**Thai Journal of Mathematics**
Volume 19 Number 2 (2021)
Pages 635–652

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

# The Method for Solving the Split Equality Variational Inequality Problem and Application

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**Abstract** Some method is introduced to solve the split equality variational inequality problem and we also apply our result to find solution of the split equality fixed problem and the null point problem of maximal monotone which are introduced by Moudafi and Al-Shemas [A. Moudafi, E. Al-Shemas, Simultaneous iterative methods for split equality problem, Trans. Math. Program. Appl. 1 (2013) 1–11] and Chang and Agarwal [S.S. Chang, R.P. Agarwal, Strong convergence theorems of general split equality problems for quasi-nonexpansive mappings, Journal of Inequalities and Applications 2014 (2014) 367], respectively.

**MSC:** 47H10; 90C33
**Keywords:** the split equality variational inequality problem; the split equality fixed problem; the null point problem of maximal monotone

Submission date: 10.09.2019 / Acceptance date: 24.08.2020

## 1. Introduction

Thoughout this paper, we use the following symbols;

i) $H$, $H_i$ are real Hilbert spaces and $C_i$ is a nonempty closed convex subset of $H_i$, for all $i = 1, 2$,

ii) " $\rightharpoonup$" and " $\rightarrow$" to denoted weak and strong convergence, respectively,

iii) $W_w(x_n) = \{x : x_{n_k} \rightharpoonup x, \exists a\ subsequence\ \{x_{n_k}\}\ of\ \{x_n\}\}$,

iv) $F(T) = \{x \in C : Tx = x\}$ is the set of solution of fixed points problem of $T : C \to C$ with $C \subseteq H$.

The split feasibility problem (SFP) was first introduced by Censor and Elfving [1] in a finite dimension on Hilbert space. The SFP is to find a point $x^* \in C_1$, and $Ax^* \in C_2$, where $A : H_1 \to H_2$ is a bounded linear operator. The set of all solution of SFP is denoted by $\Gamma_{SFP}$.

If $\Gamma_{SFP} \neq \emptyset$. It is obvious that $x^* \in \Gamma_{SFP} \Leftrightarrow x^* = P_{C_1}(I - \lambda A^*(I - P_{C_2}))x^*$.

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The SFP has been extremely popular, since it can be applied to various fields such as the modelling of inverse problem in mathematics, in radiation therapy treatment planning, neural networks and so on. This problem have been studied by many researcher, for example [2], [3], [4].

Byrne [5] is introduced the most widely algorithm to solve SFP. This algorithm generated by sequence $\{x_n\}$ such that

$$x_{n+1} = P_C\left(x_n - \gamma A^*\left(I - P_C\right)Ax_n\right),$$

for all $n \in \mathbb{N}$, where $A^*$ is adjoint operator of $A$, $L$ is spectal radius of $A^*A$ and $P_C$ is a metric projection on $H$ onto $C$. Many researchers use this algorithm as a basis to create their algorithm for solving SFP, see for example [6], [7].

To increase the potential of SFP, Moudafi [8, 9] introduced the following split equality problem. Let $H_3$ be a real Hilbert space and let $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators. *The split equality problem (SEP)* is to find $x \in C_1$ and $y \in C_2$ such that $Ax = By$. The set of all solution of SEP is denoted by $\Gamma_{SEP}$.

To solve SEP, he introduce the following iteration:

$$\begin{cases} x_{n+1} = P_{C_1}\left[x_n - \gamma_n A^*\left(Ax_n - By_n\right)\right], \\ y_{n+1} = P_{C_2}\left[y_n + \gamma_n B^*\left(Ax_n - By_n\right)\right], \end{cases} \tag{1.1}$$

where $A^*$ and $B^*$ denote the adjoint of $A$ and $B$ respectively. Under condition $\gamma_n \in (\varepsilon, \min\left\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\right\} - \varepsilon\}$ with $\lambda_A$ and $\lambda_B$ are spectral radius of $A^*A$ and $B^*B$, respectively, then the sequence $\{(x_n, y_n)\}$ generated by (1.1) converges weakly to $(\overline{x}, \overline{y}) \in \Gamma_{SEF}$.

If $B = I$ and $H_2 = H_3$, then (SEP) reduces to (SFP). The interest of SEP is to cover many situations such as decomposition methods for PDEs, applications in game theory.

Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators such that $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. *The split equality fixed point problem (SEFP)* is to find $x \in F(S)$ and $y \in F(S)$ such that $Ax = By$ which allows asymmetric and partial relations between the variables $x$ and $y$. The such problem introduced by Moudafi and Al-Shemas [8]. The set of all solutions of SEFP is denoted by $\Omega = \{(x, y) \in C_1 \times C_2 : x \in F(S), y \in F(T) \text{ and } Ax = By\}$. Zhao [10] proposed a theorem for finding the solution of SEFP as follows:

**Theorem 1.1.** *Let $H_1, H_2, H_3$ be a real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings such that $S - I$ and $T - I$ are demiclosed at 0. Suppose that $\Omega = \{x \in F(S), y \in F(T) : Ax = By\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequence generated by $x_0 \in H_1$ and $y_0 \in H_2$ and by*

$$\begin{cases} u_n = x_n - \gamma_n A^*\left(Ax_n - By_n\right), \\ x_{n+1} = \beta_n u_n + \left(1 - \beta_n\right)Su_n, \\ w_n = y_n + \gamma_n B^*\left(Ax_n - By_n\right), \\ y_{n+1} = \beta_n w_n + \left(1 - \beta_n\right)Tw_n, \ \forall n \geq 0. \end{cases}$$

*Assume that the step-size $\gamma_n$ is chosen in such a way that*

$$\gamma_n \in \left(\varepsilon, \frac{2\left\|Ax_n - By_n\right\|^2}{\left\|B^*\left(Ax_n - By_n\right)\right\|^2 + \left\|A^*\left(Ax_n - By_n\right)\right\|^2} - \varepsilon\right),$$

*for all* $n \in \Pi$ *otherwise* $\gamma_n = \gamma$ *(*$\gamma$ *begin any nonnegative value), where the index set* $\Pi = \{n : Ax_n - By_n \neq 0\}$*. Let* $\{\alpha_n\} \subset (\delta, 1-\delta), \{\beta_n\} \subset (\eta, 1-\eta)$ *for small enough* $\delta, \eta > 0$*. Then, the sequence* $\{(x_n, y_n)\}$ *converges weakly to* $(x^*, y^*) \in \Omega$*.*

**Remark 1.2.** If $H_2 = H_3$ and $B = I$ , then the split equality fixed point problem is reduces to the split common fixed point problem.

