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L-Fuzzy Relations via Proximal Spaces

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Abstract Our goal in this paper is to establish a new approach that is to define proximal spaces for lattices. To make this, we used the opinion of the generalization of interval $[0,1]$ to lattices and so we introduce L-proximal spaces. In this paper, we define the L-proximity axioms that need to be satisfied by a L-relation on proximal relator spaces. Also, L-proximity of the sets is introduced on the power set of a nonempty set. We show that the spatial Simirnov proximity measure can be generalized for L-measure and it is a Lodato L-proximity relation. L-proximity relation approach can be used to solve a few issues solution of classification problems and given examples for the problems.

MSC: 03G10; 54E05

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1. Introduction

The concepts of proximity spaces were rediscovered in the early 1950's by Efremoviĉ [1]. He gave the definition of a proximity space by axiomatically characterizing the proximity relation "A is near(proximal) B" for subsets A and B of any set X. Efremoviĉ later used the idea of proximity neighborhoods to generate proximity spaces. Some researchers have worked with weaker axioms than those of Efremoviĉ. Several articles were published on proximity spaces, and reader can find excellent list of publications in [2]. A proximity measure is a measure of the closeness of a pair of nonempty sets. The measure of degrees of proximity δ is firstly introduced by Smirnov in 1952 [3].

A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a relator on X, the ordered pair $(X, \mathcal{R})$ is called a relator space. Relator spaces are natural generalizations of ordered sets and uniform spaces [4]. In [5], Peters introduced proximal relator space $(X, \mathcal{R}_\delta)$, such that $\mathcal{R}_\delta$ is a family of proximity relations on X.

The fuzzy sets introduced by Zadeh in 1965 as a generalization of traditional set. A fuzzy set A in X is characterized by a membership function f_A which associates with each point in X a real number in the interval $[0,1]$. Also, the concept of fuzzy relation on a set was defined by Zadeh [6, 7]. Goguen generalized fuzzy sets to L-fuzzy sets in [8]. An

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L-fuzzy set is a set with a function into a partially ordered set (hereafter called a poset). This poset is denoted by L and called the fuzzy set an L-fuzzy set or an L-set. L- fuzzy binary relation R is defined on a set X which is a function from X to L [9].

In 1979, Katsaras defined fuzzy proximity spaces and proved some results that hold for ordinary proximity spaces. He studied the proximity of fuzzy sets [10]. Also some other articles in this concepts please see [11]. In 2017, Öztürk et al introduced fuzzy proximal relator spaces and defined fuzzy proximity relation to evaluate the proximity of the sets [12].

In this paper, we define the L-proximity axioms that need to be satisfied by a L-relation on proximal relator spaces. Also, L-proximity of the sets is introduced on the power set of a nonempty set. We show that the spatial Simirnov proximity measure can be generalized for L-measure and it is a Lodato L-proximity relation. L-proximity relation approach can be used to solve a few issues solution of classification problems and given examples for the problems.

2. Preliminaries

Let us take a binary relation δ defined on the power set of a nonempty set X. For all $A, B, C \in \mathcal{P}(X)$, the following axioms are considered:

(A_0) $\varnothing \not\delta A$, $\forall A \in X$.

(A_1) $A \;\delta\; B \Rightarrow B \;\delta\; A$.

(A_2) $(A \cup B) \;\delta\; C \Leftrightarrow A \;\delta\; C$ or $B \;\delta\; C$.

(A_2^*) $(A \cup B) \;\delta\; C \Leftrightarrow A \;\delta\; C$ or $B \;\delta\; C$ and $A \;\delta\; (B \cup C) \Leftrightarrow A \;\delta\; B$ or $A \;\delta\; C$.

(A_3) $A \;\delta\; B \Rightarrow A \neq \varnothing$, $B \neq \varnothing$.

(A_4) $A \cap B \neq \varnothing \Rightarrow A \;\delta\; B$.

If it satisfies (A_0), (A_1), (A_2), (A_3) and (A_4), δ is called *a basic proximity* on X.

(A_5) If, for any $A, B \subset X$, $A \not\delta B$, there exists $C, D \subset X, C \cup D = X$ such that $A \not\delta C$ and $B \not\delta D$.

If it satisfies a basic proximity axioms and (A_5), δ is called *an Efremoviĉ proximity (EF-proximity)* on X.

(A_6) $\{x\} \;\delta\; \{y\} \Rightarrow x = y$.

If δ satisfies EF-proximity axioms and (A_6), δ is called *a seperated proximity* on X.

(A_7) $A \;\delta\; B$ and $\{b\} \;\delta\; C$ $\forall b \in B \Rightarrow A \;\delta\; C$.

If δ satisfies (A_2^*), (A_3), (A_4), and (A_7), δ is called *a Leader proximity (LE-proximity)* on X [13].

If δ satisfies LE-proximity axioms and (A_1), δ is called *a Lodato proximity (LO-proximity)* on X.

(A_7^*) $\{x\} \;\delta\; B$ and $\{b\} \;\delta\; C$ $\forall b \in B \Rightarrow \{x\} \;\delta\; C$.

If δ satisfies the basic proximity axioms and (A_6) and (A_7^*), δ is called *an S-proximity* on X.

And so, the pair (X, δ) is called a basic proximity (EF-proximity, separated proximity, Leader proximity, Lodato proximity, S proximity, respectively) space.

There are some other forms of proximity relations such as *Wallman proximity, Čech proximity, quasi proximity, paraproximity, pseudo-proximity and local proximity* [2].

Let X be a set and I the unit interval. A fuzzy set in X is an element of the set of I^X of all functions μ from X into I. A binary relation δ on I^X is called a fuzzy proximity if δ satisfies the following axioms:

(FP_1) $\mu\ \delta\ \rho$ implies $\rho\ \delta\mu$,
(FP_2) $(\mu\ \vee\rho)\,\delta\ \sigma$ iff $\mu\ \delta\sigma$ or $\rho\ \delta\sigma$
(FP_3) $\mu\ \delta\ \rho$ implies $\delta\neq 0$ and $\rho\neq 0$,
(FP_4) $\mu\ \bar{\delta}\ \rho$ implies that there exists a $\rho\in I^X$ such that $\mu\ \bar{\delta}\ \rho$ and $(1-\sigma)\,\bar{\delta}\rho$,
(FP_5) $\mu\wedge\rho\neq 0$ implies $\mu\ \delta\rho$.
The pair (X,δ) is called a fuzzy proximity space [10].

