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The Convergence Results for an AK-Generalized Nonexpansive Mapping in Hilbert Spaces

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Abstract In this paper, we introduce a new class of nonexpansive type of mapping namely, AK-generalized nonexpansive mapping, which is more general than an α-nonexpansive mapping. Moreover, we obtain convergence results of the viscosity approximation method for an AK-generalized nonexpansive semigroups under some assumptions in Hilbert spaces. Furthermore, we prove a strong convergence theorem for a family of AK-generalized nonexpansive mapping in Hilbert spaces.

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1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in C : Tx = x\}$. The mapping $T : C \to C$ is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for any $x, y \in C$. In 1965, Browder [1] shown that if a nonexpansive mapping $T : H \to H$ of a Hilbert space H into itself is asymptotically regular and has at least one fixed point then, for any $x \in H$, a weak limit of a weakly convergent subsequence of the sequence of successive approximations $T^n x$ is a fixed point of T.

In 2011, Aoyama and Kohsaka [2] introduced the class of α-nonexpansive mappings in Banach spaces as follows: Let E be a Banach space and let C be a nonempty subset of

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E. A mapping $T : C \to E$ is said to be α-nonexpansive for some real number $0 \leq \alpha < 1$ if

$$||Tx - Ty|| \leq \alpha||Tx - y|| + \alpha||Ty - x|| + (1 - 2\alpha)||x - y||,$$

for all $x, y \in C$. Clearly, 0-nonexpansive maps are exactly nonexpansive maps. This mapping was generalized and extanded by many authors in several directions; see for instance [3, 4] and references therein.

One of the most interesting iteration processes is the viscosity approximation method introduced by Moudafi [5]. In 2004, Xu [6] studied such method for a nonexpansive mapping in a Hilbert space and introduced an iterative scheme for finding the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$x_1 \in C, x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 1,$$

where $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, $f : C \to C$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Then, they proved a strong convergence theorem under suitable conditions of the sequence $\{\alpha_n\}$.

Over the past few decades, the convergence theorem was extended and improved in many directions (see [7], [8]) due to its applications are desirable and can be used in real-world applications. So, many authors have been trying to construct new iterations to prove strong convergence theorems for nonexpansive semigroups; see for instance [9–11] and references therein. Especially, in 2008, Song and Xu [12] introduced the following implicit and explicit viscosity iterative schemes,

$$\begin{aligned} x_n &= \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, n \geq 1. \end{aligned}$$

Then they proved strong convergence theorems of a nonexpansive semigroup under suitable conditions. Very recently, Song *et al.* [13] proved a strong convergence theorem of the Halpern iteration for an α-nonexpansive semigroup in Hilbert spaces under suitable conditions as the following schemes,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, n \geq 1. \tag{1.1}$$

Moreover, they also proved some strong convergence theorems of Halperns iteration defined by a such iterative method for a family $\{T_n\}$ of α-nonexpansive mappings.

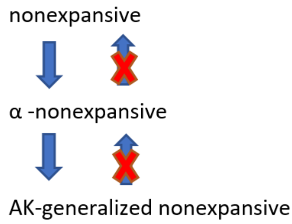
Our work improves and generalizes some of the results obtained in the above paper, we introduce a new class of nonexpansive type of mapping namely, *AK-generalized nonexpansive mapping*, which is more general than an α-nonexpansive mapping in Hilbert spaces as follow.

Definition 1.1. Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to satisfy condition (AK) (or AK-generalized nonexpansive) for some real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$ if

$$\begin{aligned} ||Tx - Ty|| \leq{}& \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Ty - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, \end{aligned} \tag{1.2}$$

for all $x, y \in C$.

Notice that the class of AK-generalized nonexpansive mappings covers several well-known mappings. For example, every α-nonexpansive mappings is an AK-generalized nonexpansive mapping and also 0-nonexpansive maps are exactly nonexpansive maps. Hence we have the following diagram.



The following example shows that the reverse implication does not hold.

