Thai Journal of Mathematics
Volume 19 Number 2 (2021)
Pages 653–663

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Critical Point Equation on 3-Dimensional Trans-Sasakian Manifolds

Dibakar Dey

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata - 700019, West Bengal, India
e-mail : deydibakar3@gmail.com

Abstract The object of the present paper is to characterize 3-dimensional trans-Sasakian manifolds satisfying the critical point equation under the condition $\phi\,\mathrm{grad}\,\alpha = \mathrm{grad}\,\beta$. Also, we present few examples which verifies our results.

MSC: 53C25; 53C15

Keywords: trans-Sasakian manifold; Einstein manifold; Ricci tensor; flat manifold; critical point equation

Submission date: 15.11.2018 / Acceptance date: 18.03.2021

1. Introduction

Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional almost contact metric manifold [1]. Then, the product $\overline{M} = M \times \mathbb{R}$ has a natural almost complex structure J, which makes $(\overline{M}, G)$ an almost Hermitian manifold, where G is the product metric. The geometry of the almost Hermitian manifold $(\overline{M}, J, G)$ dictates the geometry of the almost contact metric manifold (M, ϕ, ξ, η, g) and gives different structures on M like Sasakian structure, quasi-Sasakian structure, Kenmotsu structure and others (see [1–3]). $\mathcal{W}_4$ is the class of almost Hermitian manifolds M satisfying the identity

$$\begin{aligned}\nabla_X(F)(Y,Z) &= \frac{-1}{2(n-1)}\{\langle X,Y\rangle\delta F(Z) - \langle X,Z\rangle\delta F(Y)\\ &\quad -\langle X,JY\rangle\delta F(JZ) + \langle X,JZ\rangle\delta F(JY)\},\end{aligned}$$

where $2n$ is the real dimension of M, F is the Kaehler form and δ denotes the coderivative. Three facts about the class $\mathcal{W}_4$ are noteworthy: (1) Any manifold in $\mathcal{W}_4$ automatically has an integrable almost complex structure. (2) Any manifold locally conformally equivalent to a Kaehler manifold is in $\mathcal{W}_4$. (3) Let the Lee form θ of an almost Hermitian manifold M be defined by $\theta = \delta F \cdot J$. Suppose $M \in \mathcal{W}_4$, then M is locally or globally conformally Kaehlerian according to whether θ is closed or exact. It is known that there are sixteen different types of structures on the almost Hermitian manifold $(\overline{M}, J, G)$ (see [4])

Published by The Mathematical Association of Thailand.
Copyright © 2021 by TJM. All rights reserved.

and recently, using the structure in the class $\mathcal{W}_4$ on $(\overline{M}, J, G)$ a structure $(\phi, \xi, \eta, g, \alpha, \beta)$ on M called trans-Sasakian structure is introduced [5], which generalizes Sasakian structure and Kenmotsu structure on almost contact metric manifolds ([2], [3]), where α, β are smooth functions defined on M. Since the introduction of trans-Sasakian manifold, important contribution of Blair and Oubiña [2] and Marrero [6] have appeared to study the geometry of trans-Sasakian manifolds. In general, a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β) and trans-Sasakian manifolds of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are called a cosymplectic, a α-Sasakian and a β-Kenmotsu manifolds, respectively, provided α, $\beta \in \mathbb{R}$ [7]. Marrero [6] has shown that a trans-Sasakian manifold of dimension ≥ 5 is either a cosymplectic manifold, a α-Sasakian manifold or a β-Kenmotsu manifold. Since then, there is an attention on studying geometry of 3-dimensional trans-Sasakian manifolds only. In [8–13], authors have studied 3-dimensional trans-Sasakian manifolds with some restrictions on the smooth functions α, β appearing in the definition of trans-Sasakian manifolds. There are several examples of trans-Sasakian manifolds constructed mostly on 3-dimensional Riemannian manifolds (see [2, 6, 14]).