A proximity measure is a measure of the closeness of a pair of nonempty sets. Notice that a proximity measure is not a distance metric but instead a proximity measure is a set inclusion measure, *i.e.*, a measure of the degree that one set is included in another set. Let A,B be a nonempty subsets in a proximity space X. The *Smirnov proximity measure* $\delta(A,B)\in\{0,1\}$ is defined by

$$\delta\left(A,B\right)=\begin{cases}1, & \text{if} \quad A \text{ is close to } B,\\ 0, & \text{if} \quad A \text{ is far from } B.\end{cases}$$

There are two forms of proximity spaces: spatial and descriptive proximity. A set X endowed with descriptive proximity relation δ_Φ, that satisfies descriptive extensions of Efremoviĉ's axioms.

The set Φ contains probe functions $\varphi:X\to\mathbb{R}$ that represent features of non abstract points which have a location and features that can be measured [14]. ($\varphi\left(x\right)$ is a feature value of x in X). The description of x is defined by a feature vector $\Phi\left(a\right)$. $\mathcal{Q}\left(A\right)$ denotes the set of descriptions of points in $A\subset X$; $\mathcal{Q}\left(A\right)=\{\Phi\left(a\right):a\in A\}$. The descriptive intersection $A_\Phi\cap B$ is the set of all points with descriptions in both $\mathcal{Q}\left(A\right)$,$\mathcal{Q}\left(B\right)$. The closure of A (denoted by clA) is the set of all points near A. A is descriptively near B (denoted by $A\ \delta_\Phi B$), provided $clA\underset{\Phi}{\cap}clB\neq\varnothing$. The pair (X,δ_Φ) is called *a descriptive proximity space* [5].

Definition 2.1. Let X be a nonempty set. A *Lodato proximity* δ is a relation on $\mathscr{P}(X)$ which satisfies the following axioms for all subsets A,B,C of X:

(dP0): $\varnothing\ \not\delta_\Phi\ A, \forall A\subset X$.
(dP1): $A\ \delta_\Phi\ B\Leftrightarrow B\ \delta_\Phi\ A$.
(dP2): $A\underset{\Phi}{\cap}B\neq\varnothing\Rightarrow\ A\ \delta_\Phi\ B$.
(dP3): $A\ \delta_\Phi\ (B\cup C)\Leftrightarrow A\ \delta_\Phi\ B$ or $A\ \delta_\Phi\ C$.
(dP4): $A\ \delta_\Phi\ B$ and $\{b\}\ \delta_\Phi\ C$ for each $b\in B\ \Rightarrow A\ \delta_\Phi\ C$.

Let $\mathcal{R}_\delta$ be family of proximity relations on X, then $(X,\mathcal{R}_\delta)$ is a proximal relator space [5]. To be clear, we can consider a few proximity relations on relator space such as the Efremoviĉ proximity δ, the descriptive proximity δ_Φ, LE-proximity, LO-proximity and so on.

In general, $A\mathcal{R}B$ means that A is proximal to B according to at least one of the relations in relator $\mathcal{R}$. For example, let $(X,\mathcal{R})$ be a proximal relator space such that $\mathcal{R}=\{\delta,\delta_\Phi\}$. If $A\mathcal{R}B$, $A,B\subseteq X$ then $A\delta B$ or $A\delta_\Phi B$.

Peters redefined the Smirnov proximity measure for a relator space [15]. Let $\varepsilon>0$, $\upsilon\left(A,B\right)=\frac{|A\cap B|}{|X|}$. $\delta_{\varepsilon,\nu}\left(A,B\right)\in[0,1]$ is called spatial Smirnov proximity measure and is defined by

$$\delta_{\varepsilon,\nu}\left(A,B\right)=\begin{cases}\frac{|A\cap B|}{|X|} & ,\text{ if} \quad \varepsilon<\upsilon\left(A,B\right)\leq 1,\\ 0 & ,\text{ if} \quad \upsilon\left(A,B\right)\leq\varepsilon.\end{cases}$$

Furthermore, Peters gave the definition of descriptive part for the extended Smirnov proximity measure [15]. Let $\varepsilon_\Phi > 0$, $\upsilon(A,B) = \frac{|\Phi(A)\cap\Phi(B)|}{|\Phi(X)|}$. $\delta_{\varepsilon_\Phi,\nu}(A,B) \in [0,1]$ is called descriptive Smirnov proximity measure and defined by

$$\delta_{\varepsilon_\Phi,\nu}(A,B) = \begin{cases} \frac{|\Phi(A)\cap\Phi(B)|}{|\Phi(X)|} & \text{, if } \ \varepsilon_\Phi < \upsilon(A,B) \le 1, \\ 0 & \text{, if } \ \upsilon(A,B) \le \varepsilon_\Phi. \end{cases}$$

A partially ordered set is a $(L,\le)$, where L is a nonempty set and $\le$ is a partial order on L. For any subset X of L and $x \in X$, x is called a lower bound (upper bound) of X if $x \le a$ ($a \le x$ respectively) for all $a \in X$. A poset $(L,\le)$ is called a lattice if every nonempty finite subset of L has greatest lower bound (or infimum) and least upper bound (or supremum) in L. If $(L,\le)$ is a lattice and, for any $a,b \in L$ if we define $a \wedge b = infimum\{a,b\}$ and $a \vee b = supremum\{a,b\}$ then $\wedge$ and $\vee$ are binary operations on L which are commutative, associative and idempotent and satisfy the absorption laws $a \wedge (a \vee b) = a = a \vee (a \wedge b)$. Conversely, any algebraic system $(L,\wedge,\vee)$ satisfying the above properties becomes a lattice in which the partial order is defined by $a \le b \Leftrightarrow a = a \wedge b = b$.

A lattice $(L,\wedge,\vee)$ is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a,b,c \in L$ (equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\forall a,b,c \in L$). A lattice $(L,\wedge,\vee)$ is called a bounded lattice it has the smallest element 0 and largest element 1; there are elements 0 and 1 in L, $0 \le x \le 1$ $\forall x \in L$. A partially ordered set in which every subset has infimum and supremum is called a complete lattice. Two elements a,b of a bounded lattice $(L,\wedge,\vee,0,1)$ are complements if $a \wedge b = 0$, $a \vee b = 1$. A lattice in which each element has at a least one complement is called a complemented lattice. If each element of a lattice has precisely one complement, such a lattice is called uniquely complemented.

Let L be a lattice. Then for all $a,b,c,d \in L$,

(1) $a \le b \Rightarrow a \vee c \le b \vee c$ and $a \wedge c \le b \wedge c$,
(2) $a \le b$ and $c \le d \Rightarrow a \vee c \le b \vee d$ and $a \wedge c \le b \wedge d$.

That is, $\wedge$ and $\vee$ operations are binary operations and both are monotone with respect to the order. Let $(L,\le)$ be partially ordered set. An element $a \in L$ is called an atom if it covers some minimal element of L. A poset L is called atomic if for every element $b \in L$ that is not minimal has an atom a such that $a \le b$ [16].