Example 1.2 ([14])**.** Let $X = \{(0,0),(2,0)(0,4),(4,0),(4,5),(5,4)\}$ be a subset of $\mathbb{R}^2$ with dictionary order. Define a inner product $(X, \langle\cdot,\cdot\rangle = ||\cdot,\cdot||)$. by $||x_1, x_2|| = (|x_1| + |x_1|)^2$. Then $(X, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Define a mapping $T : X \to X$ by

$$T(0,0) = (0,0),\ T(2,0) = (0,0),\ T(0,4) = (0,0),$$
$$T(4,0) = (2,0),\ T(4,5) = (4,0),\ T(5,4) = (0,4).$$

Then, we have T is not an α-nonexpansive mapping for any $\alpha < 1$, and $x = (4,5)$ and $y = (5,4)$, but, we consider $||T(x) - x|| = 25$ and $||T(y) - y|| = 25$. Then, we have

$$\begin{aligned}||Tx - Ty|| = 64 &< \frac{3}{4}100 + \frac{1}{4}4 \\ &\leq \alpha_1 25 + \alpha_2 25 + \alpha_3 25 + \alpha_4 25 + (1 - 4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})4 \\ &\leq \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Tx - y|| \\ &\quad + (1 - 4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})||x - y||, \end{aligned} \tag{1.3}$$

where $\min\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} = \frac{16}{25}$. Thus, T is an AK-generalized nonexpansive.

Example 1.3. Let $X = [0,2]$ be a nonempty closed convex subset of a Hilbert space $(H = \mathbb{R}, \langle\cdot,\cdot\rangle = |\cdot|)$. Suppose that $T : [0,2] \to [0,2]$ be given by $Tx = \sin x + \cos x$, for all $x \in [0,2]$. Now, we consider

$$\begin{aligned}||Tx - Ty|| &= \frac{1}{2}|2\sin x + 2\cos x - 2\sin y - 2\cos y| \\ &\leq \frac{1}{2}|\sin x + \cos x - x| + \frac{1}{2}|\sin y + \cos y - y| + \frac{1}{2}|\sin x + \cos x - y| \\ &\quad + \frac{1}{2}|\sin y + \cos y - x| \\ &\leq \alpha_1|\sin x + \cos x - x| + \alpha_2|\sin y + \cos y - y| + \alpha_3|\sin x + \cos x - y| \\ &\quad + \alpha_4|\sin y + \cos y - x| + (1 - 4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})|x - y| \\ &= \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Tx - y|| \\ &\quad + (1 - 4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})||x - y||, \end{aligned} \tag{1.4}$$

where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$. Then T is an AK-generalized nonexpansive.

Our work improves and generalizes some of the results obtained in the above paper. We introduce the AK-generalized nonexpansive mapping as generalization of an α-nonexpansive mapping. We also discuss sucient and necessary conditions of some property for such mappings and obtain a convergence result of the viscosity approximation method for an AK-generalized nonexpansive semigroups under some assumptions in Hilbert spaces. Moreover, we prove a strong convergence theorem for a family of AK-generalized nonexpansive mapping in Hilbert spaces.

2. Preliminaries

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

For any $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z \rangle \geq 0, \forall z \in C.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition*, *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n \to \infty} \inf \|x_n - x\| < \lim_{n \to \infty} \inf \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1 ([15]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \in \mathbb{N},$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$\text{(1)} \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ (2)} \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n \to \infty} s_n = 0.$

Lemma 2.2. *Let H be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$$

for all $x, y \in H$.

Now, we introduce the definitions follow on the results of Song *et al.* [13].
Let E be a Banach space. An (one-parameter) AK-generalized nonexpansive semigroup is a family $\mathcal{T} = \{T(t) : t > 0\}$ of mappings $D(\mathcal{T}) = \bigcap_{t > 0} D(T(t))$ and range $R(T)$ such that

(1): $T(0)x = x$ for all $x \in D(\mathcal{T})$;
(2): $T(t + s)x = T(t)T(s)x$ for all $t, s > 0$ and $x \in D(\mathcal{T})$;
(3): for each $t > 0$, $T(t)$ is an AK-generalized nonexpansive mapping, that is,

$$\begin{aligned} ||Tx - Ty|| \leq{}& \alpha_1 ||Tx - x|| + \alpha_2 ||Ty - y|| + \alpha_3 ||Tx - y|| + \alpha_4 ||Ty - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}) ||x - y||, \end{aligned} \tag{2.1}$$

for all $x, y \in C$, $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$.