In 2015, Li and He [11] proved a strong convergence theorem to solve the split common fixed point problem for quasi-nonexpansive mappings as follows.

**Theorem 1.3.** *Let* $H_1$ *and* $H_2$ *be a real Hilbert spaces, let* $C$ *and* $K$ *be nonempty closed convex subsets of* $H_1$ *and* $H_2$*, respectively.* $T_1 : C \to H_1$ *and* $T_2 : K \to H_2$ *be two quasi-nonexpansive mappings with* $F(T_1) \neq \emptyset$ *and* $F(T_2) \neq \emptyset$*.* $A : H_1 \to H_2$ *is a bounded linear operator. Assume that* $T_1 - I$ *and* $T_2 - I$ *are demi-closed at* 0*. let* $x_0 \in C$*,*$C_0 = C$ *and* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n) T_1 z_n, \\ z_n = P_C (x_n + \lambda A^* (T_2 - I) A x_n), \\ C_{n+1} = \{x \in C_n : \|y_n - x\| \leq \|z_n - x\| \leq \|x_n - x\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \ \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

*where* $P$ *is a projection operator and* $A^*$ *denotes the adjoint of* $A$*.* $\{\alpha_n\} \subset (0, \eta) \subset (0,1), \lambda \in \left(0, \frac{1}{\|A\|^2}\right)$*. Assume that* $\Gamma = \{p \in F(T_1), Ap \in F(T_2)\} \neq \emptyset$*, then* $x_n \to x^* \in F(T_1)$ *and* $Ax_n \to Ax^* \in F(T_2)$*.*

Many author prove a strong convergence theorem for quasi-nonexpansive mapping by using the condition $I - T$ is a demi-closed mapping, where $T : C \to C$ is a quasi nonexpansive mapping, see for more detail [10], [12], [13].

The variational inequality problem $(VIP)$ is to find a point $v \in C$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0,$$

for all $y \in C$, where $C$ is a subset of $H$ and $A : C \to H$ is a mapping. The set of all solution of VIP is denoted by $VI(C, A)$. Such a problem was first introduced by Lions and Stampacchia [14]. This problem is very important in many branches of mathematics. The following is a well-known tool for solving variational inequality problem.

**Lemma 1.4.** *Let* $C$ *be a nonempty closed convex subset of Hilbert space* $H$ *and let* $A : C \to H$ *be a mapping. Then*

$$u = P_C (I - \lambda A) u \Leftrightarrow u \in VI(C, A),$$

*for all* $u \in C$ *and* $\lambda > 0$*.*

Korpelevich [15] introduce a sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$\begin{cases} y_n = P_C (I - \lambda A) x_n, \\ x_{n+1} = P_C (x_n - \lambda A y_n), \end{cases}$$

for all $n \in \mathbb{N}$. Under suitable conditions of the mapping $A$ and $\lambda \in \left(0, \frac{1}{k}\right)$. She proved that the sequence $\{x_n\}$ converges strongly to an element of $VI(C, A)$. Many research use this iteration as a model to prove their theorem for finding a solution of variational inequality problem, see for example [16], [17], [18].

The split variational inequality problems (SVIP) is to find $x^* \in C_1$ such that $\langle fx^*, x - x^*\rangle \geq 0$, for all $x \in C_1$ and such that $y^* = Ax^* \in C_2$ solves $\langle gy^*, y - y^*\rangle \geq 0$ for all $y \in C_2$, where $f : C_1 \to H_1$, $g : C_2 \to H_2$ and $A : H_1 \to H_2$ is a bounded linear operator. The set of all solution of SVIP is denoted by $\Gamma_{SVIP}$. The such problem is introduced by Censor et al. [19]. SVIP can be reduce to split minimization problem between two spaces so that the image of a solution point of one minimization problem, under a given bounded linear operator, is a solution point of another minimization problem.

Many author has been studied the split variational inequality problems, see for example, [10], [19].

By combining the concept of the split equality problem and the split variational inequality problem, we introduce a new problem which is called *the split equality variational inequality problem (SEVIP)* as follows;

Let $A^i : C_i \to H_i$ be a mapping for all $i = 1, 2$ and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. *The split equality variational inequality problem* is to find $(x, y) \in C_1 \times C_2$ such that

$$x \in VI\left(C_1, A^1\right), y \in VI\left(C_2, A^2\right) \quad \text{and } Ax = By. \tag{1.2}$$

The set of all solutions of (1.2) is denoted by

$$\begin{aligned} VI\left(C_{1,2}, sA^{1,2}\right) &= \{(x, y) \in C_1 \times C_2 : x \in VI\left(C_1, A^1\right), y \in VI\left(C_2, A^2\right) \\ &\quad \text{and } Ax = By\}. \end{aligned}$$

If $H_2 = H_3$ and $B = I$ , then SEVIP is reduced to SVIP. Another special case of the SEVIP is the split Feasibility Problem (SFP).

Motivated by above researches, the purpose of this paper is to introduce a method for finding solution of SEVIP by improving the Halpern iteration. We also apply this method to find solution of the split equality fixed problem and the null point problem of maximal monotone which are introduced by Moudafi and Al-Shemas [8] and Chang and Agarwal [20], respectively.

## 2. Preliminaries

In order to prove our main theorem. Therefore, these tools are needed.

Let $P_C$ be the metric projection of $H$ onto $C$ i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following lemma is a property of $P_C$.

**Lemma 2.1** (See [21])**.** *Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$\langle x - y, y - z\rangle \geq 0 \ \forall z \in C.$

**Lemma 2.2** (See [22])**.** *Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \ \forall n \geq 0$$

*where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions:*

(1) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

*Then* $\lim_{n\to\infty} s_n = 0$.

**Lemma 2.3** (See [23]). *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$;*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, t_n \leq t_{\tau(n)+1}.$$

*In fact,* $\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}$.

