Thai Journal of Mathematics
Volume 6 (2008) Number 2 : 271–277
www.math.science.cmu.ac.th/thaijournal

Extension of Hall's theorem and an algorithm for finding the $(1, n)$-complete matching

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Abstract : Hall's theorem provides the necessary and sufficient conditions for the existence of $(1, 1)$-complete matching in bipartite graphs. The extension of Hall's theorem provides the necessary and sufficient conditions for the existence of $(1, n)$-complete matching, with $n \geq 1$. The proof of the extension exist in some few advanced texts with more advanced language, and therefore the extension is not widely known. In this paper we propose another approach of the proof which is simpler and less involved. Also, from this, an algorithm for finding the $(1, n)$-complete matching is derived.

Keywords : Hall's theorem, bipartite graph, complete matching, algorithm.

1 Hall's theorem

Let $V_1 = \{m_1,\ m_2,\ ...,\ m_k\}$ be a set of men, and $V_2 = \{w_1,\ w_2,\ ...,\ w_l\}$, where $l \geq k$, be a set of l jobs(women, books, etc.). The set V_1 and V_2 form a bipartite graph $G(V_1,\ V_2)$. The line joining m_i in V_1 with w_j in V_2 means that the man m_i is qualified for the job w_j. Each job in V_2 requires a man to complete it, while each man m_i is required to do the same number of jobs $n \geq 1$. A man can be assigned to do only the jobs they are qualified for. Consider a graph $G(V_1,\ V_2)$. We say that there is $(1, n)$-complete matching from V_1 to V_2 if each of all men in V_1 can be assigned to do n jobs in V_2. For example, let $V_{1a} = \{m_1,\ m_2,\ m_3,\ m_4,\ m_5\}$, and $V_{2a} = \{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,\ w_7\}$. Figure 1(a) describes the bipartite graph $G(V_{1a},\ V_{2a})$. There is $(1, 1)$-complete matching from V_{1a} to V_{2a}. See bold letters in the Figure 1(a) for jobs that are assigned for the men in V_{1a}. For another example, let $V_{1b} = \{m_1,\ m_2,\ m_3,\ m_4\}$, and $V_{2b} = \{w_1,\ w_2,\ w_3,\ ...,\ w_{10}\}$. Figure 1(b) describes the bipartite graph $G(V_{1b},\ V_{2b})$. There is $(1, 2)$-complete matching from V_{1b} to V_{2b}. See bold letters in each line of Figure 1(b) for two jobs that are assigned for each man.

$G(V_{1a},\ V_{2a})$

men in V_{1a}	*Jobs that men are qualified for in* V_{2a}
m_1	$\mathbf{w_1},\ w_4$
m_2	$w_1,\ \mathbf{w_3},\ w_4$
m_3	$w_4,\ \mathbf{w_5},\ w_6,\ w_7$
m_4	$w_2,\ \mathbf{w_4}$
m_5	$w_5,\ \mathbf{w_6},\ w_7$

(1,1)-complete matching

(a)

$G(V_{1b},\ V_{2b})$

men in V_{1b}	*Jobs that men are qualified for in* V_{2b}
m_1	$\mathbf{w_1},\ w_2,\ \mathbf{w_3},\ w_7$
m_2	$\mathbf{w_4},\ \mathbf{w_5},\ w_9$
m_3	$w_7,\ \mathbf{w_8},\ \mathbf{w_9},\ w_{10}$
m_4	$w_5,\ \mathbf{w_6},\ w_9,\ \mathbf{w_{10}}$

(1,2)-complete matching

(b)

Figure 1

Let A be a non empty subset of V_1 and $\phi(A)$ be the set of all jobs in V_2 that at least one man in A is qualified to do them. The sizes or the number of elements of A and $\phi(A)$ are denoted by $|A|$ and $|\phi(A)|$ respectively. Philip Hall, in 1935, proposed a famous theorem that provides the necessary and sufficient conditions for the existence of $(1,1)$-complete matching from V_1 to V_2. Hall's theorem is usually discussed in most texts in graphs, combinatorics, or discrete mathematics. For examples, see [1], [2], [3], [5], [6], and [7].

