to construct a most complicated program. Some problems, particularly the problem of convergence of measurable functions with respects to measure lead to a generalization of notion of metric. In 1998, Czerwik [1] introduced the concept of b-metric space in which the triangle inequality is controlled by a fixed constant, as a generalization of notion of metric space. Erdal Karapinar [2] established interpolative contraction to prove existence of fixed points in Metric space. In [2], For a metric space (X, d), the self mapping $T : X \to X$ is said to be an interpolative kannan type contraction, if there are constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)],^{1-\alpha}$$

for all $x, y \in X$ with $x \neq Tx$.
Erdal Karapinar et al. [2] investigate interpolative Reich-Rus-Ćirić type contractions on partial metric space and a complete metric space. Since then, this concept has been studied by many authors, see for instance [3–7]. In this paper we generalize the celebrated fixed point theorem of interpolative Ćirić-Reich-Rus to prove the existence of fixed points in the framework of a complete b-metric space. For more studies of fixed point results for

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contraction refer [6–14] and reference therein.
Next, we will recall the basic notions of a b-metric space and contraction. The following notion will be used in the presentation.

Definition 1.1. [1] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a *b-metric* if it satisfies the following properties:

(1) $d(x, y) = 0 \iff x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

Then the triplet (X, d, s) is called a *b-metric space* with coefficient s.

Example 1.2. Let $X = [0, 2]$ and $d : X \times X \to [0, \infty)$ be defined by

$$\text{d(x,y)}= \begin{cases} (x-y)^2, & x, y \in [0,1], \\ |\frac{1}{x^2} - \frac{1}{y^2}|, & x, y \in [1,2], \\ |x-y|, & \text{otherwise.} \end{cases}$$

Then (X, d, s) is a b-metric space with $s = 2$.

Example 1.3. Let $X = \{1, 2, 3, 4\}$, Define $d : X \times X \to [0, \infty)$ as follows:
$d(n, n) = 0, \quad n = 1, 2, 3, 4;$
$d(1, 2) = d(2, 1) = 2; \qquad d(2, 3) = d(3, 2) = \frac{1}{2}; \qquad d(1, 3) = d(3, 1) = 1;$
$d(1, 4) = d(4, 1) = \frac{3}{2}; \qquad d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 3.$
then d is a b-metric space with $s = 2$.

The class of b-metric spaces is larger than that of metric spaces as there are b-metric spaces which are not a metric space, and a metric space is a b-metric space with coefficient $s = 1$. Moreover, the notion of convergent sequence, Cauchy sequence, completeness, etc. can as well be defined accordingly in b-metric spaces.

Theorem 1.4. *In a complete b-metric space (X, d, s), if $T : X \to X$ forms a Ćirić-Reich-Rus contraction mapping*

$$\frac{1}{s} d(Tx, Ty) \leq \lambda[d(x, y) + d(x, Tx) + d(y, Ty)], \tag{1.1}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{3})$, then T possesses a fixed point.

Definition 1.5. In a complete b-metric space (X, d, s), a mapping $T : X \to X$ is called an *interpolative Ćirić-Reich-Rus type contraction*, if there are constants $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ such that

$$\frac{1}{s} d(Tx, Ty) \leq \lambda[d(x, y)]^{\beta}[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha-\beta} \tag{1.2}$$

for all $x, y \notin Fix_T(X)$, where $Fix_T(X)$ denotes the set of all fixed points of T.

Definition 1.6. Let (X, d, s) be a complete b-metric space, $T : X \to X$ form Reich contractions, if it satisfies

$$d(Tx, Ty) \leq s[a[d(x, y)] + b[d(x, Tx)] + c[d(y, Ty)]] \tag{1.3}$$

there exists $a, b, c \in (0, \infty)$ such that $0 \leq a + b + c < 1$.

Definition 1.7. Let (X, d) be a complete b-metric space, if $T : X \to X$ form interpolative Reich contractions, if there exists $a, b, c \in (0, \infty)$ such that $0 \leq a + b + c < 1$, then

$$d(Tx, Ty) \leq a[d(x, y)]^{\beta}.b[d(x, Tx)]^{\alpha}.c[d(y, Ty)]^{1-\alpha-\beta} \tag{1.4}$$

where $\alpha, \beta \in (0, 1)$.

2. Main Results

In this section, we will discuss an interpolative Ćirić-Reich-Rus-type contraction and an interpolative Reich type contraction to prove existence of fixed points on a b-metric space.