Example 2.3. Let $X = [0,2]$ be a nonempty closed convex subset of a Hilbert space $(H = \mathbb{R}, \langle\cdot,\cdot\rangle = |\cdot|)$. Suppose that $T : [0,2] \to [0,2]$ be given by $Tx = 3^{-x}$, for all $x \in [0,2]$. Now, for any $t, s > 0$ and $x \in D(\mathcal{T})$;

(1) $T(0)x = 3^0 x = x$;

(2) $T(t+s)x = 3^{-(t+s)}x = 3^{-(t)}3^{-(s)}x = T(t)T(s)x$;

(3) for each $t > 0$, $T(t)$ is an AK-generalized nonexpansive mapping, that is,

$$\begin{aligned} ||Tx - Ty|| &= |3^{-x} - 3^{-y}| \\ &= \frac{1}{2}|2(3^{-x} - 3^{-y})| \\ &= \frac{1}{2}|(3^{-x} - 3^{-y}) + (3^{-x} - 3^{-y}) + x - x + y - y| \\ &= \frac{1}{2}|3^{-x} - x - 3^{-y} + y + 3^{-x} - y - 3^{-y} + x| \\ &= \frac{1}{2}|(3^{-x} - x) - (3^{-y} - y) + (3^{-x} - y) - (3^{-y} - x)| \\ &\le \frac{1}{2}|3^{-x} - x| + \frac{1}{2}|3^{-y} - y| + \frac{1}{2}|3^{-x} - y| + \frac{1}{2}|3^{-y} - x| \\ &\le \alpha_1|3^{-x} - x| + \alpha_2|3^{-y} - y| + \alpha_3|3^{-x} - y| \\ &\quad + \alpha_4|3^{-y} - x| + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})|x - y| \\ &= \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Tx - y|| \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, \end{aligned}$$

where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$.

Let $\mathcal{T} = \{T(t) : t > 0\}$ stants for one-parameter AK-generalized nonexpansive semigroup and $F(\mathcal{T}) = \bigcap_{t>0} F(T(t))$. We give the concept of the uniformly asymptotically regular as the following definitions.

Definition 2.4. An AK-generalized nonexpansive semigroup $\mathcal{T} = \{T(t) : t > 0\}$ is said to be *uniformly asymptotically regular (in short, u.a.r.)* if, for any $s \ge 0$ and any bounded subset K of $D(\mathcal{T})$,

$$\lim_{t\to\infty} \sup_{x\in K} \|T(s)(T(t)x) - T(t)x\| = 0.$$

Definition 2.5. A family $\{T_n\}$ of an AK-generalized nonexpansive mapping is said to be *uniformly asymptotically regular (in short, u.a.r.)* if, for each positive integer m and any bounded subset K of $\bigcap_n D(T_n)$,

$$\lim_{n\to\infty} \sup_{x\in K} \|T_m(T_n x) - T_n x\| = 0.$$

3. Main Results

In this section, we first study some properties of an AK-generalized nonexpansive mapping in a Hilbert space.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an AK-generalized nonexpansive mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed convex and $||Tx - p|| \le ||x - p||$ for all $x \in C$ and $p \in F(T)$.*

Proof. Since T is an AK-generalized nonexpansive mapping, for all $x \in C$ and $p \in F(T)$

$$\begin{aligned}||Tx - p|| &= ||Tx - Tp|| \\ &\leq \alpha_1||Tx - x|| + \alpha_2||Tp - p|| + \alpha_3||Tx - p|| + \alpha_4||Tp - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - p|| \\ &\leq \alpha_1(||Tx - p|| + ||p - x||) + \alpha_3||Tx - p|| + \alpha_4||p - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - p||, \qquad (3.1)\end{aligned}$$

and so

$$||Tx - p|| \leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1 - \alpha_1 - \alpha_3}||x - p|| < ||x - p||. \qquad (3.2)$$