## 3. Main Results

**Theorem 3.1.** *For every $i = 1,2,3$, let $H_i$ be a real Hilbert space and let $C_1, C_2$ be nonempty closed convex subset of $H_1$ and $H_2$, respectively. Let $A^i : C_i \to H_i$ be $\alpha_i$-inverse strongly monotone mapping for all $i = 1,2$ with $\alpha = \min\{\alpha_1, \alpha_2\}$ and let the mappings $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operator with adjoints $A^*$ and $B^*$, respectively. Suppose that $\Omega = VI\left(C_{1,2}, A^{1,2}\right)$ is a nonempty set. Let sequences $\{x_n\}$ and $\{y_n\}$ generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and*

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1}\left(I - \lambda_1 A^1\right) u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2}\left(I - \lambda_2 A^2\right) v_n, \ \text{for all } n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subseteq [0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$ and $\lambda_i \in [0, 2\alpha]$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\varepsilon, \dfrac{2}{\lambda_A + \lambda_B} - \varepsilon\right)$ for all $n \in \mathbb{N}$, where $\lambda_A, \lambda_B$ are spectral radius of $A^*A$ and $B^*B$ respectively and $\varepsilon$ is small enough.*
*Then $\{(x_n, y_n)\}$ converges strongly to $\left(\widehat{x^*}, \widehat{y^*}\right) \in \Omega$, where $\widehat{x^*} = P_{VI(C_1,A^1)}u$ and $\widehat{y^*} = P_{VI(C_2,A^2)}v$.*

*Proof.* Let $(x^*, y^*) \in \Omega$, we have

$$x^* \in VI\left(C_1, A^1\right), y^* \in VI\left(C_2, A^2\right) \text{ and } Ax^* = By^*.$$

Since $x^* \in VI\left(C_1, A^1\right)$ and $y^* \in VI\left(C_2, A^2\right)$, we have $x^* \in F\left(P_{C_1}\left(I - \lambda_1 A^1\right)\right)$ and $y^* \in F\left(P_{C_2}\left(I - \lambda_2 A^2\right)\right)$, respectively.
For every $x, y \in C_1$, we have

$$\begin{aligned} \left\|P_{C_1}\left(I - \lambda_1 A^1\right)x - P_{C_1}\left(I - \lambda_1 A^1\right)y\right\|^2 &\leq \left\|x - y - \lambda_1(A^1x - A^1y)\right\|^2 \\ &= \|x-y\|^2 - 2\lambda_1\langle x - y, A^1x - A^1y\rangle \\ &\quad + \lambda_1^2\|Ax - Ay\|^2 \end{aligned}$$

$$\begin{aligned} &= \|x-y\|^2 - 2\lambda_1\alpha\|A^1x - A^1y\|^2 \\ &\quad +\lambda_1^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 - \lambda_1(2\alpha-\lambda_1)\|A^1x-A^1y\|^2 \\ &\le \|x-y\|^2. \end{aligned}$$

Then $P_{C_1}\left(I-\lambda_1A^1\right)$ is a nonexpansive mapping. Using the same method, we have $P_{C_2}\left(I-\lambda_2A^2\right)$ is a nonexpansive mapping.
From the definition of $u_n$, we have

$$\begin{aligned} \|u_n-x^*\|^2 &= \|x_n-x^*-\gamma_nA^*(Ax_n-By_n)\|^2 \\ &= \|x_n-x^*\|^2+\gamma_n^2\|A^*(Ax_n-By_n)\|^2 \\ &\quad -2\gamma_n\langle x_n-x^*, A^*(Ax_n-By_n)\rangle \\ &\le \|x_n-x^*\|^2+\gamma_n^2\lambda_A\|Ax_n-By_n\|^2 \\ &\quad -\gamma_n\|Ax_n-Ax^*\|^2-\gamma_n\|Ax_n-By_n\|^2 \\ &\quad +\gamma_n\|Ax^*-By_n\|^2 \\ &= \|x_n-x^*\|^2-\gamma_n(1-\lambda_A\gamma_n)\|Ax_n-By_n\|^2 \\ &\quad -\gamma_n\|Ax_n-Ax^*\|^2+\gamma_n\|Ax^*-By_n\|^2 \end{aligned} \tag{3.1}$$

From the definition of $v_n$ and using the method as (3.1), we have

$$\begin{aligned} \|v_n-y^*\|^2 &\le \|y_n-y^*\|^2-\gamma_n(1-\lambda_B\gamma_n)\|Ax_n-By_n\|^2-\gamma_n\|By_n-By^*\|^2 \\ &\quad +\gamma_n\|Ax_n-By^*\|^2. \end{aligned}$$

From the last two inequalities above, we have

$$\begin{aligned} \|u_n-x^*\|^2+\|v_n-y^*\|^2 &\le \|x_n-x^*\|^2+\|y_n-y^*\|^2 \\ &\quad -\gamma_n(2-\gamma_n(\lambda_A+\lambda_B))\|Ax_n-By_n\|^2. \end{aligned} \tag{3.2}$$

From the definition of $x_n$, we have

$$\begin{aligned} \|x_{n+1}-x^*\|^2 &= \left\|\alpha_n(u-x^*)+(1-\alpha_n)\left(P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\right)\right\|^2 \\ &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\right\|^2 \\ &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2. \end{aligned}$$

Similarly, we have

$$\|y_{n+1}-y^*\|^2 \le \alpha_n\|v-y^*\|^2+(1-\alpha_n)\|v_n-y^*\|^2.$$

From above inequalities, we have

$$\begin{aligned} \|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\le \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right) \\ &\quad +(1-\alpha_n)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right) \\ &\le \max\{\|u-x^*\|^2+\|v-y^*\|^2, \|x_1-x^*\|^2 \\ &\quad +\|y_1-y\|^2\}. \end{aligned}$$

From mathematical induction, we have $\{x_n\}$, $\{y_n\}$ bounded and so are $\{u_n\}$, $\{v_n\}$. From the definitions of $\{x_n\}$ and $\{y_n\}$, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad +(1-\alpha_n)\left(\|u_n-x^*\|^2+\|v_n-y^*\|^2\right)\\
&\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad +(1-\alpha_n)\Bigg(\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\quad -\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2\Bigg)\\
&= \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad +(1-\alpha_n)\left(\|x_n-x^*\|^2+\|y_n-y^*\|^2\right)\\
&\quad -\gamma_n(1-\alpha_n)\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2.
\end{aligned}$$

It implies that

$$\begin{aligned}
\gamma_n(1-\alpha_n)\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\|Ax_n-By_n\|^2 &\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad +C_n-C_{n+1}, \qquad (3.3)
\end{aligned}$$

where $C_n=\|x_n-x^*\|^2+\|y_n-y^*\|^2$, for all $x^*\in VI\left(C_1,A^1\right)$,$y^*\in VI\left(C_2,A^2\right)$ and $n\in\mathbb{N}$.
From (3.3), we will divide the proof into two cases.