A Riemannian manifold (M, g) of dimension $n \geq 3$ with constant scalar curvature and unit volume together with a non-constant smooth potential function λ satisfying the equation

$$Hess\lambda - (S - \frac{r}{n-1}g)\lambda = S - \frac{r}{n}g \tag{1.1}$$

is called a critical point equation (in short, CPE) on M, where S is the Ricci tensor defined by $S(X, Y) = g(QX, Y)$, Q is the Ricci operator, r is the scalar curvature and $Hess\lambda$ is the Hessian of the smooth function λ.
Note that if $\lambda = 0$, then (1.1) becomes Einstein metric. Therefore, we consider only non-trivial potential function λ. In [15], Besse conjectured that the solution of the CPE is Einstein. Barros and Ribeiro [16] proved that the CPE conjecture is true for half conformally flat. In [17], Hwang proved that the CPE conjecture is also true under certain conditions on the bounds of the potential function λ. Very recently, Neto [18] deduced a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein.

Throughout the paper, we assume that the smooth functions α and β satisfy the condition

$$\phi \,\mathrm{grad}\, \alpha = \mathrm{grad}\, \beta. \tag{1.2}$$

Then it follows that

$$X\beta + (\phi X)\alpha = 0 \tag{1.3}$$

and hence, $\xi\beta = 0$.

Since trans-Sasakian manifold M generalizes a large class of almost contact metric manifolds, we consider the CPE conjecture in the frame-work of 3-dimensional trans-Sasakian manifolds of type (α, β). We proved that if (g, λ) is a non-constant solution of the critical point equation, then the manifold M is either a space of constant curvature or β-Kenmotsu or flat. We also study the CPE on M when it is complete.

2. Preliminaries

Let (M, ϕ, ξ, η, g) be a 3-dimensional almost contact metric manifold, where ϕ being a $(1,1)$-tensor field, ξ a unit vector field and η smooth 1-form dual to ξ with respect to the Riemanian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields $X, Y \in \chi(M)$, where $\chi(M)$ being the Lie algebra of smooth vector fields on M [1]. If there are smooth functions α, β on an almost contact metric manifold (M, ϕ, ξ, η, g) satisfying

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.3}$$

for any vector fields $X, Y \in \chi(M)$, then it is said to be a trans-Sasakian manifold, where ∇ is the Levi-Civita connection with respect to the metric g [2, 6, 14]. We shall denote the trans-Sasakian manifold by $(M, \phi, \xi, \eta, g, \alpha, \beta)$ and it is called trans-Sasakian manifold of type (α, β). From (2.3), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.4}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.5}$$

A trans-Sasakian manifold is said to be

- cosymplectic or co-Kaehler if $\alpha = \beta = 0$.
- quasi-Sasakian manifold if $\beta = 0$ and $\xi(\alpha) = 0$.
- α-Sasakian manifold if α is a non-zero constant and $\beta = 0$.
- β-Kenmotsu manifold if $\alpha = 0$ and β is a non-zero constant.

Therefore, trans-Sasakian manifold generalizes a large class of almost contact manifolds. For a 3-dimensional trans-Sasakian manifold (see [8]), we have

$$2\alpha\beta + \xi\alpha = 0. \tag{2.6}$$

The Ricci operator Q satisfies [8]

$$Q(\xi) = \phi(\nabla\alpha) - \nabla\beta + 2(\alpha^2 - \beta^2)\xi - g(\nabla\beta, \xi)\xi. \tag{2.7}$$

$$\begin{aligned} S(X, Y) \quad = \quad & (\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2))g(X, Y) \\ & -(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y) \\ & -(Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y) \\ &-g(Y,Z)((\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\xi \\ &-\eta(X)(\phi\,\mathrm{grad}\,\alpha-\mathrm{grad}\,\beta)+(X\beta+(\phi X)\alpha)\xi) \\ &+g(X,Z)((\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\xi \\ &-\eta(Y)(\phi\,\mathrm{grad}\,\alpha-\mathrm{grad}\,\beta)+(Y\beta+(\phi Y)\alpha)\xi) \\ &-((Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z) \\ &+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\eta(Z))X \\ &((Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z) \\ &+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Z))Y \end{aligned} \tag{2.9}$$

hold, where S is the Ricci tensor of type $(0,2)$, R is the Riemannian curvature tensor of type $(1,3)$ and r is the scalar curvature of the manifold M.
If M satisfies the condition (1.2), then equations (2.8) and (2.9) reduces to

$$S(X,Y)=(\frac{r}{2}-(\alpha^2-\beta^2))g(X,Y)-(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{2.10}$$

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2}-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y) \\ &-g(Y,Z)((\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\xi \\ &+g(X,Z)((\frac{r}{2}-3(\alpha^2-\beta^2))\eta(Y)\xi) \\ &-(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)X \\ &+(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Z)Y. \end{aligned} \tag{2.11}$$