Let $p, q \in F(T)$, $(0 \leq \lambda \leq 1)$ and set $z = \lambda p + (1 - \lambda)q$. Using the Parallelogram Law, we get

$$\begin{aligned}||\frac{z - p}{2} - \frac{Tz - p}{2}||^2 + \frac{1}{4}||z - Tz||^2 &= \frac{1}{2}||z - p||^2 + \frac{1}{2}||Tz - p||^2 \\ &\leq ||z - p||^2, \\ ||\frac{z - q}{2} - \frac{Tz - q}{2}||^2 + \frac{1}{4}||z - Tz||^2 &= \frac{1}{2}||z - q||^2 + \frac{1}{2}||Tz - q||^2 \\ &\leq ||z - q||^2.\end{aligned}$$

(3.2) implies that

$$\begin{aligned}||\frac{z + Tz}{2} - p||^2 = ||\frac{z - p}{2} + \frac{Tz - p}{2}||^2 &\leq ||z - p||^2 - \frac{1}{4}||z - Tz||^2 \\ &= (1 - \lambda)^2||p - q||^2 - \frac{1}{4}||z - Tz||^2, \\ ||\frac{z + Tz}{2} - q||^2 = ||\frac{z - q}{2} + \frac{Tz - q}{2}||^2 &\leq ||z - q||^2 - \frac{1}{4}||z - Tz||^2 \\ &= \lambda^2||p - q||^2 - \frac{1}{4}||z - Tz||^2.\end{aligned}$$

Suppose that $z \neq Tz$. Then, we have

$$||\frac{z + Tz}{2} - p||^2 < (1 - \lambda)^2||p - q||^2, \quad ||\frac{z + Tz}{2} - q||^2 < \lambda^2||p - q||^2.$$

So, we obtain that

$$||p - q|| \leq ||\frac{z + Tz}{2} - p|| + ||\frac{z + Tz}{2} - q|| < (1-\lambda)||p - q|| + \lambda||p - q|| = ||p - q||,$$

which is a contradiction and so $z = Tz$. Thus $F(T)$ is convex. Now, we show $F(T)$ is closed. Suppose that $\{x_n\} \in F(T)$ with $\lim_{n\to\infty} x_n = x$, it follows from (3.3) that $||x_n - Tx|| = ||x_n - x|| \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} x_n = Tx = x$, Thus $F(T)$ is closed. ∎

Proposition 3.2. *Let H be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an AK-generalized nonexpansive mapping. If a sequence $\{x_n\}$ in C converges weakly to $x \in C$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $x = Tx$.*

Proof. Since $\{x_n\}$ is weakly convergent, we have $\{x_n\}$ is bounded. Since

$$||Tx_n|| \leq ||Tx_n - x_n|| + ||x_n||,$$

we get $\{Tx_n\}$ is bounded. If $0 \leq \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$, then

$$\begin{aligned} ||Tx_n - Tx|| &\leq \alpha_1||Tx_n - x_n|| + \alpha_2||Tx - x|| + \alpha_3||Tx_n - x|| + \alpha_4||Tx - x_n|| \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x|| \\ &\leq \alpha_1||Tx_n - x_n|| + \alpha_2(||Tx - Tx_n|| + ||Tx_n - x_n|| + ||x_n - x||) \\ &\quad + \alpha_3(||Tx_n - x_n|| + |x_n - x||) + \alpha_4(||Tx - Tx_n|| + ||Tx_n - x_n||) \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x||. \end{aligned} \tag{3.3}$$

This implies that

$$\begin{aligned} ||Tx_n - Tx|| &\leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1-\alpha_2-\alpha_4}||x_n-x|| + \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{1-\alpha_2-\alpha_4}||Tx_n-x_n|| \\ &\leq ||x_n - x|| + \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n||. \end{aligned} \tag{3.4}$$