<u>Case i</u> Suppose that $C_{n+1}\leq C_n$ for all $n\geq n_0$ (for $n_0$ large enough).
Since a sequence $\{C_n\}$ is bounded, we get $\lim_{n\to\infty}C_n=c$ for some $c\in\mathbb{R}$.
From (3.3) and properties of $\gamma_n$ and $\alpha_n$, we have

$$\lim_{n\to\infty}\|Ax_n-By_n\|=0. \qquad (3.4)$$

From the definitions of $u_n, v_n$, we have

$$\|u_n-x_n\|=\gamma_n\|A^*(Ax_n-By_n)\| \text{ and } \|v_n-y_n\|=\gamma_n\|B^*(Ax_n-By_n)\|. \qquad (3.5)$$

From (3.4) and (3.5), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|v_n-y_n\|=0. \qquad (3.6)$$

By using properties of $P_{C_1}$, we have

$$\begin{aligned}\left\|P_{C_1}\left(I-\lambda_1 A^1\right)u_n - x^*\right\|^2 &\leq \left\|u_n - x^* - \lambda_1\left(A^1 u_n - A^1 x^*\right)\right\|^2 \\ &= \left\|u_n - x^*\right\|^2 - 2\lambda_1\langle u - x^*, A^1 u_n - A^1 x^*\rangle \\ &\quad +\lambda_1^2\left\|A^1 u_n - A^1 x^*\right\|^2 \\ &\leq \left\|u_n - x^*\right\|^2 - 2\lambda_1\alpha\left\|A^1 u_n - A^1 x^*\right\|^2 \\ &\quad +\lambda_1^2\left\|A^1 u_n - A^1 x^*\right\|^2 \\ &= \left\|u_n - x^*\right\|^2 \\ &\quad -\lambda_1\left(2\alpha - \lambda_1\right)\left\|A^1 u_n - A^1 x^*\right\|^2. \end{aligned} \tag{3.7}$$

Similarly, we have

$$\left\|P_{C_2}\left(I-\lambda_2 A^2\right)v_n - y^*\right\|^2 \leq \left\|v_n - y^*\right\|^2 - \lambda_2\left(2\alpha - \lambda_2\right)\left\|A^2 v_n - A^2 y^*\right\|^2. \tag{3.8}$$

From (3.2), (3.7) and (3.8), we have

$$\begin{aligned}\left\|P_{C_1}\left(I-\lambda_1 A^1\right)u_n - x^*\right\|^2 &+ \left\|P_{C_2}\left(I-\lambda_2 A^2\right)v_n - y^*\right\|^2 \\ &\leq \left\|u_n - x^*\right\|^2 + \left\|v_n - y^*\right\|^2 \\ &\quad -\lambda_1\left(2\alpha-\lambda_1\right)\left\|A^1 u_n - A^1 x^*\right\|^2 \\ &\quad -\lambda_2\left(2\alpha-\lambda_2\right)\left\|A^2 v_n - A^2 y^*\right\|^2 \\ &\leq \left\|x_n - x^*\right\|^2 + \left\|y_n - y^*\right\|^2 \\ &\quad -\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\left\|Ax_n - By_n\right\|^2 \\ &\quad -\lambda_1\left(2\alpha-\lambda_1\right)\left\|A^1 u_n - A^1 x^*\right\|^2 \\ &\quad -\lambda_2\left(2\alpha-\lambda_2\right)\left\|A^2 v_n - A^2 y^*\right\|^2. \end{aligned} \tag{3.9}$$

From the definition of $x_n, y_n$, we have

$$\begin{aligned}\left\|x_{n+1} - x^*\right\|^2 + \left\|y_{n+1} - y^*\right\|^2 &\leq \alpha_n\left(\left\|u - x^*\right\|^2 + \left\|v - y^*\right\|^2\right) \\ &\quad +\left(1-\alpha_n\right)\Bigg(\left\|P_{C_1}\left(I-\lambda_1 A^1\right)u_n - x^*\right\|^2 \\ &\quad +\left\|P_{C_2}\left(I-\lambda_2 A^2\right)v_n - y^*\right\|^2\Bigg) \\ &\leq \alpha_n\left(\left\|u - x^*\right\|^2 + \left\|v - y^*\right\|^2\right) \\ &\quad +\left(1-\alpha_n\right)\Bigg(\left\|x_n - x^*\right\|^2 + \left\|y_n - y^*\right\|^2 \\ &\quad -\gamma_n\left(2-\gamma_n\left(\lambda_A+\lambda_B\right)\right)\left\|Ax_n - By_n\right\|^2 \\ &\quad -\lambda_1\left(2\alpha-\lambda_1\right)\left\|A^1 u_n - A^1 x^*\right\|^2 \\ &\quad -\lambda_2\left(2\alpha-\lambda_2\right)\left\|A^2 v_n - A^2 y^*\right\|^2\Bigg). \end{aligned}$$

It implies that

$$(1-\alpha_n)\left(\lambda_1(2\alpha-\lambda_1)\left\|A^1u_n-A^1x^*\right\|^2+\lambda_2(2\alpha-\lambda_2)\left\|A^2v_n-A^2y^*\right\|^2\right)$$
$$\leq C_n-C_{n+1}+\alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right). \qquad (3.10)$$

Applications for (3.10) and $\lim_{n\to\infty}C_n=c$, we have

$$\lim_{n\to\infty}\left\|A^1u_n-A^1x^*\right\|=\lim_{n\to\infty}\left\|A^2v_n-A^2y^*\right\|=0. \qquad (3.11)$$

From the properties of $P_{C_1}$, we have

$$\begin{aligned}
\left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\right\|^2 &= \left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-P_{C_1}\left(I-\lambda_1A^1\right)x^*\right\|^2\\
&\leq \left\langle\left(I-\lambda_1A^1\right)u_n-\left(I-\lambda_1A^1\right)x^*,P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\right\rangle\\
&= \frac{1}{2}\Big(\|\left(I-\lambda_1A^1\right)u_n-\left(I-\lambda_1A^1\right)x^*\|^2+\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\|^2\\
&\quad -\|\left(I-\lambda_1A^1\right)u_n-\left(I-\lambda_1A^1\right)x^*-P_{C_1}\left(I-\lambda_1A^1\right)u_n+x^*\|^2\Big)\\
&\leq \frac{1}{2}\Big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\|^2\\
&\quad -\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n-\lambda_1\left(A^1u_n-A^1x^*\right)\|^2\Big)\\
&= \frac{1}{2}\Big(\|u_n-x^*\|^2+\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\|^2\\
&\quad -\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|^2-\lambda_1^2\|A^1u_n-A^1x^*\|^2\\
&\quad +2\lambda_1\langle u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n,A^1u_n-A^1x^*\rangle\Big). \qquad (3.12)
\end{aligned}$$

