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A Cone Generalized b-Metric Like Space over Banach Algebra and Contraction Principle

Zoran D. Mitrović[1,*], Aziz Ahmed[2] and J. N. Salunke[2]

[1] *University of Banja Luka, Faculty of Electrical Engineering, Patre 5, 78000 Banja Luka, Bosnia and Herzegovina*
e-mail : zoran.mitrovic@etf.unibl.org (Z. D. Mitrović)

[2] *School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded, India*
e-mail : azizahmed02@gmail.com (A. Ahmed); drjnsalunke@gmail.com (J. N. Salunke)

Abstract This paper introduce the notion of cone generalized b-metric like space over a Banach algebra which is a generalization of cone metric-like space over a Banach algebra, cone metric space over a Banach algebra and metric like space. Also, this prove Banach and Kannan fixed point theorems for contractive generalized Lipschitz mapping in such a space. In addition some examples are given to illustrate that our generalization is a real generalization.

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1. Introduction

The metric space was originated in the Ph.D. thesis of Maurice Fréchet [1] in 1906. After that many authors generalized this concept and obtained partial metric space, generalized metric space, rectangular metric space, K-metric space i.e. cone metric space by some authors, cone metric space, rectangular b-metric space, $b_v(s)$-metric space (see for example [2–13]). One of the generalization of metric space is given by Bakhtin [14], in 1989, known as b-metric space, where the triangular inequality is replaced by b-triangular inequality with coefficient $s \geq 1$. In 2016 Huang and Radenović [15] introduced the concept of cone b-metric space over a Banach algebra. Recently, in 2017 second and third author [16] define algebra cone generalized b-metric space over a Banach algebra by replacing the constant s with the vector, an element of a cone P with $r(s) \geq 1$.

Now, this paper defines cone generalized b-metric-like space over a Banach algebra by replacing the coefficient of b-metric, $s \geq 1$ with a vector $s \succeq e$, an element of cone P and obtain Banach and Kannan fixed point results on such space for contractive generalized Lipschitz mappings.

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*Corresponding author.

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The following definitions and results will be needed in the sequel.

In 1998, Czerwik [17] introduced the notion of a b-metric space.

Definition 1.1. Let X be a nonempty set, $s \geq 1$ and $d : X \times X \to [0, +\infty)$ a function such that for all $x, y, z \in X$,
(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
Then d is called a b-metric on X and (X, d, s) is called a b-metric space.

Let A be a real Banach algebra that is a real Banach space in which an operation of multiplication is defined subject to the following properties

(1) $(xy)z = x(yz)$,
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$,
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,
(4) $||xy|| \leq \|x\|\|y\|$,

for all $x, y, z \in A$, $\alpha \in \mathbb{R}$.

An algebra A with unit element e, is called unital algebra A, i.e. multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

A subset P of a Banach algebra A is called a cone if

(1) P is non-empty, closed and $\{\theta, e\} \subset P$,
(2) $aP + bP \subset P$, for all non-negative real numbers a and b,
(3) $P^2 = PP \subset P$,
(4) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra A. For a given cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$ and we write $x \prec y$ if $x \preceq y$ and $x \neq y$ while $x \ll y$ will stands for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$ then P is called a solid cone. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in A$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|.$$

Definition 1.2. ([18]) Let A be a Banach algebra with unit elements e and $P \subseteq A$ be a cone, P is algebra cone if $e \in P$ and $ab \in P$ for all $a, b \in P$.

Definition 1.3. ([19]) Let X be a non-empty set. Let A be a Banach algebra and $P \subseteq A$ be an algebra cone. Suppose the mapping $d\colon X \times X \to A$ satisfies

(1) $\theta \preceq d(x, y)$ for every $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$ for every $x, y \in X$,
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for every $x, y, z \in X$.

Then (X, d) is called algebra cone metric space over a Banach algebra.

Definition 1.4. ([6, 15]) Let (X, d) be a cone metric space over a Banach algebra A, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$;
(2) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$;

(3) A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

In the sequel we shall consider so-called solid cone ($intP \neq \emptyset$).

