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Some Fixed Point Results for a New Multivalued Hybrid Mapping in Geodesic Spaces

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Abstract The studies about hybrid mappings are mainly focused for single-valued mappings in Hilbert spaces. We define a new class of multivalued mappings in CAT (κ) spaces which contains the multivalued nonexpansive mappings, α-nonexpansive mappings and some hybrid mappings such as (α, β)-hybrid mappings and we study existence and convergence of this new class of mappings on CAT(κ) spaces which is more general than CAT(0) spaces and also non-Euclidean generalization of Hilbert spaces.

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1. Introduction

Fixed point theory has a large application area as mathematics, economy, computer, medicine and so on. But all of them mentioned above have different behavior according to come true real life. In the real world, events have both linear and nonlinear structures. Fixed point theory includes works about both of them. In this theory, nonlinear structures is important more than linear ones. In this sense, geodesic spaces are an example to these nonlinear structures. We also have to point out that the CAT(0) and Hilbert spaces are similar structures. On the other hand, single and multi valued mappings in the theory are important two concepts. There are huge studies about linear and nonlinear single valued mappings. But nonlinear and multivalued studies are a few in the literature.

In this study we improve and generalize the wide mapping classes defined by P. Kocourek et. al. [P. Kocourek, W. Takahashi and J.-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces, Taiwanese J. Math., 14 (2010),2497-2511]. These classes are more general than nonspreading mappings, nonexpansive mappings, hybrid mappings and multivalued mappings. We also prove some fixed point results in CAT(κ)-space for $\kappa > 0$.

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This new definition for multivalued mapping class is more general than most of multivalued mappings in literature, for example multivalued nonexpansive mapping. Along with that it is also multivalued generalization of many mapping classes of single valued hybrid mappings which are not their multivalued generalization in literature.

2. Preliminaries

Let H be a Hilbert space and $K \subseteq H, K \neq \emptyset$. Let us take T as a single valued mapping from K to H. If T satisfies

$$||Tx - Ty|| \leq ||x - y||,$$
$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2$$

and

$$3||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2 + ||Ty - x||^2$$

for all $x, y \in K$ then it called *non-expansive*, *non-spreading* [1] and *hybrid* [2] respectively. None of these classes of mappings is included in the other. In 2010, Aoyama et al. [3] defined $\lambda-$*hybrid* as follows;

$$(1 + \lambda)||Tx - Ty||^2 - \lambda||x - Ty||^2 \leq (1 - \lambda)||x - y||^2 + \lambda||Tx - y||^2$$

where $x, y \in K$ and λ is fixed real number. $\lambda-$hybrid mappings are general than nonexpansive mappings, non-spreading mappings and hybrid mappings. In 2011, Aoyama and Kohsaka [4] introduced $\alpha-$*non-expansive mappings* in Banach spaces as follows;

$$||Tx - Ty||^2 \leq (1 - 2\alpha)||x - y||^2 + \alpha||Tx - y||^2 + \alpha||x - Ty||^2$$

where $x, y \in K$ and $\alpha < 1$ is fixed. They showed that $\alpha-$non-expansive and $\lambda-$hybrid are equivalent in Hilbert spaces for $\lambda < 2$. Kocourek et al. [5], introduced more general class of mappings than the above mappings in Hilbert spaces, called $(\alpha, \beta)-$*generalized hybrid*, as follows;

$$\alpha||Tx - Ty||^2 + (1 - \alpha)||x - Ty||^2 \leq \beta||Tx - y||^2 + (1 - \beta)||x - y||^2$$

where $x, y \in K$ and α, β are fixed real numbers.

Many iterative processes to find a fixed point of multivalued mappings have been introduced in metric and Banach spaces. The well known one is defined by Nadler as generalization of Picard as follows;

$$x_{n+1} \in Tx_n.$$

A multivalued version of Mann and Ishikawa fixed point procedures goes as follow;

$$x_{n+1} \in (1 - \zeta_n)x_n + \zeta_n Tx_n$$

and

$$\begin{aligned} x_{n+1} &\in (1 - \zeta_n)x_n + \zeta_n Ty_n, \\ y_n &\in (1 - \varsigma_n)x_n + \varsigma_n Tx_n \end{aligned}$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0, 1]$.