From (2.10), we get

$$S(X,\xi)=2(\alpha^2-\beta^2)\eta(X) \tag{2.12}$$

and from (2.11), it follows that

$$R(X,Y)\xi=(\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y), \tag{2.13}$$

$$R(\xi,X)Y=(\alpha^2-\beta^2)(g(X,Y)\xi-\eta(Y)X). \tag{2.14}$$

3. CPE on 3-Dimensional Trans-Sasakian Manifolds

In this section, we study CPE on 3-dimensional trans-Sasakian manifolds under the condition (1.2). To prove our main results, we first state the followings:

Lemma 3.1. (Lemma 3.1 of [19]) *Let (g,λ) be a non-trivial solution of the CPE (1.1) on an n-dimentional Riemannian manifold M. Then the curvature tensor R can be expressed*

$$\begin{aligned} R(X,Y)D\lambda &= (X\lambda)QY-(Y\lambda)QX+(\lambda+1)(\nabla_X Q)Y \\ &-(\lambda+1)(\nabla_Y Q)X+(Xf)Y-(Yf)X, \end{aligned}$$

where D is the gradient operator and $f = -r(\frac{\lambda}{n-1} + \frac{1}{n})$.

Lemma 3.2. (Theorem 1 of [20]) *For a trans-Sasakian manifold M^n, $n > 1$, under the condition $\phi \operatorname{grad} \alpha = (n-2) \operatorname{grad} \beta$, we have*

$$\begin{aligned}&[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)]\\&= \beta S(Y,Z) - (n-1)(\alpha^2-\beta^2)\beta g(Y,Z)\\&-(n-1)(\alpha^2-\beta^2)\alpha g(Y,\phi Z) + \alpha S(Y,\phi Z).\end{aligned}$$

Now, for a 3-dimensional trans-Sasakian manifold under the condition (1.2), we can write from Lemma 3.2

$$\begin{aligned}[(\nabla_\xi Q)Y - (\nabla_Y Q)\xi] &= \beta QY - 2(\alpha^2-\beta^2)\beta Y\\&\quad +2(\alpha^2-\beta^2)\alpha\phi Y - \alpha\phi QY. \qquad (3.1)\end{aligned}$$

Lemma 3.3. ([21]) *A contact metric manifold M^{2n+1} satisfying the condition $R(X,Y)\xi = 0$ for all X, Y is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

We now prove our main results.

Theorem 3.4. *Let $(M, \phi, \xi, \eta, g, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold such that α is identically zero or nowhere vanishing satisfying the condition $\phi \operatorname{grad} \alpha = \operatorname{grad} \beta$. If (g, λ) is a non-constant solution of the critical point equation (1.1), then the manifold M is either a space of constant curvature or β-Kenmotsu or flat.*

Proof. From Lemma (3.1), we have

$$\begin{aligned}R(X,Y)D\lambda &= (X\lambda)QY - (Y\lambda)QX + (\lambda+1)(\nabla_X Q)Y\\&\quad -(\lambda+1)(\nabla_Y Q)X + (Xf)Y - (Yf)X. \qquad (3.2)\end{aligned}$$

Substituting ξ in place of X in the above equation and using (3.1), we get

$$\begin{aligned}R(\xi,Y)D\lambda &= (\xi\lambda)QY - 2(\alpha^2-\beta^2)(Y\lambda)\xi + (\lambda+1)(\beta QY\\&\quad -2(\alpha^2-\beta^2)\beta Y + 2(\alpha^2-\beta^2)\alpha\phi Y - \alpha\phi QY)\\&\quad -\frac{r}{2}(\xi\lambda)Y + \frac{r}{2}(Y\lambda)\xi. \qquad (3.3)\end{aligned}$$

Taking inner product of (3.3) with ξ and using (2.2), we obtain

$$\begin{aligned}g(R(\xi,Y)D\lambda,\xi) &= 2(\alpha^2-\beta^2)(\xi\lambda)\eta(Y) - 2(\alpha^2-\beta^2)(Y\lambda)\\&\quad -\frac{r}{2}(\xi\lambda)\eta(Y) + \frac{r}{2}(Y\lambda). \qquad (3.4)\end{aligned}$$