Lemma 1.5. ([15, 20]) *Let A be a Banach algebra with a unity e and $x \in A$, then $\lim_{x\to\infty} \|x^n\|^{1/n}$ exists and the spectral radius $r(x)$ satisfies*

$$r(x) = \lim_{x\to\infty} \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}.$$

If $r(x) < |\lambda|$ then $\lambda e - k$ is invertible in A. Moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 1.6. ([15]) *Let A be a Banach algebra with a unity e and $k \in A$. If λ is a complex constant and $r(k) < |\lambda|$ then*

$$r((\lambda e - k)^{-1}) \leq \frac{1}{|\lambda| - r(k)}.$$

Definition 1.7. ([21, 22]) If P be a solid cone in a Banach space X. A sequence $\{u_n\} \subset X$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

Proposition 1.8. ([21]) *Let P be a solid cone in a Banach space X and let $\{x_n\}$ and $\{y_n\}$ are sequences in X. If $\{x_n\}$ and $\{y_n\}$ are c-sequence and $\alpha, \beta \in P$ then $\{\alpha x_n + \beta y_n\}$ is a c-sequence.*

Proposition 1.9. ([21]) *Let P be a solid cone in a Banach space X and let $\{x_n\}$ be a sequence in P. Then the following conditions are equivalent*

(1) $\{x_n\}$ is a c-sequence;
(2) for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n \prec c$ for $n \geq n_0$;
(3) for each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $x_n \preceq c$ for $n \geq n_1$.

Proposition 1.10. ([22]) *Let (X, d) be a complete cone metric space over a Banach algebra A and let P be the underlying solid cone in A. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$ then we have*

(1) $\{d(x_n, x)\}$ is a c-sequence,
(2) for any $n, p \in \mathbb{N}, \{d(x_n, x_{n+p})\}$ is a c-sequence.

Remark 1.11. From Proposition 1.10 we have, for any $m, n \in \mathbb{N}$, $m > n$, $\{d(x_n, x_m)\}$ is a c-sequence.

Lemma 1.12. ([22]) *Let A Banach algebra A and $x, y \in A$. If x and y commutes then the following hold*

(1) $r(xy) \leq r(x)r(y)$,
(2) $r(x + y) \leq r(x) + r(y)$.

Lemma 1.13. ([15]) *Let A be a Banach algebra with a unity e and P be a underlying solid cone in A. Let $y \in A$ and $u_n = y^n$. If $r(y) < 1$ then $\{u_n\}$ is a c-sequence.*

Lemma 1.14. ([23]) *Let E is a real Banach space with a solid cone P and if $\theta \leq u \ll c$ for each $c \gg \theta$ then $u = \theta$.*

Lemma 1.15. ([23]) *Let E is a real Banach space with a solid cone P and if $\|x_n\| \to 0 (n \to \infty)$ then for any $\theta \ll c$ there exists $N \in \mathbb{N}$ such that for any $n > N$ we have $x_n \ll c$.*

Remark 1.16. ([22]) If $r(k) < 1$ then $\|k^n\| \to 0$ when $(n \to \infty)$.

2. Main Result

This section defines cone b-metric-like space over a Banach algebra.

Definition 2.1. Let X be a non-empty set, $P \subseteq A$ be a cone in a Banach algebra A and $d : X \times X \to A$ be a mapping such that for all $x, y, z \in X$ satisfies

(1) $d(x, y) = \theta$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) there exists $s \in P, e \preceq s$ such that $d(x, y) \preceq s[d(x, z) + d(z, y)]$.

Then d is called algebra cone b-metric and (X, d) is called algebra cone b-metric space over a Banach algebra (in short ACbMS-BA) with coefficient s.

Definition 2.2. Let X be a non-empty set, $P \subseteq A$ be a cone in a Banach algebra A and $d : X \times X \to A$ be a mapping such that for all $x, y, z \in X$ satisfies

(1) $d(x, y) = \theta$ implies $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) there exists $s \in P, e \preceq s$ such that $d(x, y) \preceq s[d(x, z) + d(z, y)]$.

Then d is called cone b-metric-like and (X, d) is called cone generalized b-metric-like space over a Banach algebra (in short CGbMLS-BA) with coefficient s.

A cone generalized $b-$metric-like space over a Banach algebra satisfies all of the conditions of a cone $b-$metric space over a Banach algebra except that $d(x, x)$ need not be θ for $x \in X$.
We present an example of a cone generalized b-metric-like space over a Banach algebra in the following.

