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A Simultaneous Scheme for Solving Systems of Inclusion and Equilibrium Problems in a Real Banach Space

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Abstract In this paper, we propose an iterative method for approximating a common zero of finite family of m-accretive operators and a common solution of finite family of equilibrium problems simultaneously in a real reflexive, strict convex and smooth Banach space. We prove a strong convergence theorem and also give some applications of our result to approximating solutions of other nonlinear problems in real Banach spaces. As a special case, we obtain a result for approximating the common zero of a finite family of m-accretive operators which is also a common solution of a finite family of equilibrium problems in a real reflexive, strictly convex and smooth Banach space.

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1. Introduction

Let E be a real Banach space and C be a nonempty, closed and convex subset of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\},\ \forall\ x \in E, \tag{1.1}$$

where E^* is the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*. It is well known that if E^* is strictly convex, then J is single-valued which we shall

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denote by j. Let $T : C \to C$ be a mapping, the set of fixed points of T denoted by $F(T)$ is defined by

$$F(T) = \{x \in C : Tx = x\}.$$

A mapping $T : C \to C$ is called a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$||Tx - Ty|| \leq \alpha||x - y||, \ \forall \ x, y \in C. \tag{1.2}$$

If $\alpha = 1$ in (1.2), then T is said to be nonexpansive. T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - p|| \leq ||x - p||, \quad \forall x \in C, \text{ and } p \in F(T).$$

Let C be a nonempty, closed and convex subset of a Banach space E and let T be a mapping from C into itself. A point $u \in C$ is said to be a weakly asymptotic fixed point of T [1] if there exists a sequence $\{x_n\}$ in C which converges weakly to u and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. We denote the set of all weakly asymptotic fixed points of T by $\widehat{F}(T)$. Also, a mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $F(T)$ is nonempty,
(2) $||Tu - p|| \leq ||u - p||$, $\forall p \in F(T)$, $u \in C$,
(3) $\widehat{F}(T) = F(T)$.

It is easy to see that any relatively nonexpansive mapping is quasi-nonexpansive. Also, T is said to be firmly nonexpansive-type if

$$\langle Tx - Ty, JTx - JTy\rangle \leq \langle Tx - Ty, Jx - Jy\rangle \tag{1.3}$$

for all $x, y \in C$. If E is a Hilbert space, then J is the identity operator on E and (1.3) reduces to $||Tx - Ty||^2 \leq \langle x - y, Tx - Ty\rangle$ for all $x, y \in C$. Note that the class of firmly nonexpansive-type mappings that have a nonempty set of fixed points is contained in the class of relatively nonexpansive mappings [2].

Let $\Theta : C \times C \to \mathbb{R}$ be a nonlinear bifunction, the Equilibrium Problem (EP) is to find a point $\bar{x} \in C$ such that

$$\Theta(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(\Theta)$. Numerous problems in physics, optimization, economics and fixed point theory can be formulated as finding a solution of the EP. Many researchers have proposed different iterative schemes for finding solution of EP (1.4) and related optimization problems in Hilbert and real Banach spaces, see for example [3–16] and reference therein. For solving the equilibrium problem (1.4), the following assumptions are made on the bifunction Θ:

(A1) $\Theta(x, x) = 0$ for all $x \in C$,
(A2) Θ is monotone, i.e $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$,
(A3) for all $x, y \in C$,

$$\limsup_{t\downarrow 0} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y),$$

(A4) for all $x \in C$, $\Theta(x, \cdot)$ is convex and lower semicontinuous.

A mapping $A : D(A) \subseteq E \to E$ is said to be accretive if for all $x, y \in E$, there exist $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0.$$

An operator $A : D(A) \subseteq E \to E$ is called m-accretive if it is accretive and the range of $(I + \lambda A)$ denoted by $R(I + \lambda A)$ is E for all $\lambda > 0$. Also, A is said to satisfy the range condition if $D(A) \subseteq R(I + \lambda A)$, $\forall \lambda > 0$. If A is m-accretive, we define the resolvent operator of A, $J_\lambda^A : R(I + \lambda A) \to D(A)$ by $J_\lambda^A = (I + \lambda A)^{-1}$. It is well known that J_λ^A is nonexpansive and single-valued. Also, $F(J_\lambda^A) = \mathcal{N}(A)$, where

$$\mathcal{N}(A) := \{z \in D(A) : 0 \in Az\} = A^{-1}(0).$$

In a real Hilbert space, the m-accretive operators become the maximal monotone operators.
Considerable effort have been devoted to finding the zero points of accretive operators (see, for example [17–25]). One popular technique for approximating zeros of m-accretive operators is the Proximal Point Algorithm (PPA) which generates a sequence $\{x_n\}$ by the formula:

$$x_0 \in E, \quad x_{n+1} = J_{\lambda_n}^A x_n, \; n \geq 0, \tag{1.5}$$

where $\{\lambda_n\} \subset (0, \infty)$. This method was introduced by Martinet [26].
Motivated by the PPA and the iterative method of Halpern [27], Kamimura and Takahashi [28] and Bernavides et al. [29] presented the following algorithm for approximating the zero point of an m-accretive operator in a real Hilbert space and a real reflexive Banach space E respectively: for $u \in E$, $x_0 \in E$,

$$x_{n+1} = a_n u + (1 - a_n) J_{\lambda_n}^A x_n, \; \forall n \geq 0, \tag{1.6}$$

where $a_n \in (0, 1)$. They proved that the sequence $\{x_n\}$ generated (1.6) strongly converges to the zero point of A. Furthermore, Kim and Xu [30] improved the result of Kamimura and Takahashi [28] to a uniformly smooth real Banach space. However, in 2006, Xu [31] further extended this work to a reflexive Banach space with weakly continuous duality mapping with gauge φ.
In 2000, Moudafi [32] introduced the viscosity approximation method as a generalization of the Halpern iterative method for approximating fixed point of a nonexpansive mapping T in a real Hilbert space H: for $x_0 \in H$,

$$x_{n+1} = a_n f(x_n) + (1 - a_n) T x_n, \tag{1.7}$$

where f is a contraction and $\{a_n\} \subset (0, 1)$. Based on the work of Moudafi [32], Takahashi [33] combined the PPA and the viscosity approximation method to form the following iterative scheme for approximating the zero point of an accretive operator A in a reflexive Banach space with a uniformly Gâteaux differentiable norm E: for $x_0 \in E$,

$$x_{n+1} = a_n f(x_n) + (1 - a_n) J_{\lambda_n}^A x_n, \; \forall n \geq 0. \tag{1.8}$$

Under some mild condition on the parameters $\{a_n\}$ and $\{\lambda_n\}$, he proved that the sequence $\{x_n\}$ defined by (1.8) converges strongly to a zero of the accretive operator A.
Zegeye and Shahzad [34] further studied the problem of finding a common zero of a finite family of m-accretive operators in a strictly convex and reflexive Banach space E. They presented the following algorithm and proved its strong convergence to a common zero of the m-accretive operators A_i, $i = 1, 2, \dots, N$. For $u, x_0 \in C$,

$$x_{n+1} = a_n u + (1 - a_n) S_N x_n, \; \forall n \geq 0, \tag{1.9}$$

where $S_N := a_0 I + a_1 J_{\lambda_n}^{A_1} + a_2 J_{\lambda_n}^{A_2} + \cdots + a_N J_{\lambda_n}^{A_N}$, $0 < a_i < 1$, for $i = 1, 2, \dots, N$, $\sum_{i=0}^N a_i = 1$ and $\{a_n\}$ is a real sequence which satisfy some suitable conditions.