Theorem 1 (*Hall's theorem*)

Let $G(V_1,\ V_2)$ be a bipartite graph. There exists a $(1,1)$-complete matching from V_1 to V_2 if and only if $|A| \leq |\phi(A)|$ for every subset A of V_1. ■

For example, from Figure 1(a) we can verify that $|A| \leq |\phi(A)|$ for every subset A of V_{1a}, therefore there is $(1,1)$-complete matching, as shown in Figure 1(a), from V_{1a} to V_{2a}.

2 Extension of Hall's theorem

Although Hall's theorem is well known, the theorem for the extension of Hall's theorem, see Theorem 2, is not widely known to students in graph theory. Perhaps, this is partly because the proof for the extension exist only in some few advanced texts, and in a form of more advanced language. See [4], for example. In this section, we propose another approach of the proof for the extension of Hall's theorem, i.e. the necessary and sufficient conditions for the existence of $(1, n)$-complete matching, for $n \geq 1$. The proof is simple, less involved, so ones can gain immediate access to the extension of Hall's theorem. With $n = 1$, this can also be used as another version of proof for Hall's theorem.

Theorem 2 Let $G(V_1,\ V_2)$ be a bipartite graph. There exists $(1, n)$-complete matching from V_1 to V_2 if and only if $n|A| \leq |\phi(A)|$ for every subset A of V_1.

Another Proof: First, we prove the necessary part. Suppose there exists $(1, n)$-complete matching from V_1 to V_2 but there is a subset A of V_1 such that $n|A| > |\phi(A)|$. From the definition of $(1, n)$-complete matching, a man in A will has n jobs in $\phi(A)$ to do, so all men in A need $n|A|$ jobs in $\phi(A)$ to do. Therefore, this is not possible if $n|A| > |\phi(A)|$, and so $n|A| \leq |\phi(A)|$.

Next, we prove the sufficient part. Let $n|A| \leq |\phi(A)|$ for every subset A of $V_1 = \{m_1,\ m_2,\ ...,\ m_k\}$. We shall show by mathematical induction that, for all $k \geq 1$, there is $(1, n)$-complete matching from V_1 to V_2, i.e. each of the k men in V_1 can be assigned for n jobs. When $k = 1$, $V_1 = \{m_1\}$. Since $n|\{m_1\}| \leq |\phi(\{m_1\})|$, then $n \leq |\phi(\{m_1\})|$. So, we can assign n jobs in $\phi(\{m_1\})$ for m_1. Therefore, the theorem is true when $k = 1$. Consider the case when $k \geq 2$. Assume that there is $(1, n)$-complete matching for any $k - 1$ men, we then consequently try to show that there also is $(1, n)$-complete matching for all of the k men. Let $A_1 = \{m_1,\ m_2,\ ...,\ m_{k-1}\}$ be the set of $k - 1$ men. From the assumption, we can assign n jobs for each man of A_1. From this, we shall try to assign n jobs for m_k. Here, it would be easier for the proof if we assign n jobs for m_k first.

From the condition $n|A| \leq |\phi(A)|$ for every subset A of V_1, before assigning n jobs for m_k, we have

$$n|A_1| \leq |\phi(A_1)| \tag{2.1}$$

$$n|A_1 \cup \{m_k\}| \leq |\phi(A_1 \cup \{m_k\})|. \tag{2.2}$$

We note that the number of jobs in $\phi(A_1)$ or any subset of $\phi(A_1)$ could be affected by some of the n jobs assigned for m_k, because some of these n jobs in $\phi(A_1)$ can no longer be assigned for men in A_1 . So, to complete the assumption that we can assign n jobs for each man of A_1, we need to verify that the condition (2.1), after assigning n jobs for m_k, still holds for A_1 and it's subsets. Some jobs of $\phi(\{m_k\})$ could be outside $\phi(A_1)$, while some could be inside $\phi(A_1)$. Let n_{out}, and n_{in} be the numbers of jobs of $\phi(\{m_k\})$ that are outside, and inside $\phi(A_1)$ respectively. So, we have $|\phi(\{m_k\})| = n_{out} + n_{in}$. In assigning n jobs for m_k we consider two cases: case(1) when $n_{out} \geq n$, and case(2) when $n_{out} < n$. In both cases, we try to assign, as many as possible, jobs outside $\phi(A_1)$ for m_k.