Gursoy and Karakaya [6] introduced Picard-S iteration as follows;

$$\begin{aligned} x_{n+1} &= Ty_n, \\ y_n &= (1 - \zeta_n)Tx_n + \zeta_n Tz_n, \\ z_n &= (1 - \varsigma_n)x_n + \varsigma_n Tx_n \end{aligned}$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0,1]$. Now, we give definition of the multivalued version of Picard-S iteration in CAT(κ) spaces as follows

$$\begin{aligned} x_{n+1} &= P_K(u_n), \\ y_n &= P_K((1-\zeta_n)w_n \oplus \zeta_n v_n), \\ z_n &= P_K((1-\varsigma_n)x_n \oplus \varsigma_n w_n) \end{aligned} \tag{2.1}$$

where $\{\zeta_n\}$ and $\{\varsigma_n\}$ are sequences in $[0,1]$ with $\liminf_n (1-\varsigma_n)\varsigma_n > 0$, $u_n \in Ty_n$, $v_n \in Tz_n$ and $w_n \in Tx_n$.

Let (X,d) be a metric space and take $K \subseteq X, K \neq \emptyset$. In the rest of this paper, we will use following notations; $C(X)$ for all nonempty, closed subsets of X, $CC(X)$ for all nonempty closed and convex subsets of X, $KC(X)$ for nonempty, compact and convex subsets of X and $CB(X)$ for all nonempty, closed and convex subsets of X. Let H be Hausdorff metric on $CB(X)$, defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$$

where $d(x,B) = \inf\{d(x,y); y \in B\}$. A point p is called *fixed point* of multivalued mapping T if $p \in Tp$ and the set of all fixed points of T is denoted by $F(T)$.

Let (X,d) be bounded metric space and take $x,y \in X$ and $K \subseteq X, K \neq \emptyset$. A *geodesic path* (or shortly a geodesic) joining x and y is a map $c : [0,t] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(t) = y$ and $d(c(r),c(s)) = |r-s|$ for all $r,s \in [0,t]$. In fact, c is an isometry and $d(c(0),c(t)) = t$. The image of c, $c([o,t])$ is called *geodesic segment* from x to y and it is not necessarily be unique. It is unique then it is denoted by $[x,y]$. $z \in [x,y]$ if and only if there exists $t \in [0,1]$ such that $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. The point z is denoted by $z = (1-t)x \oplus ty$. For fixed $r > 0$, the space (X,d) is called *r-geodesic space* if any two point $x,y \in X$ with $d(x,y) < r$ there is a geodesic joining x to y. if for every $x,y \in X$, there is a geodesic path then (X,d) called *geodesic space* and uniquely geodesic space if that geodesic path is unique for any pair x,y. We call a subset $K \subseteq X$ as a convex subset if it contains all geodesic segment joining any pair of points in it.

Definition 2.1 ([7])**.** Let take $\kappa \in \mathbb{R}$.

i) if $\kappa = 0$ then M^n_κ is Euclidean space $\mathbb{E}^n$,

ii) if $\kappa > 0$ then M^n_κ is obtained from the sphere $\mathbb{S}^n$ by multiplying distance function by $\frac{1}{\sqrt{\kappa}}$,

iii) if $\kappa < 0$ then M^n_κ is obtained from hyperbolic space $\mathbb{H}^n$ by multiplying distance function by $\frac{1}{\sqrt{-\kappa}}$.

In geodesic metric space (X,d), a geodesic triangle, $\Delta(x,y,z)$ consist of three point x,y,z as vertices and three geodesic segments of any pair of these points, that is, $q \in \Delta(x,y,z)$ means that $q \in [x,y] \cup [x,z] \cup [y,z]$. The triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ in M^2_κ is called *comparison triangle* for the triangle $\Delta(x,y,z)$ such that $d(x,y) = d(\overline{x},\overline{y}), d(x,z) = d(\overline{x},\overline{z})$ and $d(y,z) = d(\overline{y},\overline{z})$ and such a comparison triangle always exists provided that the perimeter $d(x,y)+d(y,z)+d(z,x) < 2D_\kappa (D_\kappa = \frac{\pi}{\sqrt{K}}$ if $\kappa > 0$ and ∞ otherwise) in M^2_κ (Lemma 2.14 in [7]). A point point $\overline{z} \in [\overline{x},\overline{y}]$ is called *comparison point* for $z \in [x,y]$ if $d(x,z) = d(\overline{x},\overline{z})$. A geodesic triangle $\Delta(x,y,z)$ in X with perimeter less than $2D_\kappa$ (and given a comparison triangle $\overline{\Delta}(\overline{x},\overline{y},\overline{z})$ for $\Delta(x,y,z)$ in M^2_κ) satisfies $CAT(\kappa)$ inequality if $d(p,q) \leq d(\overline{p},\overline{q})$ for all $p,q \in \Delta(x,y,z)$ where $\overline{p},\overline{q} \in \overline{\Delta}(\overline{x},\overline{y},\overline{z})$ are the comparison points of p,q respectively. The D_κ-geodesic metric space (X,d) is called $CAT(\kappa)$ space if every geodesic triangle in X with perimeter less than $2D_\kappa$ satisfies the $CAT(\kappa)$ inequality.