Very recently, Wang et.al. [35] introduced the following composite iterative scheme for finding a common zero of two accretive operators A and B in a uniformly convex Banach space,

$$\begin{cases} y_n = \beta_n J^B_{\lambda_n} x_n + (1-\beta_n) J^A_{\lambda_n} x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{1.10}$$

which converges weakly to a common zero of the two accretive operators A and B under some certain conditions.
Many authors have introduced several iterative methods for finding a common solution of two or more problems (see, for instance [36–39]), but, there is little effort on iterative methods for approximating distinct solutions of two or more problems simultaneously. In this paper, we propose a new simultaneous algorithm for finding a common zero of finite family of accretive operators and a common solution of finite family of equilibrium problem simultaneously in a real reflexive, strictly convex and smooth Banach space. As a special case, we obtain a result for approximating the common zero of a finite family of m-accretive operators which is also a common solution of a finite family of equilibrium problems in a real reflexive, strictly convex and smooth Banach space. This improve many recent results in literature, e.g [38, 39].

2. Preliminaries

In the sequel, we denote the weak convergence of a sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \rightharpoonup x$ and the strong convergence of $\{x_n\}$ to x by $x_n \longrightarrow x$.
A real Banach space E with $dim E \geq 2$ is called strictly convex if for any $x, y \in E$, $x \neq y$, $||x|| = ||y|| = 1$, we have $||\alpha x + (1-\alpha)y|| < 1$, $\forall\ \alpha \in (0,1)$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(t) = \inf\left\{1 - ||\frac{x+y}{2}|| : ||x|| = ||y|| = 1; t = ||x-y||\right\}.$$

It is well known that the function $\dfrac{\delta_E(t)}{t}$ is nondecreasing on $(0,2]$. The space E is said to be uniformly convex if for any $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $||x|| = ||y|| = 1$ and

$$||x-y|| \geq \epsilon \Leftrightarrow \left|\left|\frac{x+y}{2}\right|\right| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex, see for details [40–42].
The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\begin{aligned} \rho_E(r) &= \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1; ||y|| = r\right\} \\ &= \sup\left\{\frac{||x+ry|| + ||x-ry||}{2} - 1 : ||x|| = 1 = ||y||\right\}. \end{aligned}$$

The space E is said to be uniformly smooth if $\lim_{r\to 0^+} \dfrac{\rho_E(r)}{r} = 0$. Also, if E is uniformly smooth, then the dual space E^* is uniformly convex and if E is uniformly convex, E^* is uniformly smooth. It is well known that every uniformly smooth Banach space is reflexive (see [40, 43] for more details).

A continuous and strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$ is called a gauge function. The duality mapping $J_\varphi : E \to 2^{E^*}$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = ||x||\varphi(||x||), ||f^*|| = \varphi(||x||),\ \forall\ x \in E\}.$$

From [40, 44], we know that the duality mapping J_φ satisfies the following properties:

(i) $J_\varphi(-x) = -J_\varphi(x)$ and $J_\varphi(\lambda x) = sgn(\lambda)\dfrac{\varphi(|\lambda|||x||)}{\varphi(||x||)}J_\varphi(x)$, for $x \in E$ and $\lambda \in \mathbb{R}$;

(ii) if E^* is uniformly convex, then J_φ is uniformly continuous on each bounded subset of E;

(iii) the reflexivity of E and strict convexity of E^* imply that J_φ is single-valued and monotone.

In the case $\varphi(t) = t$, we call J_φ the normalized duality mapping. If $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called generalized duality mapping, where $J_q(x) = ||x||^{q-2}J(x)$, $q > 1$.

For the gauge function φ, the function $\Psi : [0, +\infty) \to [0, +\infty)$ defined by

$$\Psi(t) = \int_0^t \varphi(s)ds \tag{2.1}$$

is a continuous convex strictly increasing function on $[0, +\infty)$. Also, $J_\varphi(x) = \partial\Psi(||x||)$, $x \in E$, where ∂ denotes the subdifferential in the sense of convex analysis. This implies that for each $x \in E$, (see [44])

$$\Psi(||y||) - \Psi(||x||) \geq \langle x^*, y - x\rangle,\ \forall\ y \in E, x^* \in \partial\Psi(||x||). \tag{2.2}$$

We recall that a Banach space E has a weakly continuous duality mapping if there is a guage φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to weak* sequentially continuous (i.e, if $\{x_n\} \subset E$ weakly converges to a point $x \in E$, then $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$). It is noted in [44] that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for $1 < p < +\infty$.

Let C be a nonempty, closed and convex subset of a real Banach space E and Q be a mapping of E onto C. Then Q is said to be sunny if $Q(Q(x) + t(x - Q(x)) = Q(x)$, for all $x \in E$ and $t \geq 0$. A mapping Q of E onto E is said to be a retraction if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. In a real Hilbert space H, the sunny nonexpansive retraction of Q coincides with the metric projection P_C from H onto C.

Lemma 2.1 ([45]). *Let C be a closed and convex subset of a smooth Banach space E and D a nonempty subset of C. Let $Q : C \to D$ be a retraction and J_φ be the duality mapping with a gauge function φ. Then the following are equivalent:*

(a) Q is sunny and nonexpansive,

(b) $\langle x - Qx, J_\varphi(y - Qx)\rangle \leq 0$, for all $x \in C$ and $y \in D$.

Lemma 2.2 ([46]). *Let E be a real Banach space which has a weakly continuous duality mapping J_φ with a gauge φ. Then Ψ defined by (2.1) has the following properties:*

(a) $\Psi(||x + y||) \leq \Psi(||x||) + \langle y, J_\varphi(x + y)\rangle$, $\forall x, y \subset E$.

(b) Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then

$$\limsup_{n\to\infty} \Psi(||x_n - y||) = \limsup_{n\to\infty} \Psi(||x_n - x||) + \Psi(||x - y||),\ \forall y \in E. \tag{2.3}$$

Lemma 2.3 ([47, 48]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, \; n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ for each $n \geq 0$ satisfying the conditions:

(a) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(b) $\limsup_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\{a_n\}$ converges strongly to zero.

Lemma 2.4 ([34]). *Let C be a nonempty, closed and convex subset of a strictly convex Banach space E. Let $A_i : C \to E$, $i = 1, 2, \ldots, r$, be a finite family of m-accretive mappings with $\cap_{i=1}^r \mathcal{N}(A_i) \neq \emptyset$. Let $a_0, a_1, \ldots, a_r$ be real numbers in $(0,1)$ such that $\sum_{i=0}^r a_i = 1$ and $S_r = a_0 I + a_1 J^{A_1} + a_2 J^{A_2} + \cdots + a_r J^{A_r}$ where $J^{A_i} = (I + A_i)^{-1}$. Then, S_r is nonexpansive and $F(S_r) = \cap_{i=1}^r \mathcal{N}(A_i)$.*

Lemma 2.5 ([13]). *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let Θ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$, define a mapping $T_r^\Theta : E \to C$ as follows:*

$$T_r^\Theta x = \left\{ z \in C : \Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \right\}, \tag{2.4}$$

for all $y \in C$. Then the following hold:

(1) T_r^Θ *is single-valued,*
(2) T_r^Θ *is firmly nonexpansive-type, that is, for all $x, y \in E$,*

$$\langle T_r^\Theta x - T_r^\Theta y, JT_r^\Theta x - JT_r^\Theta y\rangle \leq \langle T_r^\Theta x - T_r^\Theta y, Jx - Jy\rangle,$$

(3) $F(T_r^\Theta) = EP(\Theta)$,
(4) $EP(\Theta)$ *is closed and convex.*

3. Main Results

In this section, we give our main results in this paper, we first give the following important lemma which will be needed in the sequel.