From (3.12), we have

$$\begin{aligned}
\left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\right\|^2 &\leq \|u_n-x^*\|^2-\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|^2\\
&-\lambda_1^2\|A^1u_n-A^1x^*\|^2+2\lambda_1\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|\|A^1u_n-A^1x^*\|. \qquad (3.13)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\left\|P_{C_2}\left(I-\lambda_2A^2\right)v_n-y^*\right\|^2 &\leq \|v_n-y^*\|^2-\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|^2\\
&-\lambda_2^2\|A^2v_n-A^2y^*\|^2+2\lambda_2\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|\|A^2v_n-A^2y^*\|. \qquad (3.14)
\end{aligned}$$

From the definitions of $x_n,y_n$, (3.13) and (3.14), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 &\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&+(1-\alpha_n)\left(\left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x^*\right\|^2+\left\|P_{C_2}\left(I-\lambda_2A^2\right)v_n-y^*\right\|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad+(1-\alpha_n)\Bigg(\|u_n-x^*\|^2+\|v_n-y^*\|^2-\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|^2\\
&\quad-\lambda_1^2\|A^1u_n-A^1x^*\|^2+2\lambda_1\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|\|A^1u_n-A^1x^*\|\\
&\quad-\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|^2-\lambda_2^2\|A^2v_n-A^2y^*\|^2\\
&\quad+2\lambda_2\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|\|A^2v_n-A^2y^*\|\Bigg)\\
&\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)\\
&\quad+(1-\alpha_n)\Bigg(\|x_n-x^*\|^2+\|y_n-y^*\|^2-\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|^2\\
&\quad-\lambda_1^2\|A^1u_n-A^1x^*\|^2+2\lambda_1\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|\|A^1u_n-A^1x^*\|\\
&\quad-\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|^2-\lambda_2^2\|A^2v_n-A^2y^*\|^2\\
&\quad+2\lambda_2\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|\|A^2v_n-A^2y^*\|\Bigg).
\end{aligned}$$

It implies that

$$\begin{aligned}
(1-\alpha_n)&\left(\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|^2+\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|^2\right)\\
&\leq \alpha_n\left(\|u-x^*\|^2+\|v-y^*\|^2\right)+C_n-C_{n+1}\\
&\quad+2\lambda_1\|u_n-P_{C_1}\left(I-\lambda_1A^1\right)u_n\|\|A^1u_n-A^1x^*\|\\
&\quad+2\lambda_2\|v_n-P_{C_2}\left(I-\lambda_2A^2\right)v_n\|\|A^2v_n-A^2y^*\|.
\end{aligned}$$

From (3.11) and $\lim_{n\to\infty}C_n=c$, we have

$$\lim_{n\to\infty}\left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-u_n\right\|=\lim_{n\to\infty}\left\|P_{C_2}\left(I-\lambda_2A^2\right)v_n-v_n\right\|=0. \tag{3.15}$$

From (3.6) and (3.15), we get

$$\lim_{n\to\infty}\left\|P_{C_1}\left(I-\lambda_1A^1\right)u_n-x_n\right\|=\lim_{n\to\infty}\left\|P_{C_2}\left(I-\lambda_2A^2\right)v_n-y_n\right\|=0. \tag{3.16}$$

Since

$$x_{n+1}-x_n=\alpha_n\left(u-x_n\right)+(1-\alpha_n)\left(P_{C_1}\left(I-\lambda_1A^1\right)u_n-x_n\right),$$

$$y_{n+1}-y_n=\alpha_n\left(v-y_n\right)+(1-\alpha_n)\left(P_{C_2}\left(I-\lambda_2A\right)v_n-y_n\right)$$

and (3.16), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.17}$$

Since $W_w\left(x_n\right),W_w\left(y_n\right)$ are nonempty sets, there exists $\widehat{x}\in C_1,\widehat{y}\in C_2$ such that $\widehat{x}\in W_w\left(x_n\right)$ and $\widehat{y}\in W_w\left(y_n\right)$.

We may assume that, there exists subsequences $\{x_{n_k}\}, \{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that

$$x_{n_k} \rightharpoonup \widehat{x} \text{ as } k \to \infty, \tag{3.18}$$

and

$$y_{n_k} \rightharpoonup \widehat{y} \text{ as } k \to \infty. \tag{3.19}$$

Next, we show that $(\widehat{x}, \widehat{y}) \in \Omega$.
From (3.6) and (3.18), we get $u_{n_k} \rightharpoonup \widehat{x}$ as $k \to \infty$.
Assume that $\widehat{x} \notin VI\left(C_1, A^1\right)$. Since $VI\left(C_1, A^1\right) = F\left(P_{C_1}\left(I - \lambda_1 A^1\right)\right)$, then we get $\widehat{x} \neq P_{C_1}\left(I - \lambda_1 A^1\right)\widehat{x}$. From Opial's condition and (3.15), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|u_{n_k} - \widehat{x}\| &< \liminf_{k\to\infty} \left\|u_{n_k} - P_{C_1}\left(I - \lambda_1 A^1\right)\widehat{x}\right\| \\
&\leq \liminf_{k\to\infty} \|u_{n_k} - P_{C_1}\left(I - \lambda_1 A^1\right) u_{n_k}\| \\
&\quad + \|P_{C_1}\left(I - \lambda_1 A^1\right) u_{n_k} - P_{C_1}\left(I - \lambda_1 A^1\right)\widehat{x}\| \\
&\leq \liminf_{k\to\infty} \|u_{n_k} - \widehat{x}\|.
\end{aligned}$$

This is a contradiction. Then $\widehat{x} \in VI\left(C_1, A^1\right)$.

From (3.6) and (3.19), we get $v_{n_k} \rightharpoonup \widehat{y}$ as $k \to \infty$. Using the methods similar to $\widehat{x} \in VI\left(C_1, A^1\right)$, we can conclude that $\widehat{y} \in VI\left(C_2, A^2\right)$.
From $A\widehat{x} - B\widehat{y} \in W_w\left(Ax_n - By_n\right)$ and weakly lower semi-continuous of normed, we have

$$\|A\widehat{x} - B\widehat{y}\| \leq \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0.$$