Throughout this paper, we use L for a complete lattice with the smallest element 0_L and the greatest element 1_L. Furthermore, we removed the word "fuzzy" in all the phrases "L-fuzzy....". Put differently, we write L-relation, L-proximity, L-subset, etc..

An L-binary relation on a set X is a map $\mathcal{R}_L : X \times X \to L$. The set of all L-binary relations on X, denoted by $\mathcal{R}_L^2(X)$, is a poset with $\le$ such that $\mathcal{R}_L, \mathcal{S}_L \in \mathcal{R}_L^2(X)$, $\mathcal{R}_L \le \mathcal{S}_L$, iff $\mathcal{R}_L(x) \le \mathcal{S}_L(x)$ $\forall x \in X \times X$.

Let $\mathcal{R}_L$ be an L-binary relation on a set X. Then,

(1) $\mathcal{R}_L$ is called L- *reflexive relation* iff $\mathcal{R}_L \ne 0_L$ and $\mathcal{R}_L(x,x) \ge \mathcal{R}_L(y,z)$ $\forall x,y,z \in X$.
(2) $\mathcal{R}_L$ is called L-*irreflexive relation* iff $\mathcal{R}_L(x,x) = 0_L$ $\forall x \in X$.
(3) $\mathcal{R}_L$ is called L- *symmetric relation* iff $\mathcal{R}_L(x,y) = \mathcal{R}_L(y,x)$ $\forall x,y \in X$.
(4) $\mathcal{R}_L$ is called L-*antisymmetric relation* iff $\mathcal{R}_L(x,y) = \mathcal{R}_L(y,x)$ implies $x = y$, $\mathcal{R}_L(x,y) > 0_L$ $\forall x,y \in X$.
(5) $\mathcal{R}_L$ is called L-*transitive relation* iff $\mathcal{R}_L(x,z) \ge \mathcal{R}_L(x,y) \wedge \mathcal{R}_L(y,z)$ for all $x,y,z \in X$.

(6) $\mathcal{R}_L$ is called *L-equivalence relation* iff $\mathcal{R}_L$ is an L-reflexive, L-symmetric and L-transitive.

(7) $\mathcal{R}_L$ is called *L-partial order* iff $\mathcal{R}_L$ is an L-reflexive, L-antisymmetric and L-transitive [9].

L-relations are represented in the form of two dimensional tables, i.e, $|X| = n$. A $n \times n$ matrix presents the L-relation $\mathcal{R}_L$:

$$\mathcal{R}_L = \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \overset{\begin{array}{ccc} x_1 & \cdots & x_n \end{array}}{\begin{bmatrix} \mathcal{R}_L(x_1, x_1) & \cdots & \mathcal{R}_L(x_1, x_n) \\ \vdots & \ddots & \vdots \\ \mathcal{R}_L(x_n, x_1) & \cdots & \mathcal{R}_L(x_n, x_n) \end{bmatrix}}.$$

3. L-Proximal Relator Spaces

Definition 3.1. Let $(X, \mathcal{R})$ be a proximal relator space and L be a lattice,

$$\begin{array}{llll} \mu_{\mathcal{R}_L}: & \mathcal{P}(X) \times \mathcal{P}(X) & \longrightarrow L \\ & (A, B) & \longmapsto \mu_{\mathcal{R}_L}(A, B) \end{array}$$

be a L-relation and A ,$B \subset X$; then

$$\mathcal{R}_{\mu_L} = \{((A, B), \mu_{\mathcal{R}_L}(A, B) \ ) \mid (A, B) \in \mathcal{P}(X) \times \mathcal{P}(X)\}$$

is called a L-proximity relation, provided it satisfies the following axioms:

For all $A, B, C \in P(X)$,

$N_{\mu_{\mathcal{R}_L}}1)$**:** $\mu_{\mathcal{R}_L}(A, \varnothing) = 0_L$.
$N_{\mu_{\mathcal{R}_L}}2)$**:** $\mu_{\mathcal{R}_L}(A, B) = \mu_{\mathcal{R}_L}(B, A)$.
$N_{\mu_{\mathcal{R}_L}}3)$**:** $\mu_{\mathcal{R}_L}(A, B) \neq 0_L$ implies $A \ \mathcal{R}_L \ B$.
$N_{\mu_{\mathcal{R}_L}}4)$**:** $\mu_{\mathcal{R}_L}(A, (B \cup C)) \neq 0_L$ implies $\mu_{\mathcal{R}_L}(A, B) \neq 0_L$ and $A \ \mathcal{R}_L \ B$ or $\mu_{\mathcal{R}_L}(A, C) \neq 0_L$ and $A \ \mathcal{R}_L \ C$.

The set of all L-proximity relations on $\mathcal{P}(X)$ is denoted by $\mathcal{P}_{\mathcal{R}_L}(X)$. $\mathcal{P}_{\mathcal{R}_L}(X)$ is a poset with "$\leq$" given by for $\mathcal{R}_{\mu_{L_1}}, \mathcal{R}_{\mu_{L_2}} \in \mathcal{P}_{\mathcal{R}_L}(X)$, $\mathcal{R}_{\mu_{L_1}} \leq \mathcal{R}_{\mu_{L_2}}$ iff $\mathcal{R}_{\mu_{L_1}}(A, B) \leq \mathcal{R}_{\mu_{L_2}}(A, B) \ \forall (A, B) \in \mathcal{P}(X) \times \mathcal{P}(X)$. Therefore, $\mu_{\mathcal{R}_L}(A, B)$ is called L-proximity measure. L-proximity relations are also shown in the form of the two dimensional tables such that $|\mathcal{P}(X)| = n$ and $n \times n$ relational matrix:

$$\mathcal{R}_{\mu_L} = \begin{array}{c} A_1 \\ \vdots \\ A_n \end{array} \overset{\begin{array}{ccc} A_1 & \cdots & A_n \end{array}}{\begin{bmatrix} \mu_{\mathcal{R}_L}(A_1, A_1) & \cdots & \mu_{\mathcal{R}_L}(A_1, A_n) \\ \vdots & \ddots & \vdots \\ \mu_{\mathcal{R}_L}(A_n, A_1) & \cdots & \mu_{\mathcal{R}_L}(A_n, A_n) \end{bmatrix}}$$

L-proximity measure $\mu_{\mathcal{R}_L}(A, B)$ is used the meaning of the sets A and B how near(L-proximal) to each other.

Let take two elements $\mu_{\mathcal{R}_\mathcal{L}}$, $\mu_{\mathcal{S}_L}$ of the set of all L-proximity relations $\mathcal{P}_{\mathcal{R}_L}(X)$. Since (A, B) can be equal to (C, D) and from axiom $(N_{\mu_{\mathcal{R}_L}}1)$ in 3.1, we would not define such a complementary L-proximity relations.