Lemma 3.1. *Let C be a nonempty, closed and convex subset of a real reflexive, smooth and strictly convex Banach space E. Let $\Theta_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of bifunctions satisfying assumptions (A1)-(A4). Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N$ be real numbers in $(0,1)$ such that $\sum_{i=0}^N \alpha_i = 1$ and $W_N = \alpha_0 I + \alpha_1 T_{r_n}^{\Theta_1} + \alpha_2 T_{r_n}^{\Theta_2} + \cdots + \alpha_N T_{r_n}^{\Theta_N}$, where $T_{r_n}^{\Theta_i}$ is as defined in (2.4) for $i = 1, 2, \ldots, N$ and $r_n > 0$. Suppose $\cap_{i=1}^N EP(\Theta_i)$ and $F(W_N)$ are nonempty. Then, W_N is nonexpansive and $F(W_N) = \cap_{i=1}^N (T_{r_n}^{\Theta_i}) = \cap_{i=1}^N EP(\Theta_i)$.*

Proof. Let $x, y \in C$, we have

$$\begin{aligned}
||W_N x - W_N y|| &= ||\alpha_0 x + \alpha_1 T_{r_n}^{\Theta_1} x + \alpha_2 T_{r_n}^{\Theta_2} x + \cdots + \alpha_N T_{r_n}^{\Theta_N} x \\
&\quad -\alpha_0 y - \alpha_1 T_{r_n}^{\Theta_1} y - \alpha_2 T_{r_n}^{\Theta_2} y - \cdots - \alpha_N T_{r_n}^{\Theta_N} y|| \\
&= ||\alpha_0 (x-y) + \alpha_1 (T_{r_n}^{\Theta_1} x - T_{r_n}^{\Theta_1} y) + \alpha_2 (T_{r_n}^{\Theta_2} x - T_{r_n}^{\Theta_2} y) \ldots \\
&+ \alpha_N (T_{r_n}^{\Theta_N} x - T_{r_n}^{\Theta_2} y)|| \\
&\leq \alpha_0 ||x-y|| + \sum_{i=1}^{N} \alpha_i ||T_{r_n}^{\Theta_i} x - T_{r_n}^{\Theta_i} y|| \\
&\leq \alpha_0 ||x-y|| + \sum_{i=1}^{N} \alpha_i ||x-y|| \\
&= ||x-y||. \qquad (3.1)
\end{aligned}$$

Hence, W_N is nonexpansive. It is easy to see that $\cap_{i=1}^N EP(\Theta_i) = \cap_{i=1}^N F(T_{r_n}^{\Theta_i}) \subseteq F(W_N)$. We now show that $F(W_N) \subseteq \cap_{i=1}^N F(T_{r_n}^{\Theta_i})$. Let $q \in F(W_n)$ and $p \in \cap_{i=1}^N F(T_{r_n}^{\Theta_i})$, then

$$\begin{aligned}
||q-p|| &= ||\alpha_0 q + \alpha_1 T_{r_n}^{\Theta_1} q + \alpha_2 T_{r_n}^{\Theta_2} q + \cdots + \alpha_N T_{r_n}^{\Theta_N} q - p|| \\
&= ||\alpha_0 (q-p) + \alpha_1 (T_{r_n}^{\Theta_1} q - p) + \alpha_2 (T_{r_n}^{\Theta_2} q - p) + \cdots + \alpha_N (T_{r_n}^{\Theta_N} q - p)|| \\
&\leq \alpha_0 ||q-p|| + \sum_{i=1}^{N} \alpha_i ||T_{r_n}^{\Theta_N} q - p|| \\
&\leq \alpha_0 ||q-p|| + \sum_{i=1}^{N} \alpha_i ||q-p|| \\
&= ||q-p||. \qquad (3.2)
\end{aligned}$$

This implies that

$$\begin{aligned}
||q-p|| &= \sum_{i=0}^{N-1} \alpha_i ||q-p|| + \alpha_N ||T_{r_n}^{\Theta_N} q - p|| \\
&= (1-\alpha_N)||q-p|| + \alpha_N ||T_{r_n}^{\Theta_N} q - p||,
\end{aligned}$$

hence

$$||q-p|| = ||T_{r_n}^{\Theta_N} q - p||.$$

Similarly, we obtain

$$||q-p|| = ||T_{r_n}^{\Theta_{N-1}} q - p|| = ||T_{r_n}^{\Theta_{N-2}} q - p|| = \cdots = ||T_{r_n}^{\Theta_1} q - p||.$$

From (3.2), we get

$$||q-p|| = \left|\left| \frac{\alpha_1}{\sum_{i=1}^N \alpha_i} (T_{r_n}^{\Theta_1} q - p) + \frac{\alpha_2}{\sum_{i=1}^N \alpha_i} (T_{r_n}^{\Theta_2} q - p) \cdots + \frac{a_N}{\sum_{i=1}^N \alpha_i} (T_{r_n}^{\Theta_N} q - p) \right|\right|.$$

By the strict convexity of E, we have

$$q - p = T_{r_n}^{\Theta_1} q - p = T_{r_n}^{\Theta_2} q - p = \cdots = T_{r_n}^{\Theta_N} q - p,$$

hence, $T_{r_n}^{\Theta_i} q = q$, for $i = 1, 2, \ldots, N$, which implies that $q \in \cap_{i=1}^N F(T_{r_n}^{\Theta_N})$. Therefore

$$F(W_N) \subseteq \cap_{i=1}^N F(T_{r_n}^{\Theta_i}).$$

Hence, we have

$$F(W_N) = \cap_{i=1}^{N} F(T_{r_n}^{\Theta_N}) = \cap_{i=1}^{N} EP(\Theta_i).$$

■

As a direct consequence of Lemma 3.1, we have the following result.

Lemma 3.2. *Let C be a nonempty, closed and convex subset of a real reflexive, smooth and strictly convex Banach space E. Let $S,T : C \to C$ be two nonexpansive mappings such that $F(S)$ and $F(T)$ are nonempty. Define $G := \frac{1}{2}(S+T)$, then G is nonexpansive and $F(G) = F(S) \cap F(T)$.*

We now present our main theorem.

Theorem 3.3. *Let E be a real reflexive, strictly convex and smooth Banach space which has a weakly continuous duality mapping J_φ with guage φ and let C be a nonempty, closed and convex subset of E. Let $f : C \to C$ be a θ_1-contraction mapping and $g : C \to C$ be a θ_2-contraction mapping such that $\theta = \max\{\theta_1, \theta_2\}$. Let $A_k : C \to E$, $k = 1, 2, \dots, N$ be a finite family of m-accretive operators and $\Theta_i : C \times C \to \mathbb{R}$, $i = 1, 2, \dots, M$ be bifunctions satisfying assumptions (A1)-(A4). Suppose $\cap_{k=1}^{N} \mathcal{N}(A_k)$ and $\cap_{i=1}^{M} EP(\Theta_i)$ are nonempty. Let $\{a_n\}$ be a real sequence in $(0,1)$ and for arbitrarily $x_0 \in C$ and $y_0 \in C$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated simultaneously by*

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1-a_n) S_N x_n, \\ y_{n+1} = a_n g(x_n) + (1-a_n) W_M y_n, \quad \forall n \geq 0, \end{cases} \tag{3.3}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{A_1} + \alpha_2 J_{\lambda_n}^{A_2} + \dots + \alpha_N J_{\lambda_n}^{A_N},$$
$$W_M = \beta_0 I + \beta_1 T_{r_n}^{\Theta_1} + \beta_2 T_{r_n}^{\Theta_2} + \dots + \beta_M T_{r_n}^{\Theta_M},$$

with $J_{\lambda_n}^{A_k} := (I + \lambda_n A_k)^{-1}$ for $0 < \alpha_k < 1$, $k = 0, 1, 2, \dots, N$, $\sum_{k=0}^{N} \alpha_k = 1$, $\lambda_n > 0$ and $T_{r_n}^{\Theta_i}$ is as defined in (2.4), $0 < \beta_i < 1$, $i = 0, 1, 2, \dots, M$, $\sum_{i=0}^{M} \beta_i = 1$ and $r_n > 0$. Suppose $\{a_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\liminf_{n\to\infty} \lambda_n > 0$, $\liminf_{n\to\infty} r_n > 0$,
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$,
(iii) $\sum_{n=0}^{\infty} |a_n - a_{n-1}| < \infty$ *or (iii*)* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to elements $\hat{x} = Q_1(f(\hat{y})) \in \cap_{k=1}^{N} \mathcal{N}(A_k)$ and $\hat{y} = Q_2(g(\hat{x})) \in \cap_{i=1}^{M} EP(\Theta_i)$ respectively, where Q_1 is the sunny nonexpansive retraction of C onto $\cap_{k=1}^{N} \mathcal{N}(A_k)$ and Q_2 is the sunny nonexpansive retraction of C onto $\cap_{i=1}^{M} EP(\Theta_i)$.