Again,

$$g(R(\xi,Y)D\lambda,\xi) = -g(R(\xi,Y)\xi,D\lambda).$$

Making use of (2.13) the above equation yields

$$g(R(\xi,Y)D\lambda,\xi) = -(\alpha^2-\beta^2)(\xi\lambda)\eta(Y) + (\alpha^2-\beta^2)(Y\lambda). \qquad (3.5)$$

Equations (3.4) and (3.5) together implies

$$(\frac{r}{2}-3(\alpha^2-\beta^2))(D\lambda-(\xi\lambda)\xi)=0, \tag{3.6}$$

which implies that either $r=6(\alpha^2-\beta^2)$ or $D\lambda=(\xi\lambda)\xi$.

Case 1: If $r=6(\alpha^2-\beta^2)$, then from (2.10), we have

$$S(X,Y)=2(\alpha^2-\beta^2)g(X,Y),$$

which implies that the manifold is Einstein. Since the dimension of the manifold is 3, therefore it becomes a space of constant curvature.

Case 2: Let $D\lambda=(\xi\lambda)\xi$. Then from (1.1), we can write

$$\nabla_X D\lambda=(\lambda+1)QX+fX, \tag{3.7}$$

where $f=-r(\frac{\lambda}{2}+\frac{1}{3})$.
Making use of $D\lambda=(\xi\lambda)\xi$ and (2.4) in (3.7), we obtain

$$\begin{aligned}(\lambda+1)QX &= [X(\xi\lambda)-\beta\xi(\lambda)\eta(X)]\xi \\ &\quad +[\beta\xi(\lambda)+r(\frac{\lambda}{2}+\frac{1}{3})]X-\alpha\xi(\lambda)\phi X.\end{aligned} \tag{3.8}$$

Comparing (3.8) with (2.10), we have the following equations:

$$\beta\xi(\lambda)+r(\frac{\lambda}{2}+\frac{1}{3})=(\lambda+1)(\frac{r}{2}-(\alpha^2-\beta^2)), \tag{3.9}$$

$$X(\xi\lambda)-\beta\xi(\lambda)\eta(X)=-(\lambda+1)(\frac{r}{2}-3(\alpha^2-\beta^2)) \tag{3.10}$$

and

$$\alpha\xi(\lambda)\phi X=0. \tag{3.11}$$

The equation (3.11) implies that either $\alpha=0$ or $\xi(\lambda)=0$.
If $\alpha=0$, then by hypothesis α is identically zero on M. Hence, from (1.3), we have $\beta=constant$. This implies that the manifold M is β-Kenmotsu.
If $\xi(\lambda)=0$, then (3.9) and (3.10) implies that $r=6(\lambda+1)(\alpha^2-\beta^2)$ and $r=6(\alpha^2-\beta^2)$, respectively. Equating these two values of r, we obtain $\alpha^2=\beta^2$ as λ is a non-constant function. Hence, from (2.13), we get $R(X,Y)\xi=0$. Since the manifold is of dimension 3, it follows from Lemma (3.3) that the manifold is flat. This completes the proof of our theorem. ∎

Remark 3.5. In [22], the authors proposed that a trans-Sasakian 3-manifold M of type $(0,\beta)$ satisfying $\nabla\beta=(\xi\beta)\xi$ is not necessarily β-Kenmotsu even when M is compact. Here β is constant. If we change the metric g by βg by homothetic transformation, this homothetic transformation gives the homothety between β-Kenmotsu manifold and the Kenmotsu manifold.

Corollary 3.6. *Let M be a 3-dimensional trans-Sasakian manifold satisfying the condition (1.2). If (g,λ) is a non-constant solution of the CPE (1.1), then the manifold M is homothetic to a Kenmotsu manifold, provided M is not of constant curvature.*

Remark 3.7. In [23], the authors proves that if the condition (1.2) holds on M, then ξ is an eigenvector of the Ricci operator. Since, here $\alpha = 0$ and $\beta =$ constant, from (2.7), it follows that

$$S(X,\xi) = -2\beta^2 g(X,\xi),$$

which implies that the characteristic vector field ξ is an eigen vector of the Ricci operator Q corresponding to the eigenvalue $-2\beta^2$.