Theorem 2.1. *Suppose a self mapping $T : X \to X$ is an interpolative Ćirić-Reich-Rus-type contraction on a complete b-metric space (X, d, s). Then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point, construct a sequence $\{x_n\}$ such that
$x_n = T^n(x_0), n \geq 0$.
If, $x_n \neq x_{n+1}$, for each $n \geq 0$.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \lambda[d(x_n, x_{n-1})]^{\beta}.[d(x_n, Tx_n)]^{\gamma}.[d(x_{n-1}, Tx_{n-1})]^{1-\gamma-\beta} \\ &= \lambda[d(x_n, x_{n-1})]^{\beta}[d(x_n, x_{n+1})]^{\gamma}.[d(x_{n-1}, x_n)]^{1-\gamma-\beta} \\ &= \lambda[d(x_n, x_{n+1})]^{\gamma}.[d(x_{n-1}, x_n)]^{1-\gamma} \\ d(x_{n+1}, x_n)^{1-\gamma} &\leq \lambda[d(x_{n-1}, x_n)].^{1-\gamma} \end{aligned}$$

Above inequality shows that $\{d(x_{n-1}, x_n)\}$ is a non increasing sequence. Now, we prove $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$.
Suppose that $\lim_{n\to\infty} d(x_{n-1}, x_n) = l$, where $l \geq 0$.
Consider,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda d(x_{n-1}, x_n) \\ &\leq \lambda^n d(x_0, x_1). \end{aligned}$$

Hence $d(x_n, x_{n+1}) = 0$ when $n \to \infty$ because $\lambda < 1$.

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.1}$$

Next we will prove that $\{x_n\}$ is a b-Cauchy sequence in (X, d, s).
We have $\lim_{n\to\infty} d(x_n, x_{n+1}) = s[d(x_n, y) + d(y, x_{n+1})] = 0 \implies \lim_{n\to\infty} d(x_n, x_n) = 0$.
Suppose that k is the smallest integer which satisfies above equation such that

$$d(x_{l_k-1}, x_{n_k}) < \epsilon.$$

By the definition of a b-metric space,
$\epsilon \leq d(x_{l_k}, x_{n_k}) \leq s[d(x_{l_k}, x_{l_k-1}) + d(x_{l_k-1}, x_{n_k})] < sd(x_{l_k}, x_{l_k-1}) + s\epsilon$.
Thus

$$\lim_{k\to\infty} d(x_{l_k}, x_{n_k}) = \epsilon,$$

which means

$$\lim_{k\to\infty} (d(x_{l_k}, x_{n_k}) - d(x_{l_{k-1}}, x_{n_{k-1}}) = s\epsilon.$$

By definition

$$d(x_{l_k}, x_{n_k}) \leq s[d(x_{l_k}, x_{l_k+1}) + d(x_{l_k+1}, x_{n_k})] + d(x_{n_k+1}, x_{n_k}),$$

and

$$d(x_{l_k+1}, x_{n_k+1}) \leq d(x_{l_k}, x_{l_k+1}) + d(x_{l_k}, x_{n_k}) + d(x_{n_k+1}, x_{n_k}).$$

From the above equations we can conclude that

$$d(x_{n_k+1}, x_{n_k}) \leq d(x_{l_k}, x_{l_k+1}), \tag{2.2}$$

taking the limit as $k \to \infty$, together with (2.1) and (2.2) we have

$$\lim_{k\to\infty} d(x_{l_k+1}, x_{n_k+1}) = \epsilon.$$

Then, there exists $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have

$$d(x_{l_k}, x_{n_k}) > \frac{\epsilon}{2} \text{ and } d(x_{l_k+1}, x_{n_k+1}) > \frac{\epsilon}{2} > 0.$$

Since T is continuous, we have from the above argument,

$$d(Tx_{l_k}, Tx_{n_k}) \leq \lambda\Big([d(x_{l_k}, x_{n_k})]^{\beta}[d(x_{l_k}, Tx_{l_k})]^{\gamma}[d(x_{n_k}, Tx_{n_k})]^{1-\gamma-\beta}\Big),$$

when $k \to \infty$,

$$\begin{aligned} d(Tx_{l_k}, Tx_{n_k}) &\leq \lambda(\epsilon) \\ &< \epsilon. \end{aligned}$$

This implies that $d(Tx_{l_k}, Tx_{n_k}) < \epsilon$ which is a contradiction, and therefore $\{x_n\}$ is a b-Cauchy sequence. Regarding the completeness of the b-metric space (X, d, s), we deduce that there is some $x \in X$ so that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Since T is continuous, we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\Big(\lim_{n\to\infty} x_n\Big) = Tx$.
Hence the theorem. ∎