Remark 3.4. Observe that the definition of $\{x_n\}$ involves $\{y_n\}$ and the definition $\{y_n\}$ invloves $\{x_n\}$ in (3.3).

Proof. First, we show that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $x^* \in F(S_N)$ and $y^* \in F(W_M)$, then we have

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||a_n f(y_n) + (1-a_n) S_N x_n - x^*|| \\ &\leq a_n ||f(y_n) - f(y^*)|| + a_n ||f(y^*) - x^*|| + (1-a_n) ||S_N x_n - x^*|| \\ &\leq a_n \theta_1 ||y_n - y^*|| + a_n ||f(y^*) - x^*|| + (1-a_n) ||x_n - x^*|| \\ &\leq a_n \theta ||y_n - y^*|| + a_n ||f(y^*) - x^*|| + (1-a_n) ||x_n - x^*||. \end{aligned} \tag{3.4}$$

Similarly, from (3.3), we have

$$||y_{n+1}-y^*|| \le a_n\theta||x_n-x^*|| + a_n||g(x^*)-y^*|| + (1-a_n)||y_n-y^*||. \tag{3.5}$$

Therefore, from (3.4) and (3.5), we obtain

$$\begin{aligned}||x_{n+1}-x^*||+||y_{n+1}-y^*|| &\le (1-a_n(1-\theta))(||x_n-x^*||+||y_n-y^*||)\\ &+ a_n(||f(y^*)-x^*||+||g(x^*)-y^*||)\\ &\le \max\left\{||x_n-x^*||+||y_n-y^*||, \frac{||f(y^*)-x^*||+||g(x^*)-y^*||}{1-\theta}\right\}\\ &\vdots\\ &\le \max\left\{||x_0-x^*||+||y_0-y^*||, \frac{||f(y^*)-x^*||+||g(x^*)-y^*||}{1-\theta}\right\}.\end{aligned}$$

This implies $\{||x_n-x^*||\}$ and $\{||y_n-y^*||\}$ are bounded. Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, $\{f(y_n)\}, \{g(x_n)\}, \{S_Nx_n\}$ and $\{U_My_n\}$ are bounded.
Furthermore, from (3.3) and Lemma 2.2(a), we get

$$\begin{aligned}\Psi(||x_{n+1}-x^*||) &= \Psi(||a_n(f(y_n)-x^*)+(1-a_n)(S_Nx_n-x^*)||)\\ &= \Psi(||a_n(f(y_n)-f(y^*)+f(y^*)-x^*)+(1-a_n)(S_Nx_n-x^*)||)\\ &\le \Psi(||a_n(f(y_n)-f(y^*))+(1-a_n)(S_Nx_n-x^*)||)\\ &+ a_n\langle f(y^*)-x^*, J_\varphi(x_{n+1}-x^*)\rangle\\ &\le a_n\theta_1\Psi(||y_n-y^*||)+(1-a_n)\Psi(||S_Nx_n-x^*||)\\ &+ a_n\langle f(y^*)-x^*, J_\varphi(x_{n+1}-x^*)\rangle\\ &\le a_n\theta\Psi(||y_n-y^*||)+(1-a_n)\Psi(||x_n-x^*||)\\ &+ a_n\langle f(y^*)-x^*, J_\varphi(x_{n+1}-x^*)\rangle.\end{aligned} \tag{3.6}$$

Similarly, we obtain

$$\begin{aligned}\Psi(||y_{n+1}-y^*||) &\le a_n\theta\Psi(||x_n-x^*||)+(1-a_n)\Psi(||y_n-y^*||)\\ &+ a_n\langle g(x^*)-y^*, J_\varphi(y_{n+1}-y^*)\rangle.\end{aligned} \tag{3.7}$$

Hence, from (3.6) and (3.7), we get

$$\begin{aligned}\Psi(||x_{n+1}-x^*||)+\Psi(||y_{n+1}-y^*||) &\le (1-a_n(1-\theta))(\Psi(||x_n-x^*||)+\Psi(||y_n-y^*||)\\ &+a_n(\langle f(y^*)-x^*, J_\varphi(x_{n+1}-x^*)\rangle\\ &+\langle g(x^*)-y^*, J_\varphi(y_{n+1}-y^*)\rangle).\end{aligned} \tag{3.8}$$

Next, we show that $||x_{n+1}-x_n|| \to 0$ and $||y_{n+1}-y_n|| \to 0$, as $n \to \infty$.
Let $K := \sup\{||f(y_n)||, ||g(x_n)||, ||S_Nx_n||, ||U_My_n|| : n \ge 1\}$. Observe that

$$\begin{aligned}||x_{n+1}-x_n|| &= ||a_nf(y_n)+(1-a_n)S_Nx_n-(a_{n-1}f(y_{n-1})+(1-a_{n-1})S_Nx_{n-1})||\\ &= a_n||f(y_n)-f(y_{n-1})||+|a_n-a_{n-1}|||f(y_{n-1})||\\ &+ (1-a_n)||S_Nx_n-S_Nx_{n-1}||\\ &\quad +|a_n-a_{n-1}|||S_Nx_{n-1}||\\ &\le a_n\theta||y_n-y_{n-1}||+(1-a_n)||x_n-x_{n-1}||+2K|a_n-a_{n-1}|.\end{aligned} \tag{3.9}$$

Similarly

$$||y_{n+1}-y_n|| \le a_n\theta||x_n-x_{n-1}||+(1-a_n)||y_n-y_{n-1}||+2K|a_n-a_{n-1}|. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}||x_{n+1}-x_n||+||y_{n+1}-y_n|| &\leq (1-a_n(1-\theta))(||x_n-x_{n-1}|| \\ &+ ||y_n-y_{n-1}||)+4K|a_n-a_{n-1}| \\ &= (1-a_n(1-\theta))(||x_n-x_{n-1}|| \\ &+ ||y_n-y_{n-1}||)+a_n(1-\theta)\delta_n, \qquad (3.11)\end{aligned}$$

where $\delta_n = \dfrac{4K|a_n-a_{n-1}|}{a_n(1-\theta)}$. We consider the following two cases:
Case I: Suppose condition (iii) is satisfied, then

$$||x_{n+1}-x_n||+||y_{n+1}-y_n|| \leq (1-a_n(1-\theta))(||x_n-x_{n-1}||+||y_n-y_{n-1}||)+\sigma_n,$$

where $\sigma_n = 4K|a_n-a_{n-1}|$ so that $\sum_{n=0}^{\infty}\sigma_n < \infty$.
Case II: Suppose condition (iii*) is satisfied. Then

$$||x_{n+1}-x_n||+||y_{n+1}-y_n|| \leq (1-a_n(1-\theta))(||x_n-x_{n-1}||+||y_n-y_{n-1}||)+\sigma_n,$$

where $\sigma_n = (1-\theta)a_n\delta_n$ so that $\sigma_n = o((1-\theta)a_n)$.
In either case, Lemma 2.3 and condition (ii) yield that

$$||x_{n+1}-x_n||+||y_{n+1}-y_n|| \to 0, \text{ as } n\to\infty,$$

this implies that

$$||x_{n+1}-x_n|| \to 0 \text{ and } ||y_{n+1}-y_n|| \to 0, \text{ as } n\to\infty. \qquad (3.12)$$