Definition 3.2. Let (X, δ) be a proximity space. (X, δ_{μ_L}) is called spatial L-proximity space, provided it satisfies the following axioms:
For all $A, B, C \subset X$,

$N_{\mu_L}1)$**:** $\mu_{\delta_L}(A, \varnothing) = 0_L$.
$N_{\mu_{\delta_L}}2)$**:** $\mu_{\delta_L}(A, B) = \mu_{\delta_L}(B, A)$.
$N_{\mu_{\delta_L}}3)$**:** $\mu_{\delta_L}(A, B) \neq 0_L$ implies $A\delta_L B$.
$N_{\mu_{\delta_L}}4)$**:** $\mu_{\delta_L}(A, (B \cup C)) \neq 0_L$ implies $\mu_{\delta_L}(A, B) \neq 0_L$ and $A\delta_L B$ or $\mu_{\delta_L}(A, C) \neq 0_L$ and $A\delta_L C$.

If it satisfies spatial L-proximity axioms and the following axiom $(N_{\mu_{\delta_L}}5)$, a L-relation μ_{δ_L} is called *a spatial Lodato L-proximity* relation.

$N_{\mu_{\delta_L}}5)$**:** $\mu_{\delta_L}(A, B) \neq 0_L$ and $\mu_{\delta_L}(b, C) \neq 0_L$ for all $b \in B$ implies $\mu_{\delta_L}(A, C) \neq 0_L$ and $A\delta_L C$.

Definition 3.3. Let $\mathcal{R}_{\mu_L}$ be an L-proximity relation on $\mathcal{P}(X)$. Then $(X, \mathcal{R}_{\mu_L})$ is called a L-proximal relator space.

Example 3.4. $(X, \mathcal{R})$ be relator space, L be lattice and $X = \{p, r, s\}$ be a set. $\mu_{\mathcal{R}_L}$ and L is defined by

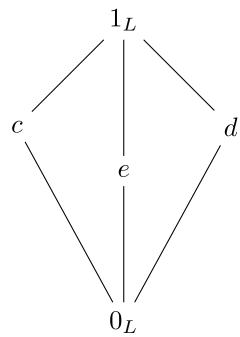

FIGURE 1. The diamond lattice M_3

and

$$\begin{array}{llll} \mu_{\mathcal{R}_L} & : \mathcal{P}(X) \times \mathcal{P}(X) & \longrightarrow & L \\ & (A, B) & \longmapsto & \mu_{\mathcal{R}_L}(A, B) \end{array}$$

with

$$\mu_{\mathcal{R}_L}(A, B) = \begin{cases} a, & A \cap B \neq \varnothing,\ A \neq B \\ 1_L, & A = B, A, B \neq \varnothing \\ 0_L, & A \cap B = \varnothing. \end{cases}$$

$$\mathcal{P}(X) = \{\varnothing, X, \{p\}, \{r\}, \{s\}, \{p, r\}, \{p, s\}, \{r, s\}\}.$$

Let take $A_1 = \varnothing$, $A_2 = X$, $A_3 = \{p\}$, $A_4 = \{r\}$, $A_5 = \{s\}$, $A_6 = \{p,r\}$, $A_7 = \{p,s\}$, $A_8 = \{r,s\}$.

$$\mathcal{R}\mu_L = \begin{array}{c} \\ A_1\\A_2\\A_3\\A_4\\A_5\\A_6\\A_7\\A_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \end{array} \\ \begin{bmatrix} 0_L & 0_L & 0_L & 0_L & 0_L & 0_L & 0_L & 0_L \\ 0_L & 1_L & a & a & a & a & a & a \\ 0_L & a & 1_L & 0_L & 0_L & a & a & 0_L \\ 0_L & a & 0_L & 1_L & 0_L & a & 0_L & a \\ 0_L & a & 0_L & 0_L & 1_L & 0_L & a & a \\ 0_L & a & a & a & 0_L & 1_L & a & a \\ 0_L & a & a & 0_L & a & a & 1_L & a \\ 0_L & a & 0_L & a & a & a & a & 1_L \end{bmatrix} \end{array}$$

It is easily seen that $\mu_{\mathcal{R}_L}$ satisfies the axioms $(N_{\mu_{\mathcal{R}_L}}1) - (N_{\mu_{\mathcal{R}_L}}4)$. Hence $(X, \mathcal{R}_{\mu_L})$ is a basic L-proximity space.

Example 3.5. Let $X = \{a,b,c,d,e,f,g,h,i\}$, $L = [0,1]$ and (X,δ) be proximity space such that basic proximity δ is defined as

$$A\ \delta\ B :\Leftrightarrow A \cap B \neq \varnothing.$$

Let $A = \{a,b,c,d,g\}$ and $B = \{d,e,f,g,h,i\}$ be subsets of X. We can show that $A\ \delta\ B$ since $A \cap B \neq \varnothing$. The L-proximity relation can be defined

$$\begin{array}{rll} \mu_{\delta_L}: & \mathcal{P}(X) \times \mathcal{P}(X) & \longrightarrow [0,1] \\ & (A,B) & \longmapsto \mu_{\delta_L}(A,B) = \frac{|A \cap B|}{|A \backslash B|} \end{array}$$

it is easily follows that

$$\begin{array}{llll} \mu_{\delta_L}(A,B) & = \frac{|A\cap B|}{|A\backslash B|} & = \frac{2}{3} & \approxeq 0.66, \\ \mu_{\delta_L}(B,A) & = \frac{|B\cap A|}{|B\backslash A|} & = \frac{2}{4} & = 0.5. \end{array}$$

Hence,

A is 0.66 L-proximal (near) to B ($A\ \delta_{0.66}\ B$),
B is 0.5 L-proximal (near) to A ($B\ \delta_{0.5}\ A$).

Since μ_{δ_L} is not symmetric, i.e, $\mu_{\delta_L}(A,B) \neq \mu_{\delta_L}(B,A)$, $(X, \mathcal{R}_{\mu_L})$ is not a L-proximity space.

Example 3.6. Let $X = \{o,p,r,s,t,v,x,y,z\}$, $L = [0,1]$ and (X,δ) be proximity space such that basic proximity δ is defined as

$$A\ \delta\ B :\Leftrightarrow A \cap B \neq \varnothing.$$

Let $A = \{o,p,r,s,t\}$, $B = \{s,t,v,x,y\}$, $C = \{t,v,x,y,z\}$, $D = \{o,p,s,t,v\}$ be subsets of X.

We can show that $A\ \delta\ B$, $B\ \delta\ C$, $A\ \delta\ C$, $A\ \delta\ D$, $B\ \delta\ D$ and $C\ \delta\ D$ since $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$, $A \cap C \neq \varnothing$, $A \cap D \neq \varnothing$, $B \cap D \neq \varnothing$ and $C \cap D \neq \varnothing$.