Theorem 2.2. *Suppose a self mapping $T : X \to X$ be an interpolative Reich-type contraction on a complete b-metric space (X, d, s). Then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point, construct a sequence $\{x_n\}$ such that $x_n = T^n(x_0), n \geq 0$.
If for some n_0, we have $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T, which ends the proof. Otherwise, $x_n \neq x_{n+1}$, for each $n \geq 0$.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq a[d(x_n, x_{n-1})]^{\beta}.b[d(x_n, x_{n+1})]^{\alpha}.c[d(x_{n-1}, x_n)]^{1-\alpha-\beta} \\ d(x_{n+1}, x_n)^{1-\alpha} &\leq a.b.c[d(x_{n-1}, x_n)]^{1-\alpha}. \end{aligned}$$

Since $a + b + c < 1$ then $abc < 1$,
then

$$d(x_{n+1}, x_n)^{1-\alpha} < d(x_{n-1}, x_n)^{1-\alpha}. \tag{2.3}$$

From (2.3) we conclude that $d(x_{n+1}, x_n) < d(x_{n-1}, x_n)$.
The sequence $\{d(x_{n-1}, x_n)\}$ forms a non increasing sequence.
Now, we will prove that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

Suppose that $\lim_{n\to\infty} d(x_{n-1}, x_n) = l$, where $l \geq 0$.
We have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq abc[d(x_{n-1}, x_n)] \\ &\leq (abc)^n d(x_0, x_1). \end{aligned}$$

Hence $d(x_n, x_{n+1}) = 0$ when $n \to \infty$ because $abc < 1$.

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.4}$$

We claim that $\{x_n\}$ is a d-Cauchy sequence in (X, d, s).
We have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
$0 \leq d_{x_n x_{n+1}} \leq d(x_n, x_{n+1}) \implies \lim_{n\to\infty} d(x_n, x_n) = 0$.
Suppose that k is the smallest integer which satisfies above equation such that

$$d(x_{l_k - 1}, x_{n_k}) < \epsilon.$$

By the definition of a b-metric space,
$\epsilon \leq d(x_{l_k}, x_{n_k}) \leq s[d(x_{l_k}, x_{l_k-1}) + d(x_{l_k-1}, x_{n_k})] < sd(x_{l_k}, x_{l_k-1}) + s\epsilon$,
thus

$$\lim_{k\to\infty} d(x_{l_k}, x_{n_k}) = \epsilon.$$

By definition

$$d(x_{l_k}, x_{n_k}) \leq s[d(x_{l_k}, x_{l_k+1}) + d(x_{l_k+1}, x_{n_k}) + d(x_{n_k+1}, x_{n_k})],$$

and

$$d(x_{l_k+1}, x_{n_k+1}) \leq s[d(x_{l_k}, x_{l_k+1}) + d(x_{l_k}, x_{n_k}) + (x_{n_k+1}, x_{n_k})].$$

From the above equations we can conclude that

$$d(x_{n_k+1}, x_{n_k}) \leq d(x_{l_k}, x_{l_k+1}), \tag{2.5}$$

taking the limit as $k \to \infty$, together with (2.4) and (2.5) we have

$$\lim_{k\to\infty} d(x_{l_k+1}, x_{n_k+1}) = \epsilon.$$

Then, there exists $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ we have

$$d(x_{l_k}, x_{n_k}) > \frac{\epsilon}{2} \text{ and } d(x_{l_k+1}, x_{n_k+1}) > \frac{\epsilon}{2} > 0.$$

From the above argument,

$$d(Tx_{l_k}, Tx_{n_k}) \leq a[d(x_{l_k}, x_{n_k})]^\beta . b[d(x_{l_k}, Tx_{l_k})]^\gamma c.[d(x_{n_k}, Tx_{n_k})]^{1-\gamma-\beta}$$

when $k \to \infty$,

$$\begin{aligned} d(Tx_{l_k}, Tx_{n_k}) &\leq abc(\epsilon) \\ &< \epsilon \end{aligned}$$

leads a contradiction, which implies that $\{x_n\}$ is a b-Cauchy sequence. Regarding the completeness of the b-metric space (X, d, s), we deduce that there is some $x \in X$ so that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Since T is continuous, we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\Big(\lim_{n\to\infty} x_n\Big) = Tx$.
Hence the theorem. ∎

Corollary 2.3. *In a complete metric space (X, d), if $T : X \to X$ is an interpolative Ćirić-Reich-Rus-type contraction, that is,*

$$d(Tx, Ty) \leq \lambda[d(x, y)]^\beta [d(x, Tx)]^\alpha [d(y, Ty)]^{1-\alpha-\beta}$$

for all $x, y \notin Fix_T(X)$, then T possesses a fixed point in X.