Theorem 3.8. *Let $(M,\phi,\xi,\eta,g,\alpha,\beta)$ be a complete 3-dimensional trans-Sasakian manifold such that α is identically zero or nowhere vanishing fulfilling the condition $\phi\,\mathrm{grad}\,\alpha = \mathrm{grad}\,\beta$. If (g,λ) is a non-constant solution of the critical point equation, then the manifold M is either β-Kenmotsu or isometric to the sphere $S^3(\frac{1}{2})$.*

Proof. Taking inner product of (3.3) with X, we obtain

$$\begin{aligned} g(R(\xi,Y)D\lambda,X) &= (\xi\lambda)S(X,Y) - 2(\alpha^2-\beta^2)(Y\lambda)\eta(X) + (\lambda+1)(\beta S(X,Y) \\ &\quad -2(\alpha^2-\beta^2)\beta g(X,Y) + 2(\alpha^2-\beta^2)\alpha g(X,\phi Y) \\ &\quad -\alpha g(X,\phi QY)) - \frac{r}{2}(\xi\lambda)g(X,Y) + \frac{r}{2}(Y\lambda)\eta(X). \end{aligned} \tag{3.12}$$

Again,

$$g(R(\xi,Y)D\lambda,X) = -g(R(\xi,Y)X,D\lambda).$$

Making use of (2.14) the above equation yields

$$g(R(\xi,Y)D\lambda,X) = -(\alpha^2-\beta^2)g(X,Y)(\xi\lambda) + (\alpha^2-\beta^2)\eta(X)(Y\lambda). \tag{3.13}$$

From (3.12) and (3.13), we get

$$\begin{aligned} &-(\alpha^2-\beta^2)g(X,Y)(\xi\lambda) + (\alpha^2-\beta^2)\eta(X)(Y\lambda) \\ = &\ (\xi\lambda)S(X,Y) - 2(\alpha^2-\beta^2)(Y\lambda)\eta(X) + (\lambda+1)(\beta S(X,Y) \\ &-2(\alpha^2-\beta^2)\beta g(X,Y) + 2(\alpha^2-\beta^2)\alpha g(X,\phi Y) \\ &-\alpha g(X,\phi QY)) - \frac{r}{2}(\xi\lambda)g(X,Y) + \frac{r}{2}(Y\lambda)\eta(X). \end{aligned} \tag{3.14}$$

Now, interchanging X and Y in (3.14) and subtracting the resulting equation from (3.14) gives

$$\begin{aligned} &(\alpha^2-\beta^2)(\eta(X)(Y\lambda) - \eta(Y)(X\lambda)) \\ = &-2(\alpha^2-\beta^2)(\eta(X)(Y\lambda) - \eta(Y)(X\lambda)) \\ &+2\alpha(\alpha^2-\beta^2)(\lambda+1)(g(\phi Y,X) - g(\phi X,Y)) \\ &+\alpha(\lambda+1)(g(\phi QX,Y) - g(\phi QY,X)) + \frac{r}{2}(\eta(X)(Y\lambda) - \eta(Y)(X\lambda)). \end{aligned} \tag{3.15}$$

From (2.10), we can see that $Q\phi = \phi Q$ on M under the condition (1.2). Using this relation, the foregoing equation yields

$$\begin{aligned} (\frac{r}{2} - 3(\alpha^2-\beta^2))(\eta(X)(Y\lambda) - \eta(Y)(X\lambda)) &= 4\alpha(\alpha^2-\beta^2)(\lambda+1)g(\phi X,Y) \\ &\quad -2\alpha(\lambda+1)g(\phi QX,Y). \end{aligned} \tag{3.16}$$

Putting $X = \phi X$ and $Y = \phi Y$ in (3.16) and using (2.2), we get

$$2\alpha(\lambda+1)(2(\alpha^2-\beta^2)g(\phi X,Y) - g(Q\phi X,Y)) = 0. \tag{3.17}$$

Replacing X by ϕX and using (2.1), we have

$$\alpha(S(X,Y) - 2(\alpha^2 - \beta^2)g(X,Y) + (2(\alpha^2 - \beta^2) - 1)\eta(X)\eta(Y)) = 0, \tag{3.18}$$

which implies that
either $\alpha = 0$ or $S(X,Y) = 2(\alpha^2 - \beta^2)g(X,Y) - (2(\alpha^2 - \beta^2) - 1)\eta(X)\eta(Y)$.

Case 1: If $\alpha = 0$, then by hypothesis α is identically zero on M. Hence, from (1.3), we have $\beta = constant$. This implies that the manifold M is β-Kenmotsu.