The L-proximity relation can be defined

$$\begin{array}{rll} \mu_{\delta_L}: & \mathcal{P}(X) \times \mathcal{P}(X) & \longrightarrow [0,1] \\ & (A,B) & \longmapsto \mu_{\delta_L}(A,B) = \frac{|A \backslash B|}{|A \cup B|}. \end{array}$$

It is easily follows that

$$\begin{aligned}
\mu_{\delta_L}(A,B) &= \tfrac{|A\setminus B|}{|A\cup B|} = \tfrac{3}{8} = 0.375\\
\mu_{\delta_L}(B,C) &= \tfrac{|B\setminus C|}{|B\cup C|} = \tfrac{1}{6} \approxeq 0.16\\
\mu_{\delta_L}(A,C) &= \tfrac{|A\setminus C|}{|A\cup C|} = \tfrac{4}{9} \approxeq 0.444\\
\mu_{\delta_L}(A,D) &= \tfrac{|A\setminus D|}{|A\cup D|} = \tfrac{1}{6} \approxeq 0.166\\
\mu_{\delta_L}(B,D) &= \tfrac{|B\setminus D|}{|B\cup D|} = \tfrac{2}{7} \approxeq 0.28\\
\mu_{\delta_L}(C,D) &= \tfrac{|C\setminus D|}{|C\cup D|} = \tfrac{3}{8} = 0.375
\end{aligned}$$

The L-proximity can be presented in this way:

$$\delta_{\mu_L} = \begin{bmatrix} 1 & 0.375 & 0.444 & 0.166\\ 0.375 & 1 & 0.16 & 0.28\\ 0.444 & 0.16 & 1 & 0.375\\ 0.166 & 0.28 & 0.375 & 1 \end{bmatrix}.$$

It is easily seen that μ_{δ_L} satisfies the axioms $(N_{\mu_{\delta_L}}1) - (N_{\mu_{\delta_L}}4)$. Hence $(X, \mathcal{R}_{\mu_L})$ is a basic L-proximity space.

Let $\delta_{1L}, \delta_{2L}, \delta_{3L}$ denote a trio of L-proximity relations and let $\mathcal{R}_{\mu_L} = \{\delta_{L_1}, \delta_{L_1}, \delta_{L_1}\}$. With the introduction of a family of L-proximity relations on X, we obtain a L-proximal relator space $(X, \mathcal{R}_{\mu_L})$ (or $X(\mathcal{R}_{\mu_L})$)[5].

For simplicity, it is considered only three L-proximity relations, namely, the Lodato L-proximity [17–19](denoted by δ_L), the descriptive Lodato proximity δ_L [20](denoted by $\delta_{L\Phi}$), an extension of the Lodato proximity, and the descriptive Lodato L-proximity [15] (denoted by $\mu_{\delta_{L\Phi}}$). Then a *L-proximal relator* $\mathcal{R}_{\mu_{\delta_{L\Phi}}}$ is defined by

$$\mathcal{R}_{\mu_{\delta_{L\Phi}}} = \{\delta_L, \delta_{L\Phi}, \mu_{\delta_{L\Phi}}\}.$$

Definition 3.7. Let (X, δ_Φ) be a Lodato proximity space. A L-relation μ_{δ_L} is called a descriptive Lodato L-proximity relation, provided it satisfies the following axioms:
For all $A, B, C \subset X$,

$N_{\mu_{\delta_{L\Phi}}}1)$: $\mu_{\delta_{L\Phi}}(A, \varnothing) = 0_L$ for all $A, B \neq \varnothing$.

$N_{\mu_{\delta_{L\Phi}}}2)$: $\mu_{\delta_{L\Phi}}(A, B) = \mu_{\delta_{L\Phi}}(B, A)$.

$N_{\mu_{\delta_{L\Phi}}}3)$: $\mu_{\delta_{L\Phi}}(A, B) \neq 0_L$ implies $A\ \delta_{L\Phi}\ B$.

$N_{\mu_{\delta_{L\Phi}}}4)$: $\mu_{\delta_{L\Phi}}(A, (B \cup C)) \neq 0_L$ implies $\mu_{\delta_{L\Phi}}(A, B) \neq 0_L$ and $A\ \delta_{L\Phi}\ B$ or $\mu_{\delta_{L\Phi}}(A, C) \neq 0_L$ and $A\ \delta_{L\Phi}\ C$.

$N_{\mu_{\delta_{L\Phi}}}5)$: $\mu_{\delta_{L\Phi}}(A, B) \neq 0_L$ and $\mu_{\delta_{L\Phi}}(b, C) \neq 0_L$ for all $b \in B$ implies $\mu_{\delta_{L\Phi}}(A, C) \neq 0_L$ and $A\ \delta_{L\Phi}\ C$.

Definition 3.8. Let δ_Φ be a descriptive Lodato L-proximity relation. Then, $(X, \delta_{\mu_{L\Phi}})$ is called descriptive Lodato L-proximity space.

Spatial Smirnov proximity measure can be generalized to spatial Smirnov proximity L-measure.

Definition 3.9. Let $\varepsilon_L \in L$ and $\varepsilon_L > 0_L$, $\upsilon(A, B) = \frac{|A\cap B|}{|X|}$. $\delta_{\varepsilon_L,\nu}(A, B) \in L$ is called spatial Smirnov proximity L-measure and defined by

$$\delta_{\varepsilon_L,\nu}(A,B) = \begin{cases} \frac{|A\cap B|}{|X|} & \text{, if } \quad \varepsilon_L < \upsilon(A,B) \leq 1_L,\\ 0_L & \text{, if } \quad \upsilon(A,B) \leq \varepsilon_L. \end{cases}$$

Theorem 3.10. *Let $(X, \mathcal{R}, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. The Smirnov proximity measure $\delta_{\varepsilon_L,\nu}$ on the Lodato proximity space is a Lodato L-proximity relation.*

Proof. We prove that $\delta_{\varepsilon_L,\nu}$ satisfies the axioms:

$N_{\mu_{\mathcal{R}_L}}.1)$ $\delta_{\varepsilon_L,\upsilon}(A,\varnothing) = 0_L$. Since $\upsilon(A,\varnothing) \leq \varepsilon_L \Rightarrow \upsilon(A,\varnothing) = \frac{|A\cap\varnothing|}{|X|} = \frac{0_L}{|X|} = 0_L$.