Then $A\widehat{x} = B\widehat{y}$. Therefore $(\widehat{x}, \widehat{y}) \in \Omega$.
It is clear that

$$\limsup_{n\to\infty} \langle u - \widehat{x^*}, x_n - \widehat{x^*}\rangle = \limsup_{k\to\infty} \langle u - \widehat{x^*}, x_{n_k} - \widehat{x^*}\rangle = \langle u - \widehat{x^*}, \widehat{x} - \widehat{x^*}\rangle \leq 0,$$

where $\widehat{x^*} = P_{VI(C_1, A^1)} u$
and

$$\limsup_{n\to\infty} \langle v - \widehat{y^*}, y_n - \widehat{y^*}\rangle = \limsup_{k\to\infty} \langle v - \widehat{y^*}, y_{n_k} - \widehat{y^*}\rangle = \langle v - \widehat{y^*}, \widehat{y} - \widehat{y^*}\rangle \leq 0,$$

where $\widehat{y^*} = P_{VI(C_2, A^2)} v$.
Next, we show that a sequence $\{(x_n, y_n)\}$ converges strongly to $\left(\widehat{x^*}, \widehat{y^*}\right) \in \Omega$, where $\widehat{x^*} = P_{VI(C_1, A^1)} u$ and $\widehat{y^*} = P_{VI(C_2, A^2)} v$.
From the definitions of $x_n$ and $y_n$, we get

$$\left\|x_{n+1} - \widehat{x^*}\right\|^2 \leq (1 - \alpha_n)\left\|x_n - \widehat{x^*}\right\|^2 + 2\alpha_n \langle u - \widehat{x^*}, x_{n+1} - \widehat{x^*}\rangle$$

and

$$\left\|y_{n+1} - \widehat{y^*}\right\|^2 \leq (1 - \alpha_n)\left\|y_n - \widehat{y^*}\right\|^2 + 2\alpha_n \langle v - \widehat{y^*}, y_{n+1} - \widehat{y^*}\rangle.$$

Then

$$\begin{aligned}\left\|x_{n+1}-\widehat{x^*}\right\|^2+\left\|y_{n+1}-\widehat{y^*}\right\|^2 \quad &\leq \quad (1-\alpha_n)\left(\left\|x_n-\widehat{x^*}\right\|^2+\left\|y_n-\widehat{y^*}\right\|^2\right)\\ &\quad +2\alpha_n\big(\langle u-\widehat{x^*},x_{n+1}-\widehat{x^*}\rangle\\ &\quad +\langle v-\widehat{y^*},y_{n+1}-\widehat{y^*}\rangle\big)\end{aligned}$$

or

$$C_{n+1}\leq(1-\alpha_n)\,C_n+2\alpha_n\eta_n, \tag{3.20}$$

where $\eta_n=\langle u-\widehat{x^*},x_{n+1}-\widehat{x^*}\rangle+\langle v-\widehat{y^*},y_{n+1}-\widehat{y^*}\rangle$, for all $n\in\mathbb{N}$.
From $\limsup_{n\to\infty}\eta_n\leq 0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$ and Lemma 2.2, we obtain

$$\lim_{n\to\infty}C_n=\lim_{n\to\infty}\left(\left\|x_n-\widehat{x^*}\right\|^2+\left\|y_n-\widehat{y^*}\right\|^2\right)=0.$$

It implies that $\{(x_n,y_n)\}$ converges strongly to $\left(\widehat{x^*},\widehat{y^*}\right)$.
Since $A\widehat{x^*}-B\widehat{y^*}\in W_w\left(Ax_n-By_n\right)$ and lower semi-continuous of normed, we have

$$\left\|A\widehat{x^*}-B\widehat{y^*}\right\|\leq\liminf_{k\to\infty}\|Ax_{n_k}-By_n\|=0.$$

Hence $A\widehat{x^*}=B\widehat{y^*}$. Therefore $\left(\widehat{x^*},\widehat{y^*}\right)\in\Omega$.

<u>Case ii</u> Suppose that $C_n$ is not monotone sequence, then there exists an integer $n_0$ such that $C_{n_0}\leq C_{n_0+1}$.
Define the integer sequence $\tau(n)$ for all $n\geq n_0$ as follows

$$\tau(n)=\max\{k\leq n:C_k<C_{k+1}\}.$$

It is clear that $\tau(n)$ is a nondecreasing with $\lim_{n\to\infty}\tau(n)=\infty$ and $C_{\tau(n)}<C_{\tau(n)+1}$.
From (3.20), we have

$$C_{\tau(n)+1}\leq\left(1-\alpha_{\tau(n)}\right)C_{\tau(n)}+2\alpha_{\tau(n)}\eta_{\tau(n)}.$$

From Lemma 2.2, we have $\lim_{n\to\infty}C_{\tau(n)}=0$. Applying (3.17), we have $\lim_{n\to\infty}C_{\tau(n)+1}=0$.
By Lemma 2.3, we have

$$C_n\leq\max\left\{C_n,C_{\tau(n)}\right\}\leq C_{\tau(n)+1}.$$

From inequality above and $\lim_{n\to\infty}C_{\tau(n)+1}=0$, we have

$$\lim_{n\to\infty}\left(\left\|x_n-\widehat{x^*}\right\|^2+\left\|y_n-\widehat{y^*}\right\|^2\right)=\lim_{n\to\infty}C_n=0.$$

It implies that $\{(x_n,y_n)\}$ converges strongly to $\left(\widehat{x^*},\widehat{y^*}\right)$. By using the same methods as case 1, we have $\left(\widehat{x^*},\widehat{y^*}\right)\in\Omega$, where $\widehat{x^*}=P_{VI(C_1,A^1)}u$ and $\widehat{y^*}=P_{VI(C_2,A^2)}v$. This complete the proof. ∎

**Remark 3.2.** *i*) If we take $A^1 \equiv A^2 \equiv 0$ in Theorem 3.1, we have

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1} (x_n - \gamma_n A^* (Ax_n - By_n)), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2} (y_n + \gamma_n B^* (Ax_n - By_n)), \text{ for all } n \geq 1, \end{cases} \tag{3.21}$$

which is modified (1.1). If $\Gamma_{SEP} = \{(x,y) \in C_1 \times C_2 : x \in C_1, y \in C_2 \textit{ and } Ax = By\} \neq \emptyset$ and $\{\alpha_n\}, \{\gamma_n\}$ are the same as in Theorem 3.1, we have $\{(x_n, y_n)\}$ generated by (3.21) converges strongly to $(x^*, y^*) \in \Gamma_{SEP}$.

*ii*) If take $H_2 \equiv H_3$ and $B \equiv I$ in Theorem 3.1, we have

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1} \left(I - \lambda_1 A^1\right) (x_n - \gamma_n A^* (Ax_n - y_n)), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2} \left(I - \lambda_2 A^2\right) ((1-\gamma_n) y_n + \gamma_n Ax_n), \text{ for all } n \geq 1. \end{cases} \tag{3.22}$$

If $\Gamma_{SVIP} = \left\{(x,y) \in C_1 \times C_2 : x \in VI\left(C_1, A^1\right), y \in VI\left(C_2, A^2\right) \textit{ and } Ax = y\right\} \neq \emptyset$ and $\{\alpha_n\}, \{\gamma_n\}, \lambda_1, \lambda_2$ are the same as in Theorem 3.1, we have $\{(x_n, y_n)\}$ generated by (3.22) converges strongly to $(x^*, y^*) \in \Gamma_{SVIP}$.