If for every $x,y,z\in X$, there is a $R\in(0,2]$ satisfying

$$d^2(x,(1-\lambda)y\oplus\lambda z)\le(1-\lambda)d^2(x,y)+\lambda d^2(x,z)-\frac{R}{2}\lambda(1-\lambda)d^2(y,z)$$

then (X,d) called $R-$*convex* [8]. Hence, (X,d) is a $CAT(0)$ space if and only if it is a $2-$*convex space.*

Lemma 2.2 ([9])**.** *Let $\kappa>0$ and (X,d) be a $CAT(\kappa)$ space with $diam(X)<\frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon\in(0,\frac{\pi}{2})$. Then (X,d) is a $R-$convex space for $R=(\pi-2\varepsilon)\tan(\varepsilon)$.*

Proposition 2.3 ([7])**.** *Let X be $CAT(\kappa)$ space. Then any ball of radius smaller than $\frac{\pi}{2\sqrt{\kappa}}$ are convex.*

Proposition 2.4 (Exercise 2.3(1), [7])**.** *Let $\kappa>0$ and (X,d) be a $CAT(\kappa)$ space with $diam(X)<\frac{D_\kappa}{2}=\frac{\pi}{2\sqrt{\kappa}}$. Then, for any $x,y,z\in X$ and $t\in[0,1]$, we have*

$$d((1-t)x\oplus ty,z)\le(1-t)d(x,z)+td(y,z).$$

Proposition 2.5 ([10])**.** *The modulus of convexity for $CAT(\kappa)$ space X (of dimension ≥ 2) and number $r<\frac{\pi}{2\sqrt{\kappa}}$ and let m denote the midpoint of the segment $[x,y]$ joining x and y defined by the modulus δ_r by sitting*

$$\delta(r,\varepsilon)=\inf\{1-\frac{1}{r}d(a,m)\}$$

where the infimum is taken over all points $a,x,y\in X$ satisfying $d(a,x)\le r$, $d(a,y)\le r$ and $\varepsilon\le d(x,y)<\frac{\pi}{2\sqrt{\kappa}}$.

Lemma 2.6 ([10])**.** *Let X be a complete $CAT(\kappa)$space with modulus of convexity $\delta(r,\varepsilon)$ and let $x\in E$. Suppose that $\delta(r,\varepsilon)$ increases with r (for a fixed ϵ) and suppose $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c\in(0,1)$, $\{x_n\}$ and $\{y_n\}$ are the sequences in X such that $\limsup_{n\to\infty}d(x_n,x)\le r$, $\limsup_{n\to\infty}d(y_n,x)\le r$ and $\lim_{n\to\infty}d((1-t_n)x_n\oplus t_ny_n,x)=r$ for some $r\ge 0$. Then $\lim_{n\to\infty}d(x_n,y_n)=0$.*

Let $\{x_n\}$ be a bounded sequence in a $CAT(\kappa)$ space X and $x\in X$. Then, with setting

$$r(x,\{x_n\})=\limsup_{n\to\infty}d(x,x_n)$$

the asymptotic radius of $\{x_n\}$ is defined by

$$r(\{x_n\})=\inf\{r(x,\{x_n\});x\in X.\},$$

the asymptotic radius of $\{x_n\}$ with respect to $K\subseteq X$ is defined by

$$r_K(\{x_n\})=\inf\{r(x,\{x_n\});x\in K.\},$$

and the asymptotic center of $\{x_n\}$ is defined by

$$A(\{x_n\})=\{x\in X:r(x,\{x_n\})=r(\{x_n\})\}$$

and let $\omega_w(x_n):=\cup A(\{x_n\})$ where union is taken on all subsequences of $\{x_n\}$.