If $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 0$,

$$\begin{aligned} ||Tx_n - Tx|| &\leq \alpha_1||Tx_n - x_n|| + \alpha_2||Tx - x|| + \alpha_3||Tx_n - x|| + \alpha_4||Tx - x_n|| \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x|| \\ &\leq \alpha_1||Tx_n - x_n|| + \alpha_2(||Tx - Tx_n|| - ||Tx_n - x_n|| + ||x_n - x||) \\ &\quad + \alpha_3(||x_n - x|| - ||Tx_n - x_n||) + \alpha_4(||Tx - Tx_n|| - ||Tx_n - x_n||) \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x_n - x||. \end{aligned} \tag{3.5}$$

This implies that

$$\begin{aligned} ||Tx_n - Tx|| &\leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1-\alpha_2-\alpha_4}||x_n-x|| + \frac{\alpha_1-\alpha_2-\alpha_3-\alpha_4}{1-\alpha_2-\alpha_4}||Tx_n-x_n|| \\ &\leq ||x_n - x|| + \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n||. \end{aligned} \tag{3.6}$$

In other cases, we obtain that

$$||Tx_n - Tx|| \leq ||x_n - x|| + \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n||. \tag{3.7}$$

Thus,

$$\limsup_{n\to\infty} ||Tx_n - Tx|| \leq \limsup_{n\to\infty} ||x_n - x||. \tag{3.8}$$

Thus, by the properties of a Hilbert space H, we have

$$\begin{aligned} ||x_n - x||^2 &= ||(x_n - Tx) + (Tx - x)||^2 \\ &= ||x_n - Tx||^2 + ||Tx - x||^2 + 2\langle x_n - Tx, Tx - x\rangle \\ &\leq (||x_n - Tx_n|| + ||Tx_n - Tx||)^2 + ||Tx - x||^2 + 2\langle x_n - Tx, Tx - x\rangle. \end{aligned}$$

Since $\{x_n\}$ weakly converges to $x \in C$, it follows from (3.8) that

$$\begin{aligned}\limsup_{n\to\infty} ||x_n - x||^2 &\le \limsup_{n\to\infty} ||Tx_n - Tx||^2 + ||Tx - x||^2 \\ &\quad + 2\limsup_{n\to\infty}\langle x_n - Tx, Tx - x\rangle \\ &\le \limsup_{n\to\infty} ||Tx_n - Tx||^2 + ||Tx - x||^2 + 2\langle x - Tx, Tx - x\rangle \\ &\le \limsup_{n\to\infty} ||x_n - x||^2 - ||Tx - x||^2\end{aligned}$$

respectively, and hence $||Tx - x||^2 \le 0$. ∎

From the proof of Proposition 3.2, we have the following lemma:

Lemma 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be an AK-generalized nonexpansive mapping. Then*

$$||Tx_n - Tx|| \le ||x_n - x|| + \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1 - \alpha_2 - \alpha_4}||Tx_n - x_n|| \tag{3.9}$$

for all $x, y \in C$.

Now, we prove a strong convergence theorem of the viscosity approximation method for an AK-generalized nonexpansive semigroup under some assumptions in a Hilbert space.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H and $\mathcal{T} = \{T(t) : t > 0\}$ be the u.a.r. semigroup of AK-generalized nonexpansive mappings from C into itself with $F(\mathcal{T}) \ne \emptyset$. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in (0, 1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)T(t_n)x_n, n \ge 1, \tag{3.10}$$

where $\{\beta_n\} \subseteq (0, 1)$ and $t_n > 0$ satisfy the following conditions:

$$\text{(i) } \lim_{n\to\infty} \beta_n = 0, \text{ (ii) } \sum_{n=1}^{\infty} \beta_n = \infty, \text{ (iii) } \lim_{n\to\infty} t_n = +\infty.$$

Then the sequence $\{x_n\}$ converge strongly to $z_0 = P_{F(\mathcal{T})}f(z_0)$.