## 4. Application

We can apply our main theorem to solve the following problems.

### 4.1. Split Equality Fixed Point Problem

In order to solve SEFP, Zhao [10] introduced the following iterative scheme:

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \beta_n u_n + (1-\beta_n) S u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = y_{n+1} = \beta_n v_n + (1-\beta_n) T w_n, \forall, \text{ for all } n \geq 0, \end{cases}$$

where $S : H_1 \to H_1, T : H_2 \to H_2$ are two quasi-nonexpansive mapping with $S - I$ and $T - I$ are demi-closed at 0. Under the suitable conditions of every parameter and $\Omega \neq \emptyset$, he prove the sequence $\{(x_n, y_n)\}$ converges weakly to an element in $\Omega$.

To use a different method of [10] for solving SEFP and a strong convergence theorem, we needed the following lemma.

**Lemma 4.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $T : C \to C$ be a nonlinear mapping with a property $\langle (I-T)x - (I-T)y, x-y \rangle \geq \widehat{\gamma} \|(I-T)x - (I-T)y\|^2$ for all $x, y \in C$ for all $\widehat{\gamma} \in (0, \frac{1}{2}]$. Then $F(T) = VI(C, I-T)$, where $F(T) \neq \emptyset$.*

*Proof.* Let the conditions hold; It is easy to see that $F(T) \subseteq VI(C, I-T)$.

Let $x_0 \in VI(C, I-T)$ and let $x^* \in F(T)$. Since $x_0 \in VI(C, I-T)$, we have

$$\langle y - x_0, (I-T) x_0 \rangle \geq 0, \tag{4.1}$$

for all $y \in C$.

From the property of $T$ and (4.1) we have

$$\begin{aligned}\widehat{\gamma}\| (I-T) x_0 - (I-T) x^* \|^2 &\leq \langle (I-T) x_0 - (I-T) x^*, x_0 - x^* \rangle \\ &= -\langle (I-T) x_0, x^* - x_0 \rangle \leq 0.\end{aligned}$$

Then,

$$\| (I-T) x_0 - (I-T) x^* \|^2 \leq 0.$$

It implies that $x_0 \in F(T)$. Hence $VI(C, I-T) \subseteq F(T)$. ∎

Let $C$ be closed convex subset of a real Hilbert space $H$. Recall the classical mapping $T : C \to C$ is called $\kappa$-strictly pseudo contractive if there exists $\kappa \in [0,1)$ such that $\|Tx - Ty\|^2 \leq \|x-y\|^2 + \kappa \|(I-T)x - (I-T)y\|^2$ for all $x \in C$.

If $\kappa = 0$, $T$ is called nonexpansive mapping.

**Theorem 4.2.** *For every $i = 1,2,3$, let $H_i$ be a real Hilbert space and let $C_1, C_2$ be nonempty closed convex subset of $H_1$ and $H_2$, respectively. For every $i = 1,2$, let $T^i : C_i \to C_i$ be a nonlinear mapping with a property $\langle (I-T^i) x - (I-T^i) y, x-y \rangle \geq \widehat{\gamma}_{T^i} \left\| (I-T^i) x - (I-T^i) y \right\|^2$ for all $x, y \in C_i$ and $\widehat{\gamma}_{T^i} \in (0, \frac{1}{2}]$, $\alpha = \min\{\widehat{\gamma}_{T^1}, \widehat{\gamma}_{T^2}\}$. Assume that $\Omega = \{(x,y) \in C_1 \times C_2 : x \in F(T^1), x \in F(T^2)$ and $Ax = By\}$ is nonempty. Let the mappings $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operator with adjoints $A^*$ and $B^*$, respectively. Let sequences $\{x_n\}$ and $\{y_n\}$ generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and*

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1} \left(I - \lambda_1 (I - T^1)\right) u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2} \left(I - \lambda_2 (I - T^2)\right) v_n, \ \textit{for all } n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subseteq [0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$ and $\lambda_i \in [0, 2\alpha]$ for all $i = 1,2$ and $\gamma_n \in (a,b) \subset \left(\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon\right)$ for all $n \in \mathbb{N}$, where $\lambda_A, \lambda_B$ are spectral radius of $A^*A$ and $B^*B$ respectively and $\varepsilon$ is small enough.*
*Then $\{(x_n, y_n)\}$ converges strongly to $\left(\widehat{x^*}, \widehat{y^*}\right) \in \Omega$, where $\widehat{x^*} = P_{F(T^1)} u$ and $\widehat{y^*} = P_{F(T^2)} v$.*

*Proof.* From Theorem 3.1 and Lemma 4.1, we can conclude the desired result. ∎

**Corollary 4.3.** *For every $i = 1,2,3$, let $H_i$ be a real Hilbert space and let $C_1, C_2$ be nonempty closed convex subset of $H_1$ and $H_2$, respectively. For every $i = 1,2$, let $T^i : C_i \to C_i$ be $\kappa_i$-strictly pseudocontractive mapping with $\alpha = \min\left\{\frac{1-\kappa_1}{2}, \frac{1-\kappa_2}{2}\right\}$. Assume that $\Omega = \{(x,y) \in C_1 \times C_2 : x \in F(T^1), x \in F(T^2)$ and $Ax = By\}$ is nonempty. Let the mappings $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operator with adjoints $A^*$ and $B^*$, respectively. Let sequences $\{x_n\}$ and $\{y_n\}$ generated by $u, x_1 \in C_1; v, y_1 \in C_2$*

*and*

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_1} \left( I - \lambda_1 (I - T^1) \right) u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1-\alpha_n) P_{C_2} \left( I - \lambda_2 (I - T^2) \right) v_n, \ \text{for all } n \geq 1, \end{cases}$$

*where* $\{\alpha_n\} \subseteq [0,1]$ *with* $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$ *and* $\lambda_i \in [0, 2\alpha]$ *for all* $i = 1,2$ *and* $\gamma_n \in (a,b) \subset \left( \varepsilon, \dfrac{2}{\lambda_A + \lambda_B} - \varepsilon \right)$ *for all* $n \in \mathbb{N}$*, where* $\lambda_A, \lambda_B$ *are spectral radius of* $A^*A$ *and* $B^*B$ *respectively and* $\varepsilon$ *is small enough.*
*Then* $\{(x_n, y_n)\}$ *converges strongly to* $\left(\widehat{x^*}, \widehat{y^*}\right) \in \Omega$*, where* $\widehat{x^*} = P_{F(T^1)}u$ *and* $\widehat{y^*} = P_{F(T^2)}v$*.*