First, consider case(1) when $n_{out} \geq n$. In this case we choose n jobs outside $\phi(A_1)$ for m_k. So, the number of jobs in $\phi(A_1)$ does not change after assigning n jobs for m_k. Therefore, condition (2.1) still holds for A_1 and all subsets A of A_1.

Next, consider case(2) when $0 \leq n_{out} < n$. In this case, for m_k, we assign all n_{out} jobs outside $\phi(A_1)$, and $n_i = n - n_{out}$ jobs inside $\phi(A_1)$. After assigning n jobs for m_k, the number of jobs in $\phi(A_1)$ will be reduced by n_i, where $1 \leq n_i \leq n$. In verifying (2.1) for A_1 and it's subsets, we remove these n_i jobs assigned for m_k from $\phi(A_1)$. The new $\phi(A_1)$ whose n_i jobs were removed shall be denoted by $\phi^*(A_1)$. So, we have $|\phi^*(A_1)| = |\phi(A_1)| - n_i$. We shall show that after assigning n

jobs for m_k, and with the new reduced values of $|\phi(A_1)|$, the condition (2.1) still holds for A_1. From (2.2), we can have

$$\begin{aligned} n|A_1| + n &\leq |\phi(A_1)| + |\phi(\{m_k\})| - |\phi(A_1) \cap \phi(\{m_k\})| \\ &= (|\phi^*(A_1)| + n_i) + (n_{out} + n_{in}) - n_{in} \\ &= |\phi^*(A_1)| + n_i + n_{out} \\ &= |\phi^*(A_1)| + n \end{aligned}$$

So, we have $n|A_1| \leq |\phi^*(A_1)|$, i.e. (2.1), with the new reduced value of $|\phi(A_1)|$, still holds for A_1. We have shown that the condition (2.1), after assigning n jobs for m_k, still holds for A_1 with $k-1$ men. By using similar arguments as above we can show that the condition (2.1), after assigning n jobs for m_k, still holds for every other subset A of A_1.

From the results of case (1) and case (2) , we have shown that, for sets V_1 of any size $k \geq 2$, with $n|A| \leq |\phi(A)|$ for all subsets A of V_1, if we can assign n jobs for each of any $k-1$ men then we can also assign n jobs for all of the k men. Hence, we conclude that, with $n|A| \leq |\phi(A)|$ for all subsets A of V_1, there exists $(1, n)$-complete matching from V_1 to V_2 for all $k \geq 1$. ■

For example, consider every subset A of V_{1b} and the corresponding $\phi(A)$ of the bipartite $G(V_{1b},\ V_{2b})$ described in Figure 1(b). See Figure 2 for details. We can see that $2|A| \leq |\phi(A)|$ for every subset A. Therefore, by Theorem 2, we shall have the existence of $(1, 2)$-complete matching from V_{1b} to V_{2b}. For another example, consider Figure 1(a), the subset $A = \{m_1,\ m_4\}$ has $\phi(A) = \{w_1,\ w_2,\ w_4\}$. Since $2|A| > |\phi(A)|$, therefore by Theorem 2, there is no $(1, 2)$-complete matching from V_{1a} to V_{2a}.