Also, it is clear from (3.3) that

$$||x_{n+1}-S_Nx_n|| \leq a_n||f(y_n)-S_Nx_n|| \to 0, \text{ as } n\to\infty, \qquad (3.13)$$

and

$$||y_{n+1}-T_My_n|| \leq a_n||g(x_n)-T_My_n|| \to 0, \text{ as } n\to\infty. \qquad (3.14)$$

Therefore, from (3.12), (3.13) and (3.14), we have

$$||x_n-S_Nx_n|| \leq ||x_n-x_{n+1}||+||x_{n+1}-S_Nx_n|| \to 0, \qquad (3.15)$$
$$||y_n-T_My_n|| \leq ||y_n-y_{n+1}||+||y_{n+1}-T_My_n|| \to 0, \qquad (3.16)$$

as $n\to\infty$.
Since E is reflexive, there exist subsequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$ of $\{x_n\}$ and $\{y_n\}$ respectively such that $x_{n_j} \rightharpoonup \bar{x}$ and $y_{n_j} \rightharpoonup \bar{y}$. Next, we show that $\bar{x} \in \cap_{k=1}^{N}\mathcal{N}(A_k)$ and $\bar{y} \in \cap_{i=1}EP(\Theta_i)$.
Since J_φ is weakly continuous, we have by Lemma 2.2(b) that

$$\limsup_{j\to\infty}\Psi(||x_{n_j}-x||) = \limsup_{j\to\infty}\Psi(||x_{n_j}-\bar{x}||)+\Psi(||x-\bar{x}||), \ \forall x\in E. \qquad (3.17)$$

Let

$$\Phi(x) = \limsup_{j\to\infty}\Psi(||x_{n_j}-x||), \ \forall x\in E,$$

it follows from (3.17) that

$$\Phi(x) = \Phi(\bar{x})+\Psi(||x-\bar{x}||), \ x\in E.$$

Since $||x_{n_j} - S_N x_{n_j}|| \to 0$ as $j \to \infty$, we get that

$$\begin{aligned}\Phi(S_N\bar{x}) &= \limsup_{j\to\infty} \Psi(||x_{n_j} - S_N\bar{x}||) = \limsup_{j\to\infty} \Psi(||S_N x_{n_j} - S_N\bar{x}||)\\ &\leq \limsup_{j\to\infty} \Psi(||x_{n_j} - \bar{x}||) = \Phi(\bar{x}). \qquad (3.18)\end{aligned}$$

On the otherhand, we note that

$$\Phi(S_N\bar{x}) = \Phi(\bar{x}) + \Psi(||S_N\bar{x} - \bar{x}||). \qquad (3.19)$$

From (3.18) and (3.19), we get

$$\Psi(||S_N\bar{x} - \bar{x}||) \leq 0.$$

Hence, $S_N\bar{x} = \bar{x}$. This implies that $\bar{x} \in F(S_N)$. Therefore, by Lemma 2.4, we obtain $\bar{x} \in \cap_{k=1}^M \mathcal{N}(A_k)$.
Similarly, we can verify that $\bar{y} \in F(W_M)$. By Lemma 3.1, we have that $\bar{y} \in \cap_{i=1}^M EP(\Theta_i)$.
Now, we show that the sequence $\{x_n\}$ converges strongly to a point $\hat{x} = Q_{\cap_{k=1}^N \mathcal{N}(A_k)} f(\hat{y})$ and the sequence $\{y_n\}$ converges strongly to a point $\hat{y} = Q_{\cap_{i=1}^M EP(\Theta_i)} g(\hat{x})$.
From (3.8), we have

$$\begin{aligned}\Psi(||x_{n+1} - \hat{x}||) + \Psi(||y_{n+1} - \hat{y}||) \leq\ & (1 - a_n(1-\theta))(\Psi(||x_n - \hat{x}||) + \Psi(||y_n - \hat{y}||))\\ &+a_n(\langle f(\hat{y}) - \hat{x}, J_\varphi(x_{n+1} - \hat{x})\rangle\\ &+\langle g(\hat{x}) - \hat{y}, J_\varphi(y_{n+1} - \hat{y})\rangle).\end{aligned}$$

This implies that

$$\Gamma_{n+1} \leq (1-\gamma_n)\Gamma_n + \sigma_n, \qquad (3.20)$$

where

$$\Gamma_n := \Psi(||x_n - \hat{x}||) + \Psi(||y_n - \hat{y}||), \quad \gamma_n := a_n(1-\theta),$$

and

$$\sigma_n := \frac{a_n(1-\theta)}{1-\theta}(\langle f(\hat{y}) - \hat{x}, J_\varphi(x_{n+1} - \hat{x})\rangle + \langle g(\hat{x}) - \hat{y}, J_\varphi(y_{n+1} - \hat{y})\rangle).$$

Choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$ respectively such that

$$\limsup_{n\to\infty} \langle f(\hat{y}) - \hat{x}, J_\varphi(x_{n+1} - \hat{x})\rangle = \lim_{k\to\infty} \langle f(\hat{y}) - \hat{x}, J_\varphi(x_{n_k+1} - \hat{x})\rangle,$$

and

$$\limsup_{n\to\infty} \langle g(\hat{x}) - \hat{y}, J_\varphi(y_{n+1} - \hat{y})\rangle = \lim_{k\to\infty} \langle g(\hat{x}) - \hat{y}, J_\varphi(y_{n_k+1} - \hat{y})\rangle.$$

Since $x_{n_k} \rightharpoonup \bar{x}$, it follows from Lemma 2.1 that

$$\begin{aligned}\limsup_{n\to\infty} \langle f(\hat{y}) - \hat{x}, J_\varphi(x_{n+1} - \hat{x})\rangle &= \lim_{k\to\infty} \langle f(\hat{y}) - \hat{x}, J_\varphi(x_{n_k+1} - \hat{x})\rangle\\ &\leq \langle f(\hat{y}) - \hat{x}, J_\varphi(\bar{x} - \hat{x})\rangle \leq 0. \qquad (3.21)\end{aligned}$$

Similarly, since $y_{n_k} \rightharpoonup \bar{y}$, it follows from (2.1) that

$$\begin{aligned}\limsup_{n\to\infty} \langle g(\hat{x}) - \hat{y}, J_\varphi(y_{n+1} - \hat{y})\rangle &= \lim_{k\to\infty} \langle g(\hat{x}) - \hat{y}, J_\varphi(y_{n_k+1} - \hat{y})\rangle\\ &\leq \langle g(\hat{x}) - \hat{y}, J_\varphi(\bar{y} - \hat{y})\rangle \leq 0. \qquad (3.22)\end{aligned}$$

Using Lemma 2.3 in (3.20), and from condition (ii), (3.21), and (3.22), we obtain that $\Gamma_n \to 0$. This implies that $||x_n - \hat{x}|| \to 0$ and $||y_n - \hat{y}|| \to 0$, as $n \to \infty$. Therefore $x_n \to \hat{x}$ and $y_n \to \hat{y}$ as $n \to \infty$. This completes the proof. ■

Remark 3.5. An example of a real sequence which satisfy the conditions of Theorem 3.3 is $a_n = \{\frac{1}{n+1}\}$.

The following consequences can be derived from Theorem 3.3:

1. Suppose $\Gamma = (\bigcap_{k=1}^{N} \mathcal{N}(A_k)) \cap (\bigcap_{i=1}^{M} EP(\Theta_i)) \neq \emptyset$. Putting $x_n = y_n$ and $g(x) = f(x)$ in Theorem 3.3 and by adding x_{n+1} and y_{n+1}, we have that

$$x_{n+1} = a_n f(x_n) + (1-a_n)\frac{(S_N + W_M)}{2} x_n, \quad \forall n \geq 0. \tag{3.23}$$

Let $G := \frac{1}{2}(S_N + W_M)$, then by Corollary 3.2, we have that G is nonexpansive. Thus, we have the following result for approximating the common zero of a finite family of m-accretive operators which is also a common solution of a finite family of equilibrium problems in a real reflexive, strictly convex and smooth Banach space.