Definition 2.7 ([11])**.** A sequence $\{x_n\}\subset X$ is said to be $\Delta-$*convergent* to $x\in X$ if x is the unique asymptotic center of all subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta-\lim_n x_n=x$ and read as x is the $\Delta-$limit of $\{x_n\}$.

Proposition 2.8 ([11]). *Let X be a complete $CAT(\kappa)$ space, $K \subseteq X$ nonempty, closed and convex, $\{x_n\}$ is a sequence in X. If $r_K(\{x_n\}) < \frac{\pi}{2\sqrt{\kappa}}$ then $A_K(\{x_n\})$ consist exactly one point.*

Lemma 2.9 ([12]). *We have the following facts.*

i) Every bounded sequence in X has a Δ-convergent subsequence,
ii) If K is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.

Lemma 2.10 ([12]). *If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = u$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.11 ([11]). *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X. Then*

i) the metric projection $P_K(x)$ of x onto K is a singleton,
ii) if $x \notin K$ and $y \in K$ with $u \neq P_K(x)$, then $\angle_{P_K(x)}(x, y) \geq \frac{\pi}{2}$,
iii) for each $y \in K$, $d(P_K(x), P_K(y)) \leq d(x, y)$.

Definition 2.12. T is called $(a_1, a_2, b_1, b_2)-$*generalized multivalued hybrid mapping* from X to $CB(X)$ if

$$a_1(x)H^2(Tx, Ty) + a_2(x)d^2(Tx, y) \leq b_1(x)d^2(x, Ty) + b_2(x)d^2(x, y)$$

is satisfied for all $x, y \in X$ where $a_1, a_2 : X \to \mathbb{R}$ and $b_1, b_2 : X \to \mathbb{R}$ with $a_1(x)+a_2(x) \geq 1$ and $b_1(x) + b_2(x) \leq 1$ for all $x \in X$.

3. Main Results

Proposition 3.1. *Let X be a complete $CAT(\kappa)$ space, K be a nonempty, closed and convex subset of X with $rad(K) < \frac{\pi}{2\sqrt{\kappa}}$, T be $(a_1, a_2, b_1, b_2)-$multivalued hybrid mapping from K to $C(K)$ with $F(T) \neq \emptyset$ and $a_1(p) \geq 0$ for all $p \in F(T)$ then $F(T)$ closed.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and $x_n \to x \in K$. Then we have

$$\begin{aligned} d^2(Tx, x_n) &\leq a_1(x)d^2(Tx, x_n) + a_2(x)d^2(Tx, x_n) \\ &\leq a_1(x)H^2(Tx, Tx_n) + a_2(x)d^2(Tx, x_n) \\ &\leq b_1(x)d^2(x, Tx_n) + b_2(x)d^2(x, x_n) \\ &\leq d^2(x, x_n) \end{aligned}$$

then taking limit on n we have

$$d(Tx, x) = 0$$

so $x \in Tx$. ∎

Theorem 3.2. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty, convex and compact subset of X, $T : K \to CC(X)$ be a $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping with $a_1(x) \geq 0$ for all $x \in K$. If $\{x_n\}$ is a sequence in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ then $F(T) \neq \emptyset$.*

Proof. Assume that $\{x_n\}$ is a sequence in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ Then since K is compact, there is convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $x_{n_i} \to z \in K$.Also, by 2.11 we can find a sequence $\{y_n\}$ such that $d(x_n, y_n) = d(x_n, Tx_n)$ for all $n \in \mathbb{N}$. Since $d(x_{n_i}, Tz) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, Tz)$ and $d(y_{n_i}, Tz) \leq d(x_{n_i}, y_{n_i}) + d(x_{n_i}, Tz)$ we have that $\limsup_{i\to\infty} d(x_{n_i}, Tz) = \limsup_{i\to\infty} d(y_{n_i}, Tz)$. Then using properties of T, we have

$$\begin{aligned} a_1(z)d^2(Tz, y_{n_i}) + a_2(z)d^2(Tz, x_{n_i}) &\leq a_1(z)H^2(Tz, Tx_{n_i}) + a_2(z)d^2(Tz, x_{n_i}) \\ &\leq b_1(z)d^2(z, Tx_{n_i}) + b_2(z)d^2(z, x_{n_i}) \\ &\leq b_1(z)[d(z, x_{n_i}) + d(x_{n_i}, Tx_{n_i})]^2 \\ &\quad +b_2(z)d^2(z, x_{n_i}) \end{aligned}$$

so we get that

$$\limsup_{i\to\infty} d(Tz, x_{n_i}) \leq \limsup_{i\to\infty} d(x_{n_i}, z) = 0.$$