A	$\phi(A)$
$\{m_1\}$	$\{w_1,\ w_2,\ w_3,\ w_7\}$
$\{m_2\}$	$\{w_4,\ w_5,\ w_9\}$
$\{m_3\}$	$\{w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_4\}$	$\{w_5,\ w_6,\ w_9,\ w_{10}\}$
$\{m_1,\ m_2\}$	$\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_7,\ w_9\}$
$\{m_1,\ m_3\}$	$\{w_1,\ w_2,\ w_3,\ w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_1,\ m_4\}$	$\{w_1,\ w_2,\ w_3,\ w_5,\ w_6,\ w_7,\ w_9,\ w_{10}\}$
$\{m_2,\ m_3\}$	$\{w_4,\ w_5,\ w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_2,\ m_4\}$	$\{w_4,\ w_5,\ w_6,\ w_9,\ w_{10}\}$
$\{m_3,\ m_4\}$	$\{w_5,\ w_6,\ w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_1,\ m_2,\ m_3\}$	$\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_1,\ m_2,\ m_4\}$	$\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,\ w_7,\ w_9,\ w_{10}\}$
$\{m_1,\ m_3,\ m_4\}$	$\{w_1,\ w_2,\ w_3,\ w_5,\ w_6,\ w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_2,\ m_3,\ m_4\}$	$\{w_4,\ w_5,\ w_6,\ w_7,\ w_8,\ w_9,\ w_{10}\}$
$\{m_1,\ m_2,\ m_3,\ m_4\}$	$\{w_1,\ w_2,\ w_3,\ w_4,\ w_5,\ w_6,\ w_7,\ w_8,\ w_9,\ w_{10}\}$

Figure 2

3 An algorithm for finding $(1, n)$-complete matching in bipartite graphs

Theorem 2 can guarantee the existence of $(1, n)$-complete matching in some bipartite graphs. Suppose we know, by Theorem 2, that a $G(V_1,\ V_2)$ contains $(1, n)$-complete matching. The next question is how to assign n jobs for all of each man in V_1. If we try to assign the jobs without proper algorithm we might face, at some steps, the situation that we can not assign n jobs for some men. For example, from Figure 2, with Theorem 2, we know that there exists $(1, 2)$-

complete matching in $G(V_{1b},\ V_{2b})$ of Figure 1(b). However, if we assign w_1, w_3 for m_1, assign w_4, w_5 for m_2, and assign w_9, w_{10} for m_3, then we can not assign 2 jobs for m_4. From this example, it is clear that, in order to assign n jobs for all of each man of V_1, it is desirable to have a proper algorithm which can assure us that we can assign the jobs for all men successfully.

The alternative proof of Theorem 2 above can help us to create such algorithm. First, we shall describe the idea of how to construct the algorithm, and, after that, explain why the algorithm works.

Given a bipartite graph $G(V_1,\ V_2)$, with the condition as in Theorem 2. Consider m_1 and $A_1 = V_1 - \{m_1\}$. From $\phi(\{m_1\})$, we shall choose n jobs for m_1. Let n_{out}, and n_{in} be the numbers of jobs of $\phi(\{m_1\})$ that are outside, and inside $\phi(A_1)$ respectively. We assign jobs for m_1, as many as possible, from outside $\phi(A_1)$. That is, when $n_{out} \geq n$, we choose n jobs outside $\phi(A_1)$ for m_1, and when $0 \leq n_{out} < n$ we assign all n_{out} jobs outside $\phi(A_1)$, and $n_i = n - n_{out}$ jobs inside $\phi(A_1)$ for m_1. Remove m_1 and the n jobs assigned for m_1 from consideration. Consider m_2 and $A_2 = V_1 - \{m_1, m_2\}$. From $\phi(\{m_2\})$, we shall choose n jobs for m_2. Let n_{out}, and n_{in} be the numbers of jobs of $\phi(\{m_2\})$ that are outside, and inside $\phi(A_2)$ respectively.We, again, assign jobs for m_2, as many as possible, from outside $\phi(A_2)$. That is, when $n_{out} \geq n$, we choose n jobs outside $\phi(A_2)$ for m_2. When $0 \leq n_{out} < n$ we assign all n_{out} jobs outside $\phi(A_2)$, and $n_i = n - n_{out}$ jobs inside $\phi(A_2)$ for m_2. Remove m_2 and the n jobs assigned for m_2 from consideration. Consider m_3 and $A_3 = V_1 - \{m_1, m_2, m_3\}$. Repeat the procedure until we have assigned n jobs for m_{k-1}. Finally, we assign n jobs for m_k from the remaining jobs in $\phi(\{m_k\})$.

Let us call the above algorithm as "Finding-complete-matching algorithm" or FICMA. Theorem 3 gives an explanation about the algorithm.