Corollary 3.6. *Let E be a real reflexive, strictly convex and smooth Banach space which has a weakly continuous duality mapping J_φ with guage φ and C be a nonempty, closed and convex subset of E. Let $f : C \to C$ be a θ-contraction mapping, $A_k : C \to E$, $k = 1, 2, \dots, N$ be a finite family of m-accretive operators and $\Theta_i : C \times C \to \mathbb{R}$, $i = 1, 2, \dots, M$ be bifunctions satisfying assumptions (A1)-(A4). Suppose $\Gamma := \cap_{k=1}^{N} \mathcal{N}(A_k) \bigcap \cap_{i=1}^{M} EP(\Theta_i) \neq \emptyset$. Let $\{a_n\}$ be a real sequence in $(0,1)$ and for arbitrarily $x_0 \in C$, let the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = a_n f(x_n) + (1-a_n)\frac{1}{2}(S_N + W_M)x_n, \quad \forall n \geq 0, \tag{3.24}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{A_1} + \alpha_2 J_{\lambda_n}^{A_2} + \dots + \alpha_N J_{\lambda_n}^{A_N},$$
$$W_M = \beta_0 I + \beta_1 T_{r_n}^{\Theta_1} + \beta_2 T_{r_n}^{\Theta_2} + \dots + \beta_M T_{r_n}^{\Theta_M},$$

with $J_{\lambda_n}^{A_k} := (I + \lambda_n A_k)^{-1}$ for $0 < \alpha_k < 1$, $k = 0, 1, 2, \dots, N$, $\sum_{k=0}^{N} \alpha_k = 1$, $\lambda_n > 0$ and $T_{r_n}^{\Theta_i}$ is as defined in (2.4), $0 < \beta_i < 1$, $i = 0, 1, 2, \dots, M$, $\sum_{i=0}^{M} \beta_i = 1$ and $r_n > 0$. Suppose $\{a_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

- *(i)* $\liminf_{n\to\infty} \lambda_n > 0$, $\liminf_{n\to\infty} r_n > 0$,
- *(ii)* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$,
- *(iii)* $\sum_{n=0}^{\infty} |a_n - a_{n-1}| < \infty$ *or (iii*)* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$.

Then, the sequence $\{x_n\}$ converge strongly to an element $\hat{x} = Q_\Gamma(f(\hat{y}))$, where Q_Γ is the sunny nonexpansive retraction of C onto Γ.

2. Putting $k = 1$ and $i = 1$ in Theorem 3.3, we have the following result.

Corollary 3.7. *Let E be a real reflexive, strictly convex and smooth Banach space which has a weakly continuous duality mapping J_φ with guage φ and C be a nonempty, closed and convex subset of E. Let $f : C \to C$ be a θ_1-contraction mapping and $g : C \to C$ be a θ_2-contraction mapping such that $\theta = \max\{\theta_1, \theta_2\}$. Let $A : C \to E$ be an m-accretive operator and $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (A1)-(A4). Suppose $\mathcal{N}(A)$ and $EP(\Theta)$ are nonempty. Let $\{a_n\}$ be a real sequence in $(0,1)$ and for arbitrarily $x_0 \in C$ and $y_0 \in C$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated simultaneously by*

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1-a_n) J_{\lambda_n}^{A} x_n, \\ y_{n+1} = a_n g(x_n) + (1-a_n) T_{r_n}^{\Theta} y_n, \quad \forall n \geq 0. \end{cases} \tag{3.25}$$

Suppose $\{a_n\}$*,* $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n > 0$*,* $\liminf_{n\to\infty} r_n > 0$*,*

(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$*,*

(iii) $\sum_{n=0}^{\infty} |a_n - a_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$*.*

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to elements* $\hat{x} = Q_1(f(\hat{y}))$ *in* $\mathcal{N}(A)$ *and* $\hat{y} = Q_2(g(\hat{x}))$ *in* $EP(\Theta)$ *respectively, where* Q_1 *is the sunny nonexpansive retraction of* C *onto* $\mathcal{N}(A)$ *and* Q_2 *is the sunny nonexpansive retraction of* C *onto* $EP(\Theta)$*.*

3. Consider the gauge function $\varphi(t) = t$ and let E be a uniformly convex real Banach space with a uniformly Gâteaux differentiable norm. In this case, the duality mapping J_φ becomes the normalized duality mapping J. Thus, the following result can be obtained from Theorem 3.3.

Corollary 3.8. *Let* E *be a uniformly convex real Banach space with a uniformly Gâteaux differentiable norm and* C *be a nonempty, closed and convex subset of* E*. Let* $f : C \to C$ *be a* θ_1*-contraction mapping and* $g : C \to C$ *be a* θ_2*-contraction mapping such that* $\theta = \max\{\theta_1, \theta_2\}$*. Let* $A_k : C \to E$*,* $k = 1, 2, \ldots, N$ *be a finite family of m-accretive operators and* $\Theta_i : C \times C \to \mathbb{R}$*,* $i = 1, 2, \ldots, M$ *be bifunctions satisfying assumptions (A1)-(A4). Suppose* $\cap_{k=1}^{N} \mathcal{N}(A_k)$ *and* $\cap_{i=1}^{M} EP(\Theta_i)$ *are nonempty. Let* $\{a_n\}$ *be a real sequence in* $(0, 1)$ *and for arbitrarily* $x_0 \in C$ *and* $y_0 \in C$*, let the sequence* $\{x_n\}$ *and* $\{y_n\}$ *be generated simultaneously by*

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1 - a_n) S_N x_n, \\ y_{n+1} = a_n g(x_n) + (1 - a_n) W_M y_n, \quad \forall n \geq 0, \end{cases} \tag{3.26}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{A_1} + \alpha_2 J_{\lambda_n}^{A_2} + \cdots + \alpha_N J_{\lambda_n}^{A_N},$$

$$W_M = \beta_0 I + \beta_1 T_{r_n}^{\Theta_1} + \beta_2 T_{r_n}^{\Theta_2} + \cdots + \beta_M T_{r_n}^{\Theta_M},$$

with $J_{\lambda_n}^{A_k} := (I + \lambda_n A_k)^{-1}$ *for* $0 < \alpha_k < 1$*,* $k = 0, 1, 2, \ldots, N$*,* $\sum_{k=0}^{N} \alpha_k = 1$*,* $\lambda_n > 0$ *and* $T_{r_n}^{\Theta_i}$ *is as defined in* (2.4)*,* $0 < \beta_i < 1$*,* $i = 0, 1, 2, \ldots, M$*,* $\sum_{i=0}^{M} \beta_i = 1$ *and* $r_n > 0$*. Suppose* $\{a_n\}$*,* $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n > 0$*,* $\liminf_{n\to\infty} r_n > 0$*,*

(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$*,*

(iii) $\sum_{n=0}^{\infty} |a_n - a_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$*.*

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to elements* $\hat{x} = \Pi_1(f(\hat{y})) \in \cap_{k=1}^{N} \mathcal{N}(A_k)$ *and* $\hat{y} = \Pi_2(g(\hat{x})) \in \cap_{i=1}^{M} EP(\Theta_i)$ *respectively, where* Π_1 *is the generalized projection of* C *onto* $\cap_{k=1}^{N} \mathcal{N}(A_k)$ *and* Π_2 *is the generalized projection of* C *onto* $\cap_{i=1}^{M} EP(\Theta_i)$*.*

4. Let E be a real Hilbert space, then the duality mapping J_φ becomes and identity operator. Thus we obtain the following direct result from Theorem 3.3.