*Proof.* Since $T^i$ is $\kappa_i$-strictly pseudocontractive mapping, for all $i = 1, 2$, we have

$$\begin{aligned} \left\|T^i x - T^i y\right\|^2 &= \left\|x - y - \left(\left(I - T^i\right)x - \left(I - T^i\right)y\right)\right\|^2 \\ &= \|x-y\|^2 - 2\langle x - y, \left(\left(I-T^i\right)x - \left(I-T^i\right)y\right)\rangle \\ &\quad + \|\left(I-T^i\right)x - \left(I-T^i\right)y\|^2 \\ &\leq \|x-y\|^2 + \kappa_i \left\|\left(I-T^i\right)x - \left(I-T^i\right)y\right\|^2, \end{aligned}$$

for all $x, y \in C$.
It implies that

$$\langle \left(I-T^i\right)x - \left(I-T^i\right)y, x-y \rangle \geq \frac{1-\kappa_i}{2} \left\|\left(I-T^i\right)x - \left(I-T^i\right)y\right\|^2,$$

where $x, y \in C_i$, for all $i = 1, 2$.
From Theorem 4.2, we can conclude the desired result. ∎

## 4.2. Null Point Problem of Maximal Monotone Operators

In 2014, Chang and Agarwal [20] introduce *null point problem related to strictly maximal monotone operators* which is to find $x^* \in M^{-1}(0), y^* \in N^{-1}(0)$ such that $Ax^* = By^*$, where $M : H_1 \to H_1$ and $N : H_2 \to H_2$ be two strictly maximal monotone operators and $C_1, C_2, H_1, H_2, H_3, A, B, A^*, B^*$ are the same as in Theorem 3.1. The set of all solution of null point problem of maximal monotone operators is denote by $\Upsilon = \{(x,y) \in C_1 \times C_2 : x^* \in M^{-1}(0), y^* \in N^{-1}(0) \text{ and } Ax = By\}$.

In order to find an element in $\Upsilon$ they introduce the following iterative scheme $\{w_n\}$;

$$w_{n+1} = P\left(\alpha_n w_n + \beta_n f(w_n) + \gamma_n K \left(I - \lambda_n G^* G\right) w_n\right), \ n \geq 0,$$

where $S : H_1 \to H_1$ and $T : H_2 \to H_2$ are one-to-one and quasi-nonexpansive mappings with $K = \begin{bmatrix} S \\ T \end{bmatrix}$ and $P = \begin{bmatrix} P_{C_1} \\ P_{C_2} \end{bmatrix}$, $G = \begin{bmatrix} A & -B \end{bmatrix}$, $G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}$ Under the suitable conditions of every parameter, we have the sequence $\{w_n\}$ converges strongly to an element in $\Upsilon$.

To prove a strong convergence theorem for finding an element in $\Upsilon$ we needed the following lemma.

**Lemma 4.4.** *Let $C$ be a nonempty closed convex subset of $H$ and let $A : C \to H$ be an $\alpha$-inverse strongly monotone mapping with $A^{-1}(0) \neq \emptyset$. Then $VI(C, A) = A^{-1}(0)$.*

*Proof.* Let $z \in A^{-1}(0)$, we have $0 = Az$. Then $\langle y - z, Az \rangle = 0$. So $z \in VI(C, A)$. Then $A^{-1}(0) \subseteq VI(C, A)$.

Let $z \in VI(C, A)$ and $z^* \in A^{-1}(0)$. Then $\langle y - z, Az \rangle \geq 0$, $\forall y \in C$ and $Az^* = 0$. Since $A$ is an $\alpha$-inverse strongly monotone mapping, we have

$$\alpha \|Az - Az^*\|^2 \leq \langle z - z^*, Az - Az^* \rangle \leq 0.$$

Then $Az = 0$. So, we have$z \in A^{-1}(0)$. Therefore $VI(C, A) \subseteq A^{-1}(0)$. ■

**Theorem 4.5.** *For every $i = 1, 2, 3$, let $H_i$ be a real Hilbert space and let $C_1, C_2$ be nonempty closed convex subset of $H_1$ and $H_2$, respectively. Let $A_i : C_i \to H_i$ be $\alpha_i$-inverse strongly monotone mapping for all $i = 1, 2$ with $\alpha = \min\{\alpha_1, \alpha_2\}$ and let the mappings $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operator with adjoints $A^*$ and $B^*$, respectively. Suppose that $\Upsilon = \{(x, y) \in C_1 \times C_2 : x^* \in A_1^{-1}(0), y^* \in A_2^{-1}(0)$ and $Ax = By\}$ is a nonempty set. Let sequences $\{x_n\}$ and $\{y_n\}$ generated by $u, x_1 \in C_1; v, y_1 \in C_2$ and*

$$\begin{cases} u_n = x_n - \gamma_n A^* (Ax_n - By_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) P_{C_1} (I - \lambda_1 A^1) u_n, \\ v_n = y_n + \gamma_n B^* (Ax_n - By_n), \\ y_{n+1} = \alpha_n v + (1 - \alpha_n) P_{C_2} (I - \lambda_2 A^2) v_n, \text{ for all } n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subseteq [0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$ and $\lambda_i \in [0, 2\alpha]$ for all $i = 1, 2$ and $\gamma_n \in (a, b) \subset \left(\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon\right)$ for all $n \in \mathbb{N}$, where $\lambda_A, \lambda_B$ are spectral radius of $A^*A$ and $B^*B$ respectively and $\varepsilon$ is small enough.*
*Then $\{(x_n, y_n)\}$ converges strongly to $\left(\widehat{x^*}, \widehat{y^*}\right) \in \Upsilon$, where $\widehat{x^*} = P_{A_1^{-1}(0)} u$ and $\widehat{y^*} = P_{A_2^{-1}(0)} v$.*

*Proof.* From Theorem 3.1 and Lemma 4.4, we can conclude the desired result. ■

## 5. Conclusion

The variational inequality problem has been extensively studied because of this method is easy to apply to solving fixed point problem, minimization problem, zero point problem, etc. In order to improve the efficiency of this method, we introduced the split equality variational inequality problem (SEVI)and the method to solve SEVI. By using our main results we have the methods for solving the split equality fixed problem and the null point problem of maximal monotone which are introduced by Moudafi and Al-Shemas [8] and Chang and Agarwal [20], respectively.

## Acknowledgements

This paper is supported by the Centre of Excellence in Mathematics Program of the Commission on Higher Education (CHE), Thailand.

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