Theorem 3 Let there exist $(1, n)$-complete matching in a bipartite graph $G(V_1,\ V_2)$.Then the algorithm FICMA can be used in assigning n jobs for all men in V_1.

Proof: Consider a given $G(V_1,\ V_2)$ with the condition $n|A| \leq |\phi(A)|$ for every subset A of V_1. After assigning n jobs for m_1 as described in the algorithm, we shall find, as in the proof of Theorem 2, that $A_1 = V_1 - \{m_1\}$ still satisfies the condition $n|A_1| \leq |\phi(A_1)|$. Also, as in the proof of Theorem 2, every subset A of A_1 shall satisfy the condition $n|A| \leq |\phi(A)|$. Therefore, after remove m_1 and the n jobs assigned for m_1 from consideration, and with the use of Theorem 2, we can claim that there still exist $(1, n)$-complete matching from A_1 to the remaining jobs in V_2. Consider m_2 and $A_2 = V_1 - \{m_1, m_2\}$. Again, after assigning n jobs for m_2 as described in the algorithm, we shall find that $A_2 = V_1 - \{m_1, m_2\}$ still satisfies the condition $n|A_2| \leq |\phi(A_2)|$. Also, every subset A of A_2 shall satisfy the condition $n|A| \leq |\phi(A)|$. Therefore, after remove m_2 and the n jobs assigned for m_2 from consideration, and with the use of Theorem 2, we can claim that there still exist $(1, n)$-complete matching from A_2 to the remaining jobs in V_2. Repeat the procedure, until, finally, after remove m_{k-1} and the n jobs assigned for m_{k-1} from consideration, and with the use of Theorem 2, we can claim that there still exist $(1, n)$-complete matching from $A_{k-1} = V_1 - \{m_1, m_2,\ ..., m_{k-1}\}$

to the remaining jobs in V_2. In fact, $A_{k-1} = \{m_k\}$, and so we can finally assign n jobs for the last man m_k. ■

For illustration, consider the bipartite $G(V_{1b},\ V_{2b})$ in Figure 1(b). Here $V_1 = \{m_1,\ m_2,\ m_3,\ m_4\}$. Consider m_1 and $A_1 = V_1 - \{m_1\} = \{m_2,\ m_3,\ m_4\}$. Here $\phi(\{m_1\}) = \{w_1,\ w_2,\ w_3,\ w_7\}$, $\phi(A_1) = \{w_4,\ w_5,\ w_6,\ w_8,\ w_9,\ w_{10}\}$. So we can assign any 2 jobs in $\phi(\{m_1\})$, say w_1,w_2 for m_1. Remove m_1 and the 2 jobs $w_1,\ w_2$ from consideration. Consider m_2 and $A_2 = V_1 - \{m_1,\ m_2\} = \{m_3,\ m_4\}$. Here $\phi(\{m_2\}) = \{w_4,\ w_5,\ w_9\}$, $\phi(A_2) = \{w_5,\ w_6,\ w_7,\ w_8,\ w_9,\ w_{10}\}$. According to the algorithm, we will try to assign jobs for m_2, as many as possible, from outside $\phi(A_2)$. Here, we can choose only w_4 from outside $\phi(A_2)$ for m_2, so we need to choose one of the jobs, say w_5, in $\phi(A_2)$ for m_2. Now, we have assigned $w_4,\ w_5$ for m_2. Remove m_2 and $w_4,\ w_5$ from consideration. Consider m_3 and $A_3 = V_1 - \{m_1,\ m_2,\ m_3\} = \{m_4\}$. Here $\phi(\{m_3\}) = \{w_7,\ w_8,\ w_9,\ w_{10}\}$, $\phi(A_3) = \{w_6,\ w_9,\ w_{10}\}$. Assign $w_7,\ w_8$ for m_3. Remove m_3 and $w_7,\ w_8$ from consideration. Consider m_4 and $\phi(\{m_4\}) = \{w_6,\ w_9,\ w_{10}\}$. We can assign $w_6,\ w_9$ for m_4. Hence, we have assigned 2 jobs for each of the four men of V_1.

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(Received 30 May 2007)

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