$N_{\mu_{\mathcal{R}_L}}.2)$ We will make evaluations for two situation:

i) Let $\delta_{\varepsilon_L,\upsilon}(A,B) = 0_L$
$$\begin{aligned} &\Rightarrow \upsilon(A,B) \leq \varepsilon_L \\ &\Rightarrow \tfrac{|A\cap B|}{|X|} \leq \varepsilon_L \\ &\Rightarrow \tfrac{|B\cap A|}{|X|} \leq \varepsilon_L \\ &\Rightarrow \upsilon(B,A) \leq \varepsilon_L \\ &\Rightarrow \delta_{\varepsilon_L,\upsilon}(B,A) = 0_L. \end{aligned}$$

Hence, $\delta_{\varepsilon_L,\upsilon}(A,B) = \delta_{\varepsilon_L,\upsilon}(B,A)$.

ii) Let $\delta_{\varepsilon_L,\upsilon}(A,B) \neq 0_L$
$$\begin{aligned} &\Rightarrow \varepsilon_L < \upsilon(A,B) \leq 1_L \\ &\Rightarrow \varepsilon_L < \upsilon(B,A) \leq 1_L \\ &\Rightarrow \delta_{\varepsilon_L,\upsilon}(B,A) \neq 0_L. \end{aligned}$$

Thus, we get $\delta_{\varepsilon_L,\upsilon}(A,B) = \delta_{\varepsilon_L,\upsilon}(B,A)$.

$N_{\mu_{\mathcal{R}_L}}.3)$
$\delta_{\varepsilon_L,\upsilon}(A,B) \neq 0_L \Rightarrow$
$\varepsilon_L < \upsilon(A,B) \leq 1_L \Rightarrow \varepsilon_L < \frac{|A\cap B|}{|X|} \leq 1_L \Rightarrow A\ \delta_L\ B$.

$N_{\mu_{\mathcal{R}_L}}.4)$
$$\begin{aligned} &\delta_{\varepsilon_L,\upsilon}(A,(B\cup C)) \neq 0_L \\ &\Rightarrow \varepsilon_L < \upsilon(A,(B\cup C)) \leq 1_L \\ &\Rightarrow \varepsilon_L < \tfrac{|A\cap(B\cup C)|}{|X|} \leq 1_L \\ &\Rightarrow \varepsilon_L < \tfrac{|(A\cap B)\cup(A\cap C)|}{|X|} \leq 1_L \\ &\Rightarrow \varepsilon_L < \tfrac{|(A\cap B)|}{|X|} \leq 1_L \text{ or } \varepsilon_L < \tfrac{|(A\cap C)|}{|X|} \leq 1_L \\ &\Rightarrow \delta_{\varepsilon_L,\upsilon}(A,B) \neq 0_L \text{ and } A\ \delta_L\ B \text{ or} \\ &\quad\ \delta_{\varepsilon_L,\upsilon}(A,C) \neq 0_L \text{ and } A\ \delta_L\ C. \end{aligned}$$

$N_{\mu_{\mathcal{R}_L}}.5)$

Let $\delta_{\varepsilon_L,\upsilon}(A,B) \neq 0_L$ and $\delta_{\varepsilon_L,\upsilon}(b,C) \neq 0_L$, for all $b \in B$.
Then, $\varepsilon_L < \upsilon(A,B) \leq 1_L$ and $\varepsilon_L < \upsilon(b,C) \leq 1_L$,
for all $b \in B$,
$\Rightarrow \varepsilon_L < \frac{|A\cap B|}{|X|} \leq 1_L$ and $\varepsilon_L < \frac{|\{b\}\cap C|}{|X|} \leq 1_L$, for all $b \in B$,
$\Rightarrow \varepsilon_L < \frac{|(A\cap C)|}{|X|} \leq 1_L$
$\Rightarrow \delta_{\varepsilon_L,\upsilon}(A,C) \neq 0_L$ and $A\ \delta_L\ C$.

Hence the Smirnov proximity measure $\delta_{\varepsilon_L,\nu}$ is a Lodato L-proximity relation. ∎

Definition 3.11. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. The extended Smirnov proximity measure has a descriptive counterpart. Let $\varepsilon_L \in L$ and $\varepsilon_L > 0_L$, $\upsilon(A,B) = \frac{|\Phi(A)\cap\Phi(B)|}{|\Phi(X)|}$. $\delta_{\varepsilon_{L\Phi},\nu}(A,B) \in L$ is called descriptive Smirnov proximity L-measure and defined by

$$\delta_{\varepsilon_{L\Phi},\nu}(A,B) = \begin{cases} \frac{|\Phi(A)\cap\Phi(B)|}{|\Phi(X)|} & \text{, if } \varepsilon_L\Phi < \upsilon(A,B) \leq 1_L, \\ 0_L & \text{, if } \upsilon(A,B) \leq \varepsilon_{L\Phi}. \end{cases}$$

Theorem 3.12. *Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. The descriptive Smirnov proximity measure $\delta_{\varepsilon_{L\Phi},\nu}$ on the descriptive Lodato proximity space is a descriptive Lodato L-proximity relation.*

Proof. The proof is the similar with the proof of Theorem 3.10. ∎

Definition 3.13. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. Then,

$$m(\mathcal{R}_{\mu_L}) = \bigvee_{A,B\in\mathcal{P}(X)} \mathcal{R}_{\mu_L}(A,B)$$

is called the supremum of L-proximity relation $\mathcal{R}_{\mu_L}$. $m(\mathcal{R}_{\mu_L})$ gives the supremum proximity grade in the family of sets.

Example 3.14. From Example 3.4, $m(\mathcal{R}_{\mu_L}) = 1_L$.

Definition 3.15. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. Then we get a relationship of two sets $A, B \in \mathcal{P}(X)$ that proximal to each other with a grade larger than or equal to $\varepsilon_L \in L, \varepsilon_L \neq 1_L$, that is, $\mu_{\mathcal{R}_L}(A,B) \geq \varepsilon_L$, as ε_L-proximal.

Definition 3.16. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. Then,

$$\mathcal{C}_{\varepsilon_L} = \{A \in \mathcal{P}(X) \,|\, \mu_{\mathcal{R}_L}(A,B) \geq \varepsilon_L, B \in \mathcal{P}(X), \varepsilon_L \in L, \varepsilon_L \neq 1_L\}$$

is called set of ε_L-proximal sets.

We can easily obtain the set of L-proximal sets which are L-proximal to all non-empty sets with a grade larger than or equal to $\varepsilon_L \in L, \varepsilon_L \neq 1_L$. Hence, we can classify the subsets of X using some $\varepsilon_L \in L, \varepsilon_L \neq 1_L$. If we consider empty set $B = \varnothing$, then $\mathcal{C}_{\varepsilon_L} = \varnothing$.

Definition 3.17. Let $\mathcal{R}_{\mu_L}$ be an L-proximity relation on $\mathcal{P}(X)$.

(1) $\mathcal{R}_{\mu_L}$ is called *reflexive L- proximity relation* if $\mathcal{R}_{\mu_L} \neq 0_L$ and $\mathcal{R}_{\mu_L}(A,A) \geq \mathcal{R}_{\mu_L}(B,C)$ for all $A, B, C \in \mathcal{P}(X)$.
The set of all reflexive L-proximity relation on $\mathcal{P}(X)$ is denoted by $\mathcal{R}^r_{\mu_L}$.