Case 2: If $S(X,Y) = 2(\alpha^2 - \beta^2)g(X,Y) - (2(\alpha^2 - \beta^2) - 1)\eta(X)\eta(Y)$, then comparing it with (2.10), we obtain

$$\frac{r}{2} - (\alpha^2 - \beta^2) = 2(\alpha^2 - \beta^2) \tag{3.19}$$

and

$$\frac{r}{2} - 3(\alpha^2 - \beta^2) = 2(\alpha^2 - \beta^2) - 1. \tag{3.20}$$

Equating the values of the scalar curvature r obtained from (3.19) and (3.20), we have

$$\alpha^2 - \beta^2 = \frac{1}{2}. \tag{3.21}$$

Therefore, the scalar curvature r and the Ricci tensor S is given by

$$r = 3 \quad and \quad S(X,Y) = g(X,Y). \tag{3.22}$$

Substituting (3.22) into (1.1), we get

$$\nabla^2\lambda = -\frac{1}{2}\lambda g.$$

We now apply Tashiro's theorem [24] that states "If a complete Riemannian manifold M^n of dimension ≥ 2 admits a special concircular field ρ satisfying $\nabla\nabla\rho = (-c^2\rho + b)g$, then it is isometric to a sphere $S^n(c^2)$" to conclude that M is isometric to the sphere $S^3(\frac{1}{2})$. This completes the proof of our theorem. ∎

4. Examples

Example 4.1. In [25], the authors have constructed an example of a 3-dimensional trans-Sasakian manifold. They consider $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = e^{-z}(\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}), \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,$$

$$g(e_1,e_2) = g(e_2,e_3) = g(e_1,e_3) = 0.$$

Let η be the 1-form defined by $\eta(X) = g(X,e_3)$ for any vector field X. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then we have

$$\phi^2(X) = -X + \eta(X)e_3,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y. For $e_3 = \xi$, they have shown that $(M^3, \phi, \xi, \eta, g)$ forms a trans-Sasakian manifold of type (α, β), where $\alpha = \frac{1}{2}e^{-2z}$ and $\beta = 1$. Then it follows that $\phi\,\mathrm{grad}\,\alpha = -e^{-2z}\phi e_3 = 0 = \mathrm{grad}\,\beta$. Note that, α is nowhere vanishing. Thus the existence of trans-Sasakian manifolds of type (α, β) satisfying (1.2) is verified.

Example 4.2. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \;\; e_2 = z\frac{\partial}{\partial y}, \;\; e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_2, \;\; \phi(e_2) = e_1, \;\; \phi(e_3) = 0.$$

Then we have

$$\phi^2(X) = -X + \eta(X)e_3,$$
$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y. Hence the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M, where $e_3 = \xi$. Now, after calculating we have

$$[e_1, e_3] = -e_1, \;\; [e_1, e_2] = 0 \;\; \textit{and} \;\; [e_2, e_3] = -e_2.$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) - g(Z,[X,Y]). \end{aligned}$$

By Koszul's formula we get

$$\nabla_{e_1}e_1 = e_3, \;\; \nabla_{e_1}e_2 = 0, \;\; \nabla_{e_1}e_3 = -e_1,$$
$$\nabla_{e_2}e_1 = 0, \;\; \nabla_{e_2}e_2 = e_3, \;\; \nabla_{e_2}e_3 = -e_2,$$
$$\nabla_{e_3}e_1 = 0, \;\; \nabla_{e_3}e_2 = 0, \;\; \nabla_{e_3}e_3 = 0.$$

From the above we found that $\alpha = 0$, $\beta = -1$ and $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold. Notice that, α is identically zero here.
The Riemannian curvature tensor is given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Therefore, we have

$$R(e_1, e_2)e_1 = e_2, \;\; R(e_1, e_2)e_2 = -e_1, \;\; R(e_1, e_2)e_3 = 0,$$
$$R(e_2, e_3)e_1 = 0, \;\; R(e_2, e_3)e_2 = e_3, \;\; R(e_2, e_3)e_3 = -e_2,$$
$$R(e_1, e_3)e_1 = e_3, \;\; R(e_1, e_3)e_2 = 0, \;\; R(e_1, e_3)e_3 = -e_1.$$

From the expression of the above curvature tensor we obtain

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2.$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6.$$

The CPE (1.1) is given by

$$Hess\lambda - (S - \frac{r}{2}g)\lambda = S - \frac{r}{3}g.$$

Now, tracing the above equation we have

$$\nabla^2\lambda = 3\lambda.$$

Therefore, the required function λ is given by the above Poison equation which satisfies the CPE. Notice that the condition (1.2) is satisfied and the manifold is a β-Kenmotsu manifold. Also from the expressions of the curvature tensor it follows that the manifold is a space of constant curvature -1. Therefore, Theorem 3.4 is verified.