Proof. Firstly, we show that the sequence $\{x_n\}$ is bounded. Let $x^* = P_{F(\mathcal{T})}f(x_0)$. From Lemma 3.1, then $\|T(t)x - x^*\| \le \|x - x^*\|$ for all $x \in C$ and $t > 0$. From the definition of x_n, we get

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \beta_n\|f(x_n) - x^*\| + (1 - \beta_n)\|T(t_n)x_n - x^*\| \\ &\le \beta_n\|f(x_n) - x^*\| + (1 - \beta_n)\|x_n - x^*\| \\ &\le \beta_n\|f(x_n) - f(x^*)\| + \beta_n\|f(x^*) - x^*\| + (1 - \beta_n)\|x_n - x^*\| \\ &\le \beta_n\gamma\|x_n - x^*\| + (1 - \beta_n)\|x_n - x^*\| + \beta_n\|f(x^*) - x^*\| \\ &= (1 - \beta_n(1 - \gamma))\|x_n - x^*\| + \beta_n\|f(x^*) - x^*\|.\end{aligned}$$

By mathematical induction, we obtain that

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\right\}, \forall n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is bounded and so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$.
From the condition $\lim_{n\to\infty}\beta_n = 0$, we have

$$\lim_{n\to\infty}\|x_{n+1} - T(t_n)x_n\| = \lim_{n\to\infty}\beta_n\|f(x_n) - T(t_n)x_n\| = 0. \tag{3.11}$$

Since $\{T(t) : t > 0\}$ is the u.a.r. AK-generalized nonexpansive semigroup, then for $s > 0$,

$$\lim_{n\to\infty}\|T(s)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n\to\infty}\sup_{x\in K}\|T(s)T(t_n)x - T(t_n)x\| = 0. \tag{3.12}$$

where K is any bounded subset of C containing $\{x_n\}$.
From the definition of a AK-generalized nonexpansive and Lemma 3.3, we have

$$\begin{aligned}||T(s)(T(t_n)x_n) - T(s)x_{n+1}|| \leq& ||T(t_n)x_n - x_{n+1}||\\ &+\frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1-\alpha_2-\alpha_4}||T(s)(T(t_n)x_n) - T(t_n)x_n||.\end{aligned} \tag{3.13}$$

Hence, for all $s > 0$,

$$\begin{aligned}\|x_{n+1} - T(s)x_{n+1}\| \leq& \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(s)(T(t_n)x_n)\|\\ &+ \|T(s)(T(t_n)x_n) + T(s)x_{n+1}\|\\ \leq& \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(s)(T(t_n)x_n)\|\\ &+ ||x_{n+1} - T(t_n)x_n||\\ &+ \frac{\alpha_1 + |\alpha_2| + |\alpha_3| + |\alpha_4|}{1-\alpha_2-\alpha_4}||T(t_n)x_n - T(s)(T(t_n)x_n)||\\ \leq& 2\|x_{n+1} - T(t_n)x_n\|\\ &+ \left(1 + \frac{\alpha_1+|\alpha_2|+|\alpha_3|+|\alpha_4|}{1-\alpha_2-\alpha_4}\right)\|T(t_n)x_n - T(s)(T(t_n)x_n)\|.\end{aligned} \tag{3.14}$$

From (3.11), (3.12), and (3.14), we have

$$\lim_{n\to\infty}\|x_{n+1} - T(s)x_{n+1}\| = 0 \tag{3.15}$$

for all $s > 0$.
Next, we show that $\lim_{n\to\infty}\sup\langle f(z_0) - z_0, x_n - z_0\rangle \leq 0$ where $z_0 = P_{F(\mathcal{T})}f(z_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{k\to\infty}\langle f(z_0) - z_0, x_{n_k} - z_0\rangle. \tag{3.16}$$