Then

$$d(z, Tz) \leq d(z, x_{n_i}) + d(x_{n_i}, Tz)$$

implies that

$$d(z, Tz) \leq \limsup_{i\to\infty} d(z, x_{n_i}) + \limsup_{i\to\infty} d(x_{n_i}, Tz) = 0.$$

Hence we get that $z \in Tz$. ∎

Example 3.3. Let $X = [2, 10]$ with usual metric and $T : X \to C(X)$ be multivalued mapping defined by

$$Tx = \begin{cases} \{2\}, & x \in [2,5]; \\ [3, \frac{4x^3+1}{x^3+1}], & x \in (5,8]. \end{cases}$$

We will show that T is a $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping with $a_1(x) = \frac{2x+2}{x+1}, a_2(x) = \frac{-x-1}{x+1}, b_1(x) = \frac{x}{x+1}, b_2(x) = \frac{1}{x+2}$ for all $x \in X$.

Case 1: if $x, y \in [2, 5]$, it is obvious.

Case 2: if $x \in [2, 5], y \in (5, 8]$, then we have that $H^2(Tx, Ty) \leq 4,9 < d^2(Tx, y), 0 < d^2(x, Ty)$ and so

$$\begin{aligned} \frac{2x+2}{x+1}H^2(Tx, Ty) &\leq \frac{4x+8}{x+1} \\ &\leq \frac{9x+9}{x+1} + \frac{x+1}{x+1}d^2(x, Ty) + \frac{1}{x+2}d^2(x, y) \\ &\leq \frac{x+1}{x+1}d^2(Tx, y) + \frac{x+1}{x+1}d^2(x, Ty) + \frac{1}{x+1}d^2(x, y). \end{aligned}$$

Case 3: if $x, y \in (5, 8]$, then we have that $H^2(Tx, Ty) \leq 1$, $1 < d^2(Tx, y)$, $1 < d^2(x, Ty)$ and so

$$\begin{aligned}
\frac{2x+2}{x+1} H^2(Tx, Ty) &\leq \frac{2x+2}{x+1} \\
&\leq \frac{x+1}{x+1} + \frac{x+1}{x+1} + \frac{1}{x+1} d^2(x, y) \\
&\leq \frac{x+1}{x+1} d^2(Tx, y) + \frac{x+1}{x+1} d^2(x, Ty) + \frac{1}{x+1} d^2(x, y).
\end{aligned}$$

Thus T is a $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping with fixed point, $T(2) = \{2\}$.

Theorem 3.4. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty, convex and compact subset of X and $T : K \to CC(X)$ be a $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping with $a_1(x) \geq 0$ for all $x \in K$. If $\{x_n\}$ is a sequence in K with $\Delta - \lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $z \in K$ and $z \in T(z)$.*

Proof. It can be shown similar to proof of Theorem 3.2. ■

Lemma 3.5. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty, convex and compact subset of X and $T : K \to CC(X)$ be $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping with $a_1(x) \geq 0$ for all $x \in K$. If $\{x_n\}$ is a sequence in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, p)\}$ converges for all $p \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$ and $\omega_w(x_n)$ include exactly one point.*

Proof. Let take $u \in \omega_w(x_n)$ then there exist subsequence $\{u_n\}$ of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$.Then By Lemma 2.9 there exist subsequence $\{v_n\}$ of $\{u_n\}$ with $\Delta - \lim_{n\to\infty} v_n = v \in K$. Then by Theorem 3.4 we have $v \in F(T)$ and by Lemma 2.10 we conclude that $u = v$, hence we get $\omega_w(x_n) \subseteq F(T)$. Let take subsequence $\{u_n\}$ of $\{x_n\}$with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Because of $v \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges, so by Lemma 2.10 we have $x = u$, this means that $\omega_w(x_n)$ include exactly one point. ■