Acknowledgements

The author is thankful to the anonymous referee for his/her valuable suggestions that have improved the article. The author is supported by the Council of Scientific and Industrial Research, India (File no: 09/028(1010)/2017-EMR-1) in the form of Senior Research Fellowship.

References

[1] D.E. Blair, Contact Manifold in Riemannian Geometry, Lecture Notes on Mathematics, Vol. 509, Springer, Berlin, 1976.

[2] D.E. Blair, J.A. Oubiña, Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Math. 34 (1990) 199–207.

[3] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J. 24 (1972) 93–103.

[4] A. Gray, M.H. Luis, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. 123 (1980) 35–58.

[5] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan. 14 (1962) 333–340.

[6] J.C. Marrero, The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl. 162 (1992) 77–86.

[7] D. Janssens, L. Vancheke, Almost contact structures and curvature tensors, Kodai Math. J. 4 (1981) 1–27.

[8] U.C. De, M.M. Tripathi, Ricci tensor in 3-dimensional trans-Sasakian manifolds, Kyungpook Math. J. 43 (2003) 247–255.

[9] U.C. De, A. Sarkar, On three-dimensional trans-Sasakian manifolds, Extr. Math. 23 (2008) 265–277.

[10] S. Deshmukh, M.M. Tripathi, A note on trans-Sasakian manifolds, Math. Slov. 63 (2013) 1361–1370.

[11] S. Deshmukh, Trans-Sasakian manifolds homothetic to Sasakain manifolds, Mediterr. J. Math. 13 (2016) 2951–2958.

[12] J.S. Kim, R. Prasad, M.M. Tripathi, On generalized Ricci-recurrent trans-Sasakian manifolds, J. Korean Math. Soc. 39 (2002) 953–961.

[13] V.F. Kirichenko, On the geometry of nearly trans-Sasakian manifolds (Russian), Dokl Akad. Nauk 397 (2004) 733–736.

[14] J.A. Oubiña, New classes of almost contact metric structures, Publ. Math. Debrecen 32 (1985) 187–193.

[15] A. Besse, Einstein Manifolds, Springer, New York, 2008.

[16] A. Barros, E.Jr. Ribeiro, Critical point equation on four dimensional compact manifolds, Math. Nachr. 287 (2014) 1618–1623.

[17] S. Hwang, Critical points of the total scalar curvature functionals on the space of metrics of constant scalar curvature, Manuscr. Math. 103 (2000) 135–142.

[18] B.L. Neto, A note on critical point metrics of the total scalar curvature functional, J. Math. Anal. Appl. 424 (2015) 1544–1548.

[19] A. Ghosh, D.S. Patra, The critical point equation and contact geometry, J. Geom. 108 (2017) 185–194.

[20] C.S. Bagewadi, D.G. Prakasha, Venkatesha, Conservative projective curvature tensor on trans-Sasakian manifolds with respect to semi-symmetric metric connection, An. St. Univ. Ovidius Constanta 15 (2017) 5–18.

[21] D.E. Blair, Two remarks on contact metric structure, Tohoku Math. J. 29 (1977) 319–324.

[22] Y. Wang, W. Wang, A remark on trans-Sasakian 3-manifolds, Rev. Un. Mat. Argentina 60 (2019) 257–264.

[23] R. Prasad, V. Srivastava, Some results on trans-Sasakian manifolds, Math. Vesniik 65 (2013) 346–352.

[24] Y. Tashiro, Complete Riemannian manifolds and some vector fields, Trans. Am. Math. Soc. 117 (1965) 251–275.

[25] T. Dutta, N. Basu, A. Bhattacharyya, Almost conformal Ricci solitons on 3-dimensional trans-Sasakian manifolds, Hacet. J. Math. Stat. 45 (2016) 1379–1392.