Example 2.3. Let $A = \{a = (a_{ij})_{3\times 3} : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ and

$$\|a\| = \frac{1}{3} \sum_{1\leq i,j\leq 3} |a_{ij}|.$$

Take a cone $P = \{a \in A : a_{ij} \geq 0, 1 \leq i, j \leq 3\}$ in A. Let $X = \{1, 2, 3\}$. Define a mapping $d : X \times X \to A$ by and

$$d(1,1) = \begin{pmatrix} 3 & 2 & 2 \\ 4 & 5 & 3 \\ 5 & 7 & 4 \end{pmatrix},\ d(2,2) = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ d(3,3) = \begin{pmatrix} 7 & 6 & 3 \\ 8 & 4 & 5 \\ 7 & 3 & 2 \end{pmatrix},$$

$$d(1,2) = d(2,1) = \begin{pmatrix} 5 & 2 & 4 \\ 4 & 7 & 3 \\ 3 & 5 & 3 \end{pmatrix},\ d(3,1) = d(1,3) \begin{pmatrix} 5 & 4 & 4 \\ 6 & 12 & 5 \\ 7 & 5 & 3 \end{pmatrix},$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 9 & 5 & 6 \\ 6 & 10 & 6 \\ 7 & 9 & 5 \end{pmatrix},$$

Then (X, d) is a cone generalized b-metric-like space over a Banach algebra with coefficient $s = 2I$, where I is identity matrix.

Now we shall prove Banach fixed point theorem in setting of cone generalized b-metric space over a Banach algebra.

Theorem 2.4. *Let (X, d) be a complete cone generalized b-metric-like space over a Banach algebra A with coefficient $s \succeq e$. Let $k \in P$ such that k commutes with s and $r(k) \in [0, 1)$. Suppose that the mapping $T : X \to X$ satisfies generalized Lipschitz condition*

$$d(Tx, Ty) \preceq kd(x, y), \tag{2.1}$$

for all $x, y \in X$, then T has a unique fixed point in X.

Proof. We use some ideas from [11]. Let $x_0 \in X$ be arbitrary and $x_n = Tx_{n-1} = T^n x_0$. Then from (2.1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq kd(x_{n-1}, x_n) \\ &\preceq k^2 d(x_{n-2}, x_{n-1}) \\ &\cdots \\ &\preceq k^n d(x_0, x_1). \end{aligned}$$

Similarly, as above, we get it

$$d(x_{m+p}, x_{n+p}) \preceq k^p d(x_m, x_n), \tag{2.2}$$

for all $m, n, p \in \mathbb{N}$.

Next, as it is $r(k) < 1$ we get it to exist $p_0 \in \mathbb{N}$ such that

$$r(s)^2 r(k)^{p_0} < 1. \tag{2.3}$$

Now we have

$$\begin{aligned} d(x_m, x_n) &\preceq s(d(x_m, x_{m+p_0}) + d(x_{m+p_0}, x_n)) \\ &\preceq s[d(x_m, x_{m+p_0}) + s(d(x_{m+p_0}, x_{n+p_0}) + d(x_{n+p_0}, x_n))] \\ &\preceq sd(x_m, x_{m+p_0}) + s^2(d(x_{m+p_0}, x_{n+p_0}) + d(x_{n+p_0}, x_n)). \end{aligned}$$

From (2.2) we obtain

$$\begin{aligned} d(x_m, x_{m+p_0}) &\preceq k^m d(x_0, x_{p_0}), \\ d(x_{m+p_0}, x_{n+p_0}) &\preceq k^{p_0} d(x_m, x_n), \\ d(x_{n+p_0}, x_n) &\preceq k^n d(x_{p_0}, x_0). \end{aligned}$$

Based on the above, we conclude that it is valid

$$d(x_m, x_n) \preceq sk^m d(x_0, x_{p_0}) + s^2 k^{p_0} d(x_m, x_n) + s^2 k^n d(x_{p_0}, x_0).$$

From here we get,

$$(e - s^2 k^{p_0}) d(x_m, x_n) \preceq sk^m d(x_0, x_{p_0}) + s^2 k^n d(x_{p_0}, x_0).$$

Since k commutes with s, by Lemma 1.6 and Lemma 1.12 we obtain that

$$r(s^2 k^{p_0}) \leq r(s)^2 r(k)^{p_0} < 1,$$

and that $(e - s^2k^{p_0})$ is invertible. Therefore, we conclude that it is valid

$$d(x_n, x_m) \preceq (e - s^2k^{p_0})^{-1}[sk^m d(x_0, x_{p_0}) + s^2k^n d(x_{p_0}, x_0)]$$

Now Lemma 1.13 implies that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$

$$\begin{aligned} d(Tx^*, x^*) &\preceq s(d(Tx^*, Tx_n) + d(Tx_n, x^*)) \\ &\preceq skd(x^*, x_n) + sd(x_{n+1}, x^*) \end{aligned}$$

From Definition 1.7, Propositions 1.8, 1.9 and 1.10 we have

$$d(Tx^*, x^*) \preceq u_n,$$

where $u_n = skd(x^*, x_n) + sd(x_{n+1}, x^*)$ is a c-sequence in cone P. Hence for each $c \gg \theta$ we have $d(Tx^*, x^*) \preceq c$ so by Lemma 1.14 we obtain

$$d(Tx^*, x^*) = \theta.$$

From Definition 2.2 we obtain that x^* is a fixed point of T.