Theorem 3.6. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty, convex and compact subset of X and $T : K \to CC(X)$ be a $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping with $F(T) \neq \emptyset$, $Tp = \{p\}$ for all $p \in F(T)$ and $a_1(x) \geq 1$ for all $x \in K$. If $\{x_n\}$ be a sequence in K defined by (2.1) with $\liminf_n (1 - \varsigma_n)\varsigma_n > 0$, then it have a $\Delta-$limit which in $F(T)$.*

Proof. Let $p \in F(T)$ then for all $x \in C$ we have

$$\begin{aligned}
d^2(Tx, p) &\leq a_1(x) d^2(Tx, p) + a_2(x) d^2(Tx, p) \\
&\leq a_1(x) H^2(Tx, Tp) + a_2(x) d^2(Tx, p) \\
&\leq b_1(x) d^2(x, p) + b_2(x) d^2(x, p) \\
&\leq b_1(x) d^2(x, p) + b_2(x) d^2(x, p) \\
&\leq d^2(x, p).
\end{aligned}$$

If $a_2(x) \geq 0$ for all $x \in C$ then we have that

$$\begin{aligned} a_1(x)H^2(Tx,Tp) &\leq b_1(x)d^2(x,p)+b_2(x)d^2(x,p)-a_2(x)d^2(Tx,p) \\ &\leq d^2(x,p). \end{aligned}$$

So we have

$$H^2(Tx,Tp) \leq \frac{1}{a_1(x)}d^2(x,p) \leq d^2(x,p).$$

And now if $a_2(x) \leq 0$ for all $x \in C$ then since $a_1(x)+a_2(x) \geq 1$ implies $1 \geq \frac{1}{a_1(x)} - \frac{a_2(x)}{a_1(x)}$ so we have that

$$\begin{aligned} a_1(x)H^2(Tx,Tp) &\leq b_1(x)d^2(x,p)+b_2(x)d^2(x,p)-a_2(x)d^2(Tx,p) \\ &\leq d^2(x,p)-a_2(x)d^2(Tx,p) \end{aligned}$$

which implies that

$$\begin{aligned} H^2(Tx,Tp) &\leq \frac{1}{a_1(x)}d^2(x,p)-\frac{a_2(p)}{a_1(p)}d^2(Tx,p) \\ &\leq \frac{1}{a_1(x)}d^2(x,p)-\frac{a_2(x)}{a_1(x)}d^2(x,p) \\ &= (\frac{1}{a_1(x)}-\frac{a_2(x)}{a_1(x)})d^2(p,x) \\ &\leq d^2(p,x). \end{aligned}$$

Hence we have that $H(Tp,Tx) \leq d(p,x)$.

$$\begin{aligned} d(z_n,p) &= d(P_K((1-\varsigma_n)x_n \oplus \varsigma_n w_n),P_K(p)) \\ &= d(P_K((1-\varsigma_n)x_n \oplus \varsigma_n w_n),p) \\ &\leq d((1-\varsigma_n)x_n \oplus \varsigma_n w_n,p) \\ &\leq (1-\varsigma_n)d(x_n,p)+\varsigma_n d(w_n,p) \\ &\leq (1-\varsigma_n)d(x_n,p)+\varsigma_n d(w_n,Tp) \\ &\leq (1-\varsigma_n)d(x_n,p)+\varsigma_n H(Tx_n,Tp) \\ &\leq (1-\varsigma_n)d(x_n,p)+\varsigma_n d(x_n,p) \\ &\leq d(x_n,p) \end{aligned}$$

and

$$\begin{aligned} d(y_n,p) &= d(P_K((1-\zeta_n)w_n \oplus \zeta_n v_n,p) \\ &\leq d(P_K((1-\zeta_n)w_n \oplus \zeta_n v_n,P_K(p)) \\ &\leq d((1-\zeta_n)w_n \oplus \zeta_n v_n),p) \\ &\leq (1-\zeta_n)d(w_n,p)+\zeta_n d(v_n,p) \\ &\leq (1-\zeta_n)d(w_n,Tp)+\zeta_n d(v_n,Tp) \\ &\leq (1-\zeta_n)H(Tx_n,p)+\zeta_n H(Tz_n,Tp) \\ &\leq (1-\zeta_n)d(x_n,p)+\zeta_n d(z_n,p) \\ &\leq d(x_n,p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(P_K(u_n), P_K(p)) \\ &\leq d(u_n, p) \\ &\leq H(Ty_n, Tp) \\ &\leq d(y_n, p) \end{aligned}$$