Corollary 2.4. *Let (X, d, s) be a complete b-metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq s\lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha}$$

for all $x, y \notin Fix_T(X)$, where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$. Then, T possesses a fixed point in X.

Proof is similar to Theorem 2.1.

3. Applications

We scrutinize Theorem 2.1, using the following examples.

Example 3.1. Let $X = \{1, 2, 3, 4\}$ be a set endowed with a b-metric defined as,

$d(x,y)$	1	2	3	4
1	0	2	$\frac{1}{3}$	$\frac{3}{2}$
2	2	0	$\frac{1}{2}$	3
3	1	$\frac{1}{2}$	0	3
4	$\frac{3}{2}$	3	3	0

We define a self mapping T on X by T: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}$

T is not a Ćirić-Reich-Rus contraction. Because

$$\begin{aligned} d(T1, T3) &\leq \lambda[d(1, 3) + d(1, 1) + d(3, 3)] \\ d(1, 3) &\leq \frac{1}{3}\lambda \end{aligned}$$

above inequality does not hold for $\lambda \in [0, \frac{1}{3})$.
When we use an interpolative Ćirić-Reich-Rus contraction,
Let us take $x, y \notin Fix_T(X)$.

Case(1): $x = y = 3$

$$\begin{aligned} \frac{1}{s}d(T3, T3) &\leq \lambda[d(3, 3)]^{\beta}[d(3, T3)]^{\alpha}[d(3, T3)]^{1-\alpha-\beta} \\ d(1, 1) &\leq s\lambda[d(3, 1)]^{\beta}[d(3, 1)]^{\alpha}[d(3, 1)]^{1-\alpha-\beta} \end{aligned}$$

above inequality holds for all $\alpha, \beta \in (0, 1)$ and $0.4 \leq \lambda < 1$ with $s = 2$.

Case(2): $x = 3, y = 4$

$$\begin{aligned} d(T3, T4) &\leq s\lambda[d(3, 4)]^{\beta}[d(3, T3)]^{\alpha}[d(4, T4)]^{1-\alpha-\beta} \\ d(3, 3) &\leq s\lambda[d(3, 4)]^{\beta}[d(3, 3)]^{\alpha}[d(4, 3)]^{1-\alpha-\beta} \end{aligned}$$

holds for all $\beta \in (0, 1)$ and $\alpha = 0.5$, with $0.86 \leq \lambda < 1$ and $s = 2$.

Case(3): $x = 4, y = 4$

$$\begin{aligned} d(T4, T4) &\leq \lambda[d(4, 4)]^{\beta}[d(4, T4)]^{\alpha}[d(4, T4)]^{1-\alpha-\beta} \\ d(3, 3) &\leq \lambda[d(4, 4)]^{\beta}[d(4, 3)]^{\alpha}[d(4, 3)]^{1-\alpha-\beta} \end{aligned}$$

holds for all $\alpha \in (0, 1)$ and $\beta = 0.2$, with $0.85 \leq \lambda < 1$, with $s = 2$.
Thus, the self mapping T is an interpolative Ćirić-Reich-Rus type contraction and $1, 5$ are

required fixed points. In the setting of Ćirić-Reich-Rus type contraction, λ is restricted to $[0, \frac{1}{3})$ But here λ lies between $(0, 1)$.

Example 3.2. Let $\mathcal{M} = \left\{ \begin{pmatrix} 2p & 4p \\ p & 0 \end{pmatrix} : p \in \mathcal{R} \right\}$ and $d : \mathcal{M} \times \mathcal{M} \to \mathcal{R}_0^+$ be defined as $d(x, y) = |tr(x - y)^2|$. The triplet $(\mathcal{M}, d, s)$ forms a complete b- metric space. Let $T : \mathcal{M} \to \mathcal{M}$ be defined by $T(x) = Bx$ where $B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ with $\alpha, \beta < \frac{31}{64}$ and $s = 2$, satisfies Theorem 2.2, then T has a unique fixed point.

4. Conclusion

The novelty of this paper is conjucture of interpolative Ćirić-Reich-Rus type contractions and interpolative Reich type contractions in complete b-metric spaces to ensure existence of fixed points, when we take $\gamma = 1, \beta = 0$ then Theorem 2.2 reduces to a general interpolative Kannan contraction.

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