Let y^* be the another fixed point of T. Then we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq kd(x^*, y^*).$$

So,

$$(e - k)d(x^*, y^*) \preceq \theta \text{ and } d(x^*, y^*) = \theta$$

again from Definition 2.2 we have $x^* = y^*$. Hence T has a unique fixed point in X. ∎

Now, we shall prove Kannan fixed point theorem in the framework of cone generalized b-metric space over a Banach algebra.

Theorem 2.5. *Let (X, d) be a complete cone generalized b-metric-like space over a Banach algebra A with coefficient $s \succeq e$. Let $k_1, k_2, s \in P$ commutes and $r(k_1 + k_2) \leq \min\{1, \frac{2}{r(s)}\}$. If the mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \preceq k_1 d(Tx, x) + k_2 d(Ty, y), \tag{2.4}$$

for all $x, y \in X$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and $x_n = Tx_{n-1} = T^n x_0$, for all $n \in \mathbb{N}$.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\preceq k_1 d(Tx_n, x_n) + k_2 d(Tx_{n-1}, x_{n-1}) \\ &\preceq k_1 d(x_{n+1}, x_n) + k_2 d(x_n, x_{n-1}). \end{aligned}$$

So, we have

$$(e - k_1)d(x_{n+1}, x_n) \preceq k_2 d(x_n, x_{n-1}). \tag{2.5}$$

Also we have,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq k_1 d(Tx_{n-1}, x_{n-1}) + k_2 d(Tx_n, x_n) \\ &\preceq k_1 d(x_n, x_{n-1}) + k_2 d(x_{n+1}, x_n). \end{aligned}$$

Therefore,

$$(e - k_2)d(x_{n+1}, x_n) \preceq k_1 d(x_n, x_{n-1}). \tag{2.6}$$

Add up equation (2.5) and (2.6), we have

$$(2e - k_1 - k_2)d(x_n, x_{n+1}) \preceq (k_1 + k_2)d(x_n, x_{n-1})).$$

Let $k = k_1 + k_2$ so $r(k) = r(k_1 + k_2) < 1 < 2$ so by Lemma 1.5 $(2e - k)$ is invertible.

$$d(x_n, x_{n+1}) \preceq k(2e - k)^{-1}d(x_{n-1}, x_n)$$

Let $h = k(2e - k)^{-1}$, since k commutes with $(2e - k)^{-1}$, we have

$$r(h) = r((2e - k)^{-1}k) \leq r((2e - k)^{-1})r(k) = \frac{r(k)}{2 - r(k)} < 1. \tag{2.7}$$

Therefore,

$$d(x_n, x_{n+1}) \preceq h^n d(x_0, x_1). \tag{2.8}$$

Now, for $n, m \in \mathbb{N}$ with $m > n$, from (2.4) and (2.8), we have,

$$\begin{aligned} d(x_n, x_m) &\preceq k_1 d(x_n, x_{n-1}) + k_2 d(x_m, x_{m-1})) \\ &\preceq k_1 h^{n-1} d(x_1, x_0) + k_2 h^{m-1} d(x_1, x_0) \\ &\preceq (k_1 h^{n-1} + k_2 h^{m-1}) d(x_1, x_0). \end{aligned}$$

Since k_1 and k_2 commutes we obtain (see (2.7))

$$r((k_1 + k_2)h^{n-1}) \leq r(k)r(h)^{n-1} \leq \frac{r(k)^n}{2 - r(k)}.$$

This implies that it is $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Therefore, we obtain

$$\begin{aligned} d(Tx^*, x^*) &\preceq s(d(Tx^*, Tx_n) + d(Tx_n, x^*)) \\ &\preceq s(k_1 d(Tx^*, x^*) + k_2 d(Tx_n, x_n)) + sd(Tx_n, x^*). \end{aligned}$$

So,

$$(e - sk_1)d(Tx^*, x^*) \preceq sk_2 d(Tx_n, x_n) + sd(Tx_n, x^*). \tag{2.9}$$

Now,

$$\begin{aligned} d(x^*, Tx^*) &\preceq s(d(x^*, Tx_n) + d(Tx_n, Tx^*)) \\ &\preceq sd(x^*, Tx_n) + s(k_1 d(Tx_n, x_n) + k_2 d(Tx^*, x^*)). \end{aligned}$$