so, $d(x_{n+1}, p) \leq d(y_n, p) \leq d(x_n, p)$ implies $\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(y_n, p)$ exists. Let us say, $\lim_{n\to\infty} d(x_n, p) = k$. Since $d(w_n, p) \leq d(x_n, p)$ and $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$, we have that $\limsup_{n\to\infty} d(w_n, p) \leq k, \limsup_{n\to\infty} d(v_n, p) \leq k$ and

$$\begin{aligned} d(y_n, p) &= d(P_K((1-\zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq d((1-\zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq (1-\zeta_n)d(w_n, p) + \zeta_n d(v_n, p) \\ &\leq (1-\zeta_n)d(x_n, p) + \zeta_n d(z_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

which implies that that $\lim_{n\to\infty} d((1-\zeta_n)w_n \oplus \zeta_n v_n, p) = k$, so by Lemma 2.6, we have that $\lim_{n\to\infty} d(w_n, v_n) = 0$. And again from

$$\begin{aligned} d(y_n, p) &= d(P_K((1-\zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq d((1-\zeta_n)w_n \oplus \zeta_n v_n, p) \\ &\leq (1-\zeta_n)d(w_n, p) + \zeta_n d(v_n, p) \\ &\leq (1-\zeta_n)(d(w_n, v_n) + d(v_n, p)) + \zeta_n d(v_n, p) \\ &\leq (1-\zeta_n)d(w_n, v_n) + d(v_n, p) \end{aligned}$$

we have that $k \leq \liminf_{n\to\infty} d(v_n, p)$ and since $d(v_n, p) \leq d(z_n, p) \leq d(x_n, p)$, we have that $\lim_{n\to\infty} d(z_n, p) = k$. By $R-$convexivity, we have

$$\begin{aligned} d^2(z_n, p) &= d^2(P_K((1-\varsigma_n)x_n \oplus \varsigma_n w_n), P_p) \\ &\leq d^2((1-\varsigma_n)x_n \oplus \varsigma_n w_n, p) \\ &\leq (1-\varsigma_n)d^2(x_n, p) + \varsigma_n d^2(w_n, p) - \frac{R}{2}(1-\varsigma_n)\varsigma_n d^2(x_n, w_n) \\ &\leq (1-\varsigma_n)d^2(x_n, p) + \varsigma_n d^2(x_n, p) - \frac{R}{2}(1-\varsigma_n)\varsigma_n d^2(x_n, w_n) \\ &\leq d^2(x_n, p) - \frac{R}{2}(1-\varsigma_n)\varsigma_n d^2(x_n, w_n) \end{aligned}$$

which implies that

$$\frac{R}{2}(1-\varsigma_n)\varsigma_n d^2(x_n, w_n) \leq d^2(x_n, p) - d^2(z_n, p).$$

Since $\lim_{n\to\infty}(d^2(x_n, p) - d^2(z_n, p)) = 0$ and $\liminf_n(1-\varsigma_n)\varsigma_n > 0$, therefore we get $\lim_{n\to\infty} d(x_n, w_n) = 0$ and hence $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. So, by Lemma 3.5, $\{x_n\}$ has $\Delta-$limit which in $F(T)$. ∎

Theorem 3.7. *Let $\kappa > 0$ and X be a complete $CAT(\kappa)$ space with $diam(X) \leq \frac{\pi-\varepsilon}{2\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. Let K be a nonempty, convex and compact subset of X and $T : K \to CC(X)$ be a continuous $(a_1, a_2, b_1, b_2)-$generalized multivalued hybrid mapping.*

If $\{x_n\}$ *is a sequence in* K *defined by (2.1) with* $\liminf_n (1-\varsigma_n)\varsigma_n > 0$ *and* $(\zeta_n) \subset (0,1)$ *then* $\{x_n\}$ *is strongly convergent to an element of* $F(T)$

Proof. By Theorem 3.6, we have that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Since K is compact there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, say $\lim_{i\to\infty} x_{n_i} = z$. Then we have

$$d(z, Tz) \leq d(z, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Tz)$$

and taking limit on i, continuity of T implies that $z \in Tz$. ■

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