(2) $\mathcal{R}_{\mu_L}$ is called *irreflexive L-proximity relation* if $\mathcal{R}_{\mu_L}(A,A) = 0_L$ for all $A \in \mathcal{P}(X)$.
The set of all irreflexive L-proximity relation on $\mathcal{P}(X)$ is denoted by $\mathcal{R}^i_{\mu_L}$.

(3) $\mathcal{R}_{\mu_L}$ is called *symmetric L-proximity relation* if $\mathcal{R}_{\mu_L}(A,B) = \mathcal{R}_{\mu_L}(B,A)$ $\forall A, B \in \mathcal{P}(X)$.
The set of all symmetric L-proximity relation on $\mathcal{P}(X)$ is denoted by $\mathcal{R}^s_{\mu_L}$.

(4) $\mathcal{R}_{\mu_L}$ is called *transitive L-proximity relation* if $\mathcal{R}_{\mu_L}(A,C) \geq \mathcal{R}_{\mu_L}(A,B) \wedge \mathcal{R}_{\mu_L}(B,C)$ for all $A, B, C \in \mathcal{P}(X)$.
The set of all transitive L-proximity relation on $\mathcal{P}(X)$ is denoted by $\mathcal{R}^t_{\mu_L}$.

(5) $\mathcal{R}_{\mu_L}$ is called *equivalence L-proximity relation* if $\mathcal{R}_{\mu_L}$ is an reflexive L-proximity, symmetric L-proximity and transitive L-proximity.

The set of all equivalence L-proximity relation on $\mathcal{P}(X)$ is denoted by $\mathcal{R}^e_{\mu_L}$.

Example 3.18. Let δ_{μ_L} be a L-proximity relation given as the Example 3.6:

$$\delta_{\mu_L} = \begin{bmatrix} 1 & 0.375 & 0.55 & 0.1 \\ 0.375 & 1 & 0.16 & 0.28 \\ 0.55 & 0.16 & 1 & 0.375 \\ 0.1 & 0.28 & 0.375 & 1 \end{bmatrix}.$$

It is easily that δ_{μ_L} is a reflexive and symmetric L-proximity relation. Since $\delta_{\mu_L}(A,A) \geq \delta_{\mu_L}(B,C)$ for all $A,B,C \in \mathcal{P}(X)$.

Example 3.19. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space.
Then, $X = \{a,b,c\}$, $L = \{0_L, \alpha_1, \alpha_2, 1_L | 0_L < \alpha_1 < \alpha_2 < 1_L\}$ and
$\mu_{\mathcal{R}_L} = \{((\{a\},\{a\}),0_L),((\{a\},\{c\}),\alpha_1),((\{a,b\},\{a,b\}),0_L),((\{a,b\},\{a,c\}),\alpha_2)\}$.
Then, $\mu_{\mathcal{R}_L}$ is an irreflexive and transitive L-proximity relation.

Example 3.20. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space. Then, $X = \{x,y\}$, $L = \{0_L, \beta, 1_L | 0_L < \beta < 1_L\}$ and

$$\begin{aligned} \mu_{\mathcal{R}_{\mathcal{L}}} = \Big\{ & ((\{x\},\{x\}),1_L),((\{x\},\{y\}),\beta),((x,\{\{x\},\{y\}\}),0_L), \\ & ((\{y\},\{x\}),0_L),((\{y\},\{y\}),1_L),((\{y\},\{x,y\}),0_L), \\ & ((\{x,y\},\{x\}),0_L),((\{x,y\},\{y\}),0_L),((\{x,y\},\{x,y\}),0_L) \Big\}. \end{aligned}$$

Then, $\mu_{\mathcal{R}_L}$ is a reflexive and transitive L-proximity relation.

Let $\mathcal{R}_{\mu_L}$ be an L-proximity relation on $\mathcal{P}(X)$ and $\mathcal{R}_L$ be a binary relation on $\mathcal{P}(X)$.

(1) $\mathcal{R}^r_{\mu_L}$ is a sub poset of $\mathcal{P}_{\mathcal{R}_L}(X)$.
(2) $\mathcal{R}^i_{\mu_L}$ is a sub poset of $\mathcal{P}_{\mathcal{R}_L}(X)$.
(3) $\mathcal{R}^s_{\mu_L}$ is a sub poset of $\mathcal{P}_{\mathcal{R}_L}(X)$.
(4) $\mathcal{R}^t_{\mu_L}$ is a sub poset of $\mathcal{P}_{\mathcal{R}_L}(X)$.
(5) $\mathcal{R}^e_{\mu_L}$ is a sub poset of $\mathcal{P}_{\mathcal{R}_L}(X)$.

Proposition 3.21. *Let $\mathcal{R}_{\mu_L}$ be a L-proximity relation.*

$$\begin{array}{llll} \chi_{R_L}: & \mathcal{P}(X) \times \mathcal{P}(X) & \longrightarrow & L \\ & (A,B) & \longmapsto & \chi_{R_L}(A,B) \end{array}$$

is defined as

$$\chi_{R_L}(A,B) = \begin{cases} 1_L, & \text{if } (A,B) \in \mathcal{R}_{\mu_L}, \\ 0_L, & \text{if } (A,B) \notin \mathcal{R}_{\mu_L}. \end{cases}$$

Hence, the followings are true.

(1) $\mathcal{R}_{\mu_L}$ is a reflexive relation iff χ_{R_L} is a reflexive L-proximity relation.
(2) $\mathcal{R}_{\mu_L}$ is an irreflexive relation iff χ_{R_L} is an irreflexive L-proximity relation.
(3) Let $\mathcal{R}_{\mu_L}$ is a L-proximity relation iff χ_{R_L} is a symmetric L-proximity relation.

(4) $\mathcal{R}_{\mu_L}$ is a transitive relation iff χ_{R_L} is a transitive L-proximity relation.

Proof. (1) ($\Rightarrow$) Let $\mathcal{R}_{\mu_L}$ is a reflexive relation. Hence, it satisfies $(A,A) \in \mathcal{R}_{\mu_L}$ for each $A \in \mathcal{P}(X)$. χ_{R_L} must satisfy the condition $\chi_{R_L}(A,A) \geq \chi_{R_L}(B,C)$ to be reflexive L-proximity relation for all $A,B,C \in \mathcal{P}(X)$.
i) If $(A,A) \in \mathcal{R}_{\mu_L}$ and $(B,C) \in \mathcal{R}_{\mu_L} \Rightarrow 1_L \geq 1_L$.
ii)If $(A,A) \in \mathcal{R}_{\mu_L}$ and $(B,C) \notin \mathcal{R}_{\mu_L} \Rightarrow 1_L \geq 0_L$.