Corollary 3.9. *Let* E *be a real Hilbert space and* C *be a nonempty, closed and convex subset of* E*. Let* $f : C \to C$ *be a* θ_1*-contraction mapping and* $g : C \to C$ *be a* θ_2*-contraction mapping such that* $\theta = \max\{\theta_1, \theta_2\}$*. Let* $A_k : C \to E$*,* $k = 1, 2, \ldots, N$ *be a finite family of m-accretive operators and* $\Theta_i : C \times C \to \mathbb{R}$*,* $i = 1, 2, \ldots, M$ *be bifunctions satisfying assumptions (A1)-(A4). Suppose* $\cap_{k=1}^{N} \mathcal{N}(A_k)$ *and* $\cap_{i=1}^{M} EP(\Theta_i)$ *are nonempty.*

Let $\{a_n\}$ be a real sequence in $(0,1)$ and for arbitrarily $x_0 \in C$ and $y_0 \in C$, let the sequence $\{x_n\}$ and $\{y_n\}$ be generated simultaneously by

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1-a_n)S_N x_n, \\ y_{n+1} = a_n g(x_n) + (1-a_n)W_M y_n, \quad \forall n \geq 0, \end{cases} \tag{3.27}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{A_1} + \alpha_2 J_{\lambda_n}^{A_2} + \cdots + \alpha_N J_{\lambda_n}^{A_N},$$

$$W_M = \beta_0 I + \beta_1 T_{r_n}^{\Theta_1} + \beta_2 T_{r_n}^{\Theta_2} + \cdots + \beta_M T_{r_n}^{\Theta_M},$$

with $J_{\lambda_n}^{A_k} := (I + \lambda_n A_k)^{-1}$ for $0 < \alpha_k < 1$, $k = 0,1,2,\ldots,N$, $\sum_{k=0}^N \alpha_k = 1$, $\lambda_n > 0$ and $T_{r_n}^{\Theta_i}$ is as defined in (2.4), $0 < \beta_i < 1$, $i = 0,1,2,\ldots,M$, $\sum_{i=0}^M \beta_i = 1$ and $r_n > 0$. Suppose $\{a_n\}$, $\{\lambda_n\}$ and $\{r_n\}$ satisfy the following conditions:

- *(i)* $\liminf_{n\to\infty} \lambda_n > 0$, $\liminf_{n\to\infty} r_n > 0$,
- *(ii)* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^\infty a_n = +\infty$,
- *(iii)* $\sum_{n=0}^\infty |a_n - a_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to elements $\hat{x} = P_1(f(\hat{y})) \in \cap_{k=1}^N \mathcal{N}(A_k)$ and $\hat{y} = P_2(g(\hat{x})) \in \cap_{i=1}^M EP(\Theta_i)$ respectively, where P_1 is the metric projection of C onto $\cap_{k=1}^N \mathcal{N}(A_k)$ and P_2 is the metric projection of C onto $\cap_{i=1}^M EP(\Theta_i)$.

4. Applications

In this section, we give some applications of our result to approximating solutions of other nonlinear problems.

4.1. Convex Minimization Problem and Equilibrium Problem:

Let C be a nonempty, closed and convex subset of a real Hilbert space E and let $\phi : C \to \mathbb{R}$ be a proper, convex and lower semicontinous function. The minimization problem can be formulated as finding a point $x \in C$ such that

$$\phi(x) \leq \phi(y), \quad \forall y \in C. \tag{4.1}$$

We denote the set of solution of (4.1) by $MP(\phi)$. It is well known that the subdifferential $\partial\phi$ is m-accretive and its resolvent operator $J_\lambda^{\partial\phi}$ define by

$$J_\lambda^{\partial\phi} = \underset{u\in C}{argmin}\Big\{\phi(u) + \frac{1}{2}||x-u||^2\Big\}, \quad \forall x \in E$$

is nonexpansive and single-valued. Also $F(J_\lambda^{\partial\phi}) = MP(\phi)$. Setting $A_k = \partial\phi_k$, $k = 1,2,\ldots,N$ in Theorem 3.3, we obtain the following result for approximating a common solution of finite family of minimization problem and a common solution of a finite family of equilibrium problems simultaneously in a real Hilbert space.

Theorem 4.1. *Let E be a real Hilbert space and C be a nonempty, closed and convex subset of E. Let $f : C \to C$ be a θ_1-contraction mapping and $g : C \to C$ be a θ_2-contraction mapping such that $\theta = \max\{\theta_1, \theta_2\}$. Let $\phi_k : C \to E$, $k = 1,2,\ldots,N$ be a finite family of proper, convex and lower semicontinuous functions and $\Theta_i : C \times C \to \mathbb{R}$, $i = 1,2,\ldots,M$ be bifunctions satisfying assumptions (A1)-(A4). Suppose $\cap_{k=1}^N MP(\phi_k)$*

and $\cap_{i=1}^{M} EP(\Theta_i)$ *are nonempty. Let* $\{a_n\}$ *be a real sequence in* $(0,1)$ *and for arbitrarily* $x_0 \in C$ *and* $y_0 \in C$, *let the sequence* $\{x_n\}$ *and* $\{y_n\}$ *be generated simultaneously by*

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1-a_n)S_N x_n, \\ y_{n+1} = a_n g(x_n) + (1-a_n)W_M y_n, \quad \forall n \geq 0, \end{cases} \tag{4.2}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{\partial\phi_1} + \alpha_2 J_{\lambda_n}^{\partial\phi_2} + \cdots + \alpha_N J_{\lambda_n}^{\partial\phi_N},$$
$$W_M = \beta_0 I + \beta_1 T_{r_n}^{\Theta_1} + \beta_2 T_{r_n}^{\Theta_2} + \cdots + \beta_M T_{r_n}^{\Theta_M},$$

with $0 < \alpha_k < 1$, $k = 0,1,2,\dots,N$, $\sum_{k=0}^{N} \alpha_k = 1$, $\lambda_n > 0$ *and* $0 < \beta_i < 1$, $i = 0,1,2,\dots,M$, $\sum_{i=0}^{M} \beta_i = 1$ *and* $r_n > 0$. *Suppose* $\{a_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty} \lambda_n > 0$, $\liminf_{n\to\infty} r_n > 0$,
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$,
(iii) $\sum_{n=0}^{\infty} |a_n - a_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to elements* $\hat{x} = Q_1(f(\hat{y})) \in \cap_{k=1}^{N} MP(\phi_k)$ *and* $\hat{y} = Q_2(g(\hat{x})) \in \cap_{i=1}^{M} EP(\Theta_i)$ *respectively, where* Q_1 *is the sunny nonexpansive retraction of* C *onto* $\cap_{k=1}^{N} MP(\phi_k)$ *and* Q_2 *is the sunny nonexpansive retraction of* C *onto* $\cap_{i=1}^{M} EP(\Theta_i)$.

4.2. Fixed Point of Psudocontractive Mapping and Equilibrium Problem:

Let C be a nonempty, closed and convex subset of a real Banach space E which admits a weakly continuous duality mapping J_φ. A mapping $T : C \to C$ is said to be pseudocontractive if for all $x, y \in C$, there exists $j(x-y) \in J_\varphi(x-y)$ such that

$$\langle Tx - Ty, j_\varphi(x-y) \rangle \geq 0.$$

It is well known that the class of pseudocontractive mapping is more general than the class of nonexpansive mapping. Moreso, the class of accretive operator $A : C \to E$ is said to be pseudocontractive if $T := I - A$ is accretive (see [40]). Also, A is m-accretive if and only if $T = I - A$ is continuous pseudocontractive. In this case, the resolvent operator $J_\lambda^T : E \to D(T)$ is defined by $J_\lambda^T = (2I - \lambda T)^{-1}$. Putting $T_k = I - A_k$ in Theorem 3.3, we obtain the following result for approximating common fixed point of finite family of continuous pseudocontractive mappings and common solution of equilibrium problem simultaneously in real Banach space.