Without loss of generality, we may assume $\{x_{n_k}\} \rightharpoonup \omega$ for some $\omega \in C$. By Lemma 3.1 and (3.15), we have $\omega \in F(T(s))$. Since s is arbitrary, then $\omega \in F(\mathcal{T})$.
Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in F(\mathcal{T})$. By (3.16) and the properties of the metric projection, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle &= \lim_{k\to\infty}\langle f(z_0) - z_0, x_{n_k} - z_0\rangle\\ &= \langle f(z_0) - z_0, \omega - z_0\rangle\\ &\leq 0.\end{aligned} \tag{3.17}$$

Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{F(\mathcal{T})}f(z_0)$. By Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\beta_n(f(x_n) - z_0) + (1-\beta_n)(T(t_n)x_n - z_0)\|^2 \\
&\leq \|(1-\beta_n)(T(t_n)x_n - z_0)\|^2 + 2\beta_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&= (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\|f(x_n) - f(z_0)\|\|x_{n+1} - z_0\| \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\beta_n)^2\|x_n - z_0\|^2 + 2\beta_n\gamma\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\beta_n)^2\|x_n - z_0\|^2 + \beta_n\gamma\|x_n - z_0\|^2 + \beta_n\gamma\|x_{n+1} - z_0\|^2 \\
&\quad + 2\beta_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

It implies that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq \frac{(1-\beta_n)^2 + \beta_n\gamma}{1-\beta_n\gamma}\|x_n - z_0\|^2 + \frac{2\beta_n}{1-\beta_n\gamma}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&= \left(1 - \frac{2\beta_n(1-\gamma)}{1-\beta_n\gamma}\right)\|x_n - z_0\|^2 + \frac{2\beta_n(1-\gamma)}{1-\beta_n\gamma}\left(\frac{\beta_n}{2(1-\gamma)}\|x_n - z_0\|^2\right. \\
&\quad \left. + \frac{1}{1-\gamma}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\right).
\end{aligned}$$

From the conditions (i),(ii), (3.17), and Lemma 2.1, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}f(z_0)$. This completes the proof. ∎

By continuing in the same direction as in Theorem 3.4, we obtian the following theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H and $\{T_n\}$ be a family of u.a.r. AK-generalized nonexpansive mappings from C into itself with $\mathcal{F} = F(T_n) \neq \emptyset$. Let $f : C \to C$ be a contraction mapping with coefficient $\gamma \in (0,1)$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$ and*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)T_n x_n, n \geq 1, \tag{3.18}$$

where $\{\beta_n\} \subseteq (0,1)$ and $t_n > 0$ satisfy the following conditions:

$$\text{(i) } \lim_{n\to\infty} \beta_n = 0, \text{ (ii) } \sum_{n=1}^{\infty} \beta_n = \infty.$$

Then the sequence $\{x_n\}$ converge strongly to $z_0 = P_{\mathcal{F}}f(z_0)$.

Proof. Put the terms $T(t_n)$ and $T(s)$ with the terms T_n and T_m in Theorem 3.4. Using Definition 2.5 and the same method of proof in Theorem 3.4, we have the desired conclusion. ∎

Remark 3.6. In this work, we introduce a new class of nonexpansive type of mapping namely, AK-generalized nonexpansive mapping, which is more general than an α-nonexpansive mapping. Moreover, we obtain convergence results of the viscosity approximation method for an AK-generalized nonexpansive semigroups under some assumptions in Hilbert spaces. However, we should like remark the following:

(1) The main theorem of Song *et al.* [13] gave a strong convergence theorem of the Halpern iteration for an α-nonexpansive semigroups in Hilbert spaces by using the iterative scheme (1.1), while the main theorem of this paper give a convergence result for an AK-generalized nonexpansive semigroup by using the iterative scheme (3.10) which is improve and extend than the main results of Song *et al.* [13].

(2) The class of mappings studied in this paper is more general than the class of mappings studied in Aoyama and Kohsaka [2].

(3) We studied the AK-generalized nonexpansive mapping in Hilbert spaces as Aoyama and Kohsaka [2], Muangchoo-in *et al.* [3] and Ariza-Ruiz *et al.* [4] studied α-nonexpansive mappings in Banach space. Moreover, Xu [6] investigated a nonexpansive mapping in Hilbert spaces.

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