Therefore,

$$(e - sk_2)d(Tx^*, x^*) \preceq sk_1 d(Tx_n, x_n) + sd(Tx_n, x^*). \tag{2.10}$$

From (2.9) and (2.10), we get,

$$\begin{aligned} (2e - sk_1 - sk_2)d(Tx^*, x^*) &\preceq s(k_1 + k_2)d(Tx_n, x_n) + 2sd(Tx_n, x^*) \\ (2e - sk)d(Tx^*, x^*) &\preceq skd(Tx_n, x_n) + 2sd(Tx_n, x^*) \\ (2e - sk)d(Tx^*, x^*) &\preceq skd(x_{n+1}, x_n) + 2sd(x_{n+1}, x^*) \end{aligned}$$

From Definition 1.7, Propositions 1.8, 1.9, and 1.10 we have

$$(2e - sk)d(Tx^*, x^*) \preceq u_n,$$

where

$$u_n = skd(x_{n+1}, x_n) + 2sd(x_{n+1}, x^*)$$

is a c-sequence in cone P. Since $r(k) < \frac{2}{r(s)}$ we have that $r(sk) < 2$ and $2e - sk$ is invertible, hence for each $c \gg \theta$ we have

$$d(Tx^*, x^*) \preceq (2e - sk)^{-1} u_n \preceq c.$$

So, by Lemma 1.14, we have

$$d(Tx^*, x^*) = \theta.$$

Therefore,

$$d(Tx^*, x^*) = \theta,$$

and from Definition 2.2 we obtain that x^* is a fixed point of T.

Let y^* be the another fixed point of T.

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\preceq k_1 d(Tx^*, x^*) + k_2 d(Ty^*, y^*) \\ &\preceq \theta. \end{aligned}$$

This implies that $d(x^*, y^*) = \theta$ so, from Definition 2.2 we obtain $x^* = y^*$. Hence T has a unique fixed point in X. ∎

It is easy to prove the following Theorems by Theorems 2.4 and 2.5, so we omits the proofs.

Theorem 2.6. *Let (X, d) be a complete algebra cone metric space over a Banach algebra. If the mapping $T : X \longrightarrow X$ satisfies*

$$d(Tx, Ty) \preceq kd(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$ then T has a unique fixed point in X.

Theorem 2.7. *Let (X, d) be a complete algebra cone metric space over a Banach algebra. If the mapping $T : X \longrightarrow X$ satisfies*

$$d(Tx, Ty) \preceq k_1 d(Tx, x) + k_2 d(Ty, y),$$

for all $x, y \in X$, where $k_1, k_2 \geq 0$ such that $k_1 + k_2 < 1$ then T has a unique fixed point in X.

We present an example to illustrate Theorem 2.4.

Example 2.8. Consider Example 2.3, Let $T : X \to X$ be a mapping define by $T1 = T2 = 2, T3 = 1$ and let $k = \begin{pmatrix} 0.5 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.6 \\ 0.1 & 0.4 & 0.2 \end{pmatrix} \in P$ then $r(k) = 0.933 \in [0, 1)$. By simple calculations we can see that all the conditions of Theorem 2.4 are satisfied. The point x=2 is the unique fixed point of T.

From Theorems 2.4 and 2.5 we obtain the following results in b-metric spaces.

Theorem 2.9. ([24], Theorem 2.1) *Let (X, d, s) be a complete b-metric space. If the mapping $T : X \longrightarrow X$ satisfies*

$$d(Tx, Ty) \preceq kd(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$ then T has a unique fixed point in X.

Theorem 2.10. *Let (X, d, s) be a complete b-metric space. If the mapping $T : X \longrightarrow X$ satisfies*

$$d(Tx, Ty) \preceq k_1 d(Tx, x) + k_2 d(Ty, y),$$

for all $x, y \in X$, where $k_1, k_2 \geq 0$ such that $k_1 + k_2 < \min\{1, \frac{2}{s}\}$ then T has a unique fixed point in X.

Remark 2.11. 1. Theorem 2.9 is a proper generalization and improvement of Theorem 3.13. in [25] in the sense that the range of Lipschitzian constant λ is increased from $[0, \frac{1}{s})$ to the interval $[0, 1)$ and the metric function d it does not have to be continuous.
2. Note that the Theorem of Kannan (see [26]) holds in b-metric space if $s \leq 2$.

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