Theorem 4.2. *Let* E *be a real reflexive, strictly convex and smooth Banach space which has a weakly continuous duality mapping* J_φ *with guage* φ *and* C *be a nonempty, closed and convex subset of* E. *Let* $f : C \to C$ *be a* θ_1*-contraction mapping and* $g : C \to C$ *be a* θ_2*-contraction mapping such that* $\theta = \max\{\theta_1, \theta_2\}$. *Let* $T_k : C \to E$, $k = 1,2,\dots,N$ *be a finite family of continuous pseudocontractive mappings and* $\Theta_i : C \times C \to \mathbb{R}$, $i = 1,2,\dots,M$ *be bifunctions satisfying assumptions (A1)-(A4). Suppose* $\cap_{k=1}^{N} F(T_k)$ *and* $\cap_{i=1}^{M} EP(\Theta_i)$ *are nonempty. Let* $\{a_n\}$ *be a real sequence in* $(0,1)$ *and for arbitrarily* $x_0 \in C$ *and* $y_0 \in C$, *let the sequence* $\{x_n\}$ *and* $\{y_n\}$ *be generated simultaneously by*

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1-a_n)S_N x_n, \\ y_{n+1} = a_n g(x_n) + (1-a_n)W_M y_n, \quad \forall n \geq 0, \end{cases} \tag{4.3}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{A_1} + \alpha_2 J_{\lambda_n}^{A_2} + \cdots + \alpha_N J_{\lambda_n}^{A_N},$$
$$W_M = \beta_0 I + \beta_1 T_{r_n}^{\Theta_1} + \beta_2 T_{r_n}^{\Theta_2} + \cdots + \beta_M T_{r_n}^{\Theta_M},$$

with $J_{\lambda_n}^{T_k} := (2I + \lambda_n T_k)^{-1}$ *for* $0 < \alpha_k < 1$, $k = 0, 1, 2, \ldots, N$, $\sum_{k=0}^{N} \alpha_k = 1$, $\lambda_n > 0$ *and* $T_{r_n}^{\Theta_i}$ *is as defined in* (2.4), $0 < \beta_i < 1$, $i = 0, 1, 2, \ldots, M$, $\sum_{i=0}^{M} \beta_i = 1$ *and* $r_n > 0$. *Suppose* $\{a_n\}$, $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

- *(i)* $\liminf_{n\to\infty} \lambda_n > 0$, $\liminf_{n\to\infty} r_n > 0$,
- *(ii)* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$,
- *(iii)* $\sum_{n=0}^{\infty} |a_n - a_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|a_n - a_{n-1}|}{a_n} = 0$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to elements* $\hat{x} = Q_1(f(\hat{y})) \in \cap_{k=1}^{N} F(T_k)$ *and* $\hat{y} = Q_2(g(\hat{x})) \in \cap_{i=1}^{M} EP(\Theta_i)$ *respectively, where* Q_1 *is the sunny nonexpansive retraction of* C *onto* $\cap_{k=1}^{N} F(T_k)$ *and* Q_2 *is the sunny nonexpansive retraction of* C *onto* $\cap_{i=1}^{M} EP(\Theta_i)$.

4.3. Inclusion Problem and Monotone Variational Inequality Problem:

Let C be a nonempty, closed and convex subset of a real Banach space E. Suppose B is a monotone operator from C into E. The variational inequality problem is formulated as finding a point $u \in C$ such that

$$\langle Bu, x - u\rangle \geq 0, \quad x \in C. \tag{4.4}$$

The set of solution of (4.4) is denoted by $VIP(C, B)$. It is known that if $\Theta(x, y) = \langle Bx, y - x\rangle$ for all $x, y \in C$, then the EP (1.4) is equivalent to the VIP (4.4). The following lemma can be found in [49].

Lemma 4.3. *Let* C *be a closed and convex subset of a smooth, strictly convex and reflexive Banach space* E. *Let* $B : C \to E^*$ *be a smooth, strictly convex and reflexive Banach space. For* $r > 0$ *and* $x \in E$, *define the mapping*

$$S_r^B x = \Big\{w \in C : \langle Bw, z - w\rangle + \frac{1}{r}\langle y - w, Jw - Jx\rangle \geq 0, \quad \forall z \in C\Big\}. \tag{4.5}$$

Then the following hold:

- *(1)* S_r^B *is single-valued,*
- *(2) for all* $x, y \in E$,
$$\langle S_r^B x - S_r^B y, JS_r^B x - JS_r^B y\rangle \leq \langle S_r^B x - S_r^B y, Jx - Jy\rangle,$$
- *(3)* $F(S_r^B) = VIP(C, B)$,
- *(4)* $VIP(C, B)$ *is closed and convex.*

It is clear that putting $\Theta_i(x, y) = \langle B_i x, y - x\rangle$, $i = 1, 2, \ldots, N$ in Theorem 3.3, we obtain the following result for approximating common solution of finite family of inclusion problems and common solution of finite family of equilibrium problems simultaneously.

Theorem 4.4. *Let* E *be a real reflexive, strictly convex and smooth Banach space which has a weakly continuous duality mapping* J_φ *with guage* φ *and* C *be a nonempty, closed and convex subset of* E. *Let* $f : C \to C$ *be an* θ_1*-contraction mapping and* $g : C \to C$ *be an* θ_2*-contraction mapping such that* $\theta = \max\{\theta_1, \theta_2\}$. *Let* $A_k : C \to E$, $k = 1, 2, \ldots, N$ *be a finite family of* m*-accretive operators and* $B_i : C \to E^*$, $i = 1, 2, \ldots, M$ *be continuous* *monotone operators. Suppose* $\cap_{k=1}^{N}\mathcal{N}(A_k)$ *and* $\cap_{i=1}^{M}VIP(C,A_i)$ *are nonempty. Let* $\{a_n\}$ *be a real sequence in* $(0,1)$ *and for arbitrarily* $x_0 \in C$ *and* $y_0 \in C$*, let the sequence* $\{x_n\}$ *and* $\{y_n\}$ *be generated simultaneously by*

$$\begin{cases} x_{n+1} = a_n f(y_n) + (1-a_n)S_N x_n, \\ y_{n+1} = a_n g(x_n) + (1-a_n)W_M y_n, \quad \forall n \geq 0, \end{cases} \tag{4.6}$$

where

$$S_N = \alpha_0 I + \alpha_1 J_{\lambda_n}^{A_1} + \alpha_2 J_{\lambda_n}^{A_2} + \cdots + \alpha_N J_{\lambda_n}^{A_N},$$

$$W_M = \beta_0 I + \beta_1 S_{r_n}^{B_1} + \beta_2 S_{r_n}^{B_2} + \cdots + \beta_M S_{r_n}^{B_M},$$

with $J_{\lambda_n}^{A_k} := (I+\lambda_n A_k)^{-1}$ *for* $0 < \alpha_k < 1$*,* $k = 0,1,2,\ldots,N$*,* $\sum_{k=0}^{N}\alpha_k = 1$*,* $\lambda_n > 0$ *and* $S_{r_n}^{A_i}$ *is as defined in* (4.5)*,* $0 < \beta_i < 1$*,* $i = 0,1,2,\ldots,M$*,* $\sum_{i=0}^{M}\beta_i = 1$ *and* $r_n > 0$*. Suppose* $\{a_n\}$*,* $\{\lambda_n\}$ *and* $\{r_n\}$ *satisfy the following conditions:*

(i) $\liminf_{n\to\infty}\lambda_n > 0$, $\liminf_{n\to\infty} r_n > 0$,

(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = +\infty$,

(iii) $\sum_{n=0}^{\infty}|a_n - a_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|a_n-a_{n-1}|}{a_n} = 0$.

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to elements* $\hat{x} = Q_1(f(\hat{y})) \in \cap_{k=1}^{N}\mathcal{N}(A_k)$ *and* $\hat{y} = Q_2(g(\hat{x})) \in \cap_{i=1}^{M}VIP(C,A_i)$ *respectively, where* Q_1 *is the sunny nonexpansive retraction of* C *onto* $\cap_{k=1}^{N}\mathcal{N}(A_k)$ *and* Q_2 *is the sunny nonexpansive retraction of* C *onto* $\cap_{i=1}^{M}VIP(C,A_i)$*.*

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