In these two situations, χ_{R_L} is a reflexive L-proximity relation.
($\Leftarrow$) Let χ_{R_L} is a reflexive L-proximity relation. $\chi_{R_{\mathcal{L}}}$ satisfy the condition $\chi_{R_L}(A,A) \geq \chi_{R_L}(B,C)$ for all $A,B,C \in \mathcal{P}(X)$. This inequality is possible for three situations.
i) $1_L \geq 1_L \Rightarrow \chi_{R_{\mathcal{L}}}(A,A) \geq \chi_{R_L}(B,C) \Rightarrow (A,A) \in \mathcal{R}_{\mu_L} \forall A \in \mathcal{P}(X)$.
ii) $1_L \geq 0_L \Rightarrow \chi_{R_L}(A,A) \geq \chi_{R_L}(B,C) \Rightarrow (A,A) \in \mathcal{R}_{\mu_L} \forall A \in \mathcal{P}(X)$.
iii) $0_L \geq 0_L$. This is not true since χ_{R_L} is a reflexive L-proximity relation and L is not a empty set.
Other cases can be proved similarly from definitions. ∎

Proposition 3.22. *Let $\mathcal{R}_{\mu_L}$ be an L-proximity relation on $\mathcal{P}(X)$, and L be a complete lattice. Take any family of L-proximity binary relation $\mathcal{R}_{\mu_{i_L}}$. Then, the followings are true.*

(1) If $\mathcal{R}_{\mu_{i_L}}$ is an L-reflexive relation for every $i \in I$, then $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}$ is a reflexive L-proximity relation, whenever it is not the L-empty set 0_L or whenever L is unique atomic.
(2) If $\mathcal{R}_{\mu_{i_L}}$ is an L-irreflexive relation for every $i \in I$, then $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}$ is an irreflexive L-proximity relation.
(3) Let $\mathcal{R}_{\mu_{i_L}}$ be an L-proximity relation for every $i \in I$, then $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}$ is a symmetric L-proximity relation.
(4) If $\mathcal{R}_{\mu_{i_L}}$ is an L-transitive relation for every $i \in I$, then $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}$ is a transitive L-proximity relation.
(5) If $\mathcal{R}_{\mu_{i_L}}$ is an L-equivalence relation for every $i \in I$, then $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}$ is an equivalence L-proximity relation, whenever it is not the L-empty set 0_L or whenever L is unique atomic.

Proof. (1) Let $\mathcal{R}_{\mu_{i_L}}$ be a L-reflexive relation for every $i \in I$ and L is not empty set 0_L or L is unique atomic. $\wedge$ operation is binary operation and monotone with respective to the order.
$\mathcal{R}_{\mu_{i_L}}$ be a L-reflexive relation for every $i \in I$. Then, it satisfies the condition $\mathcal{R}_{\mu_{i_L}}(A,A) \geq \mathcal{R}_{\mu_{i_L}}(B,C)$ for each $i \in I$ and for all $A,B,C \in \mathcal{P}(X)$. Since $\wedge$ monotone with respective to the order, it can be written $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}(A,A) \geq \bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}(B,C)$.
Hence, $\bigwedge_{i\in I} \mathcal{R}_{\mu_{i_L}}$ is a reflexive L-proximity relation.
Other cases can be proved similarly. ∎

In the Proposition 3.22 for (1) to be reflexive L-proximity relation, L may not be empty set as giving the following example:

Example 3.23. Let $(X, \mathcal{R}_{\mu_L})$ be a L-proximal relator space.
Then, $X = \{x, y\}$, $L = \{0_L, \beta_1, \beta_2, 1_L | 0_L < \beta_1, \beta_2 < 1_L; \beta_1 || \beta_2\}$ and

$$\mathcal{R}_{1L} = \{((\{x\},\{x\}),\beta_1),((\{y\},\{y\}),\beta_2),((\{x\},\{y\}),0_L),((\{y\},\{x\}),0_L),((\{x,y\},\{x,y\}),0_L)\}.$$
$$\mathcal{R}_{2L} = \{((\{x\},\{x\}),\beta_2),((\{y\},\{y\}),\beta_1),((\{x\},\{y\}),0_L),((\{y\},\{x\}),0_L),((\{x,y\},\{x,y\}),0_L)\}.$$

Then, $\mathcal{R}_{1L}$ and $\mathcal{R}_{2L}$ are reflexive L-proximity relations, but $\mathcal{R}_{1\mathcal{L}} \wedge \mathcal{R}_{2L}$ is not a reflexive L-proximity relation since it is the the L-empty set 0_L.

Proposition 3.24. *Let $\mathcal{R}_{\mu_L}$ be an L-proximity relation on $\mathcal{P}(X)$, and L be a complete lattice. Take any family of L-proximity binary relation $\mathcal{R}_{\mu_{i_L}}$. Then, the followings are true.*

(1) If $\mathcal{R}_{\mu_{i_L}}$ is an L-reflexive relation for every $i \in I$, then $\bigvee_{i \in I} \mathcal{R}_{\mu_{i_L}}$ is an reflexive L-proximity relation.
(2) If $\mathcal{R}_{\mu_{i_L}}$ is an L-irreflexive relation for every $i \in I$, then $\bigvee_{i \in I} \mathcal{R}_{\mu_{i_L}}$ is an irreflexive L-proximity relation.
(3) Let $\mathcal{R}_{\mu_{i_L}}$ be an L-proximity relation for every $i \in I$, then $\bigvee_{i \in I} \mathcal{R}_{\mu_{i_L}}$ is an symmetric L-proximity relation.

Proof. (1) Let $\mathcal{R}_{\mu_{i_L}}$ be a L-reflexive relation for every $i \in I$. $\vee$ operation is binary operation and monotone with respective to the order. $\mathcal{R}_{\mu_{i_L}}$ be a L-reflexive relation for every $i \in I$. Then, it satisfies the condition $\mathcal{R}_{\mu_{i_L}}(A, A) \geq \mathcal{R}_{\mu_{i_L}}(B, C)$ for each $i \in I$ and for all $A, B, C \in \mathcal{P}(X)$. Since $\vee$ monotone with respective to the order, it can be written $\bigvee_{i \in I} \mathcal{R}_{\mu_{i_L}}(A, A) \geq \bigvee_{i \in I} \mathcal{R}_{\mu_{i_L}}(B, C)$.

Hence, $\bigvee_{i \in I} \mathcal{R}_{\mu_{i_L}}$ is a reflexive L-proximity relation.

Other cases can be proved similarly. ∎

4. Conclusions

As a result of this paper, we established a new approach lattices and we introduced L-proximal spaces. We defined the L-proximity axioms. Also, we showed that the spatial Simirnov proximity measure can be generalized for L-measure and it is a Lodato L-proximity relation. L-proximity relation approach contributes to solve some problems in application such as classification problems.

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