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Existence of the Relatively Compact Fundamental Domain for Hypergroups

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Abstract In this paper, we prove that every locally compact hypergroup has a relatively compact Borel fundamental domain corresponding to its lattice, and give some examples and applications.

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1. Introduction

Hypergroups, as extensions of locally compact groups, were introduced in a series of papers by C. F. Dunkl, R. I. Jewett and R. Spector in the 70's decade. The results on hypergroups, in addition to the group case, include so many structures in harmonic analysis such as double coset spaces and polynomial hypergroups (for more details and examples see [1, 2]). In the last decade, the theory of frame and wavelet has been extended in harmonic analysis on locally compact groups. A key tool in this field is the *fundamental domain*. If H is a closed subgroup of a locally compact group G, a Borel subset V of G is called a fundamental domain if it intersects each left coset of H in exactly one point. Recently, applying the fundamental domain, some sufficient conditions for a class of functions to form a Bessel sequence or a frame have been derived in [3]. In [4], the first and third authors have studied the theory of frames on locally compact hypergroups (see also [5]), and here we intend to show the existence of fundamental domain for hypergroups; see Definition 2.6. This could be used in studying some basic notions such as Zak transform on hypergroups.

Let X be a locally compact Hausdorff space. We denote by $\mathcal{M}(X)$ the space of all regular complex Borel measures on X and by δ_x the Dirac measure at a point x in X. The support of a measure μ in $\mathcal{M}(X)$ is denoted by $\mathrm{supp}(\mu)$. Throughout this paper, K is a locally compact hypergroup with the convolution $* : \mathcal{M}(K) \times \mathcal{M}(K) \to \mathcal{M}(K)$,

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involution $x \mapsto x^-$ from K onto K, and the identity element e. For definition and basic properties of hypergroups we refer to the Jewett's paper [2], in which hypergroup is called *convo*. The dual of K is denoted by $\hat{K}$. In spite of the group case, convolution of two Dirac measure is not necessarily a Dirac measure. Also, $\hat{K}$ is not necessarily a hypergroup. If $\hat{K}$ with the complex conjugation as involution and poinwise product, i.e.

$$\xi(x)\eta(x) = \int_{\hat{K}} \chi(x) d(\delta_\xi * \delta_\eta)(\chi), \qquad (x \in K \text{ and } \xi, \eta \in \hat{K}),$$

as convolution is a hypergroup, then K is called a strong hypergroup.

If A and B are subsets of K, then we denote

$$A * B := \bigcup_{\substack{x \in A \\ y \in B}} \operatorname{supp}(\delta_x * \delta_y),$$

and $x * A := \{x\} * A$ for each $x \in K$.

Let H be a closed subhypergroup of K. Then, $K/H := \{x * H : x \in K\}$ equipped with the quotient topology is a locally compact space, and the natural projection $q : K \to K/H$, $x \mapsto x * H$, is continuous and open. In general, K/H is not a hypergroup. By [6, Proposition 1.8], if a normal subhypergroup H of K is of compact type, then K/H is a hypergroup.

2. Main Results

In this section, first we initiate the concept of uniform lattice for hypergroups and then by some technical lemmas we prove that each lattice has a fundamental domain that is relatively compact and its corresponding section mapping is Borel measurable.

Definition 2.1. A *(uniform) lattice* in a locally compact commutative hypergroup K is a discrete subhypergroup L of K for which K/L is compact.

Example 2.2. Let $\mathbb{Z}_+^*$ be the one-point compactification of $\mathbb{Z}_+ := \{0, 1, 2, ...\}$, and p be a prime number. A convolution, with ∞ as the identity, is defined on $\mathcal{M}(\mathbb{Z}_+^*)$ by

$$\delta_m * \delta_n := \begin{cases} \delta_{\min\{n,m\}}, & n \neq m, \\ \frac{p-2}{p-1}\delta_n + \sum_{k=1}^{\infty} \frac{1}{p^k}\delta_{k+n}, & n = m, \end{cases}$$

where $m, n \in \mathbb{Z}_+$. Then, $\mathbb{Z}_+^*$ is a Hermitian hypergroup. Also, we have $\widehat{\mathbb{Z}_+^*} = \{\chi_n : n = 0, 1, 2, ...\}$, where $\chi_n(m) := 1$ for all $m \geq n$ or $m = \infty$, $\chi_n(m) := \frac{-1}{p-1}$ for $m = n - 1$, and $\chi_n(m) := 0$ for all $m \leq n - 2$.

$\widehat{\mathbb{Z}_+^*} \cong \mathbb{Z}_+$ is again a Hermitian hypergroup, with 0 as the identity, and the convolution defined by

$$\delta_n * \delta_m := \begin{cases} \delta_{\max\{n,m\}}, & n \neq m, \\ \frac{1}{p^{n-1}(p-1)}\delta_0 + \sum_{k=1}^{n-1} p^{k-n}\delta_k + \frac{p-2}{p-1}\delta_n, & n = m. \end{cases}$$

For more details see [7]. For each $n \in \mathbb{N}$, $L_n := \{n, n+1, n+2, ...\}$ is a discrete subhypergroup of $\mathbb{Z}_+^*$, and the qoutient space $\mathbb{Z}_+^*/L_n = \{\{1\}, \{2\}, ..., \{n-1\}, L_n\}$ is finite, and so it is compact. Then, for each $n \in \mathbb{N}$, L_n is a lattice for $\mathbb{Z}_+^*$.

Also, every (discrete) subhypergroup of $\widehat{\mathbb{Z}}_+^* \cong \mathbb{Z}_+$ is in the form $K_n := \{0, 1, 2, ..., n\}$, for some $n \in \mathbb{N}$. Since $\mathbb{Z}_+/K_n = \{K_n, n+1, n+2, ...\}$ is infinite, $\mathbb{Z}_+/K_n$ is not compact. Hence, the hypergroup $\mathbb{Z}_+$ has no lattice.

Remark 2.3. Let $\alpha_0 \in K^*$ be a positive semicharacter on the commutative hypergroup $(K, *)$. Then,

$$\mu \bullet \nu := \alpha_0((\alpha_0^{-1}\mu) * (\alpha_0^{-1}\nu)) \qquad (\mu, \nu \in \mathcal{M}_c(K))$$

extends uniquely to a bilinear convolution on $\mathcal{M}(K)$, and $(K, \bullet)$ becomes a commutative hypergroup with the identity and involution of $(K, *)$ [8]. The hypergroup $(K, \bullet)$ is called *deformation* of $(K, *)$ corresponding to α_0.

Now, let L be a lattice for $(K, *)$. We show that L is a lattice for $(K, \bullet)$ too. For any $x, y \in L$ we have

$$(\delta_x \bullet \delta_y)(f) = \alpha_0^{-1}(x)\alpha_0^{-1}(y) \int_K f(t)\alpha_0(t) d(\delta_x * \delta_y)(t).$$

This shows that $\{x\} \bullet \{y\} = \{x\} * \{y\} \subseteq L$, and so L is a subhypergroup of $(K, \bullet)$ too. On the other hand, for each $x \in K$, $\{x\} \bullet L = \bigcup_{y \in L}\{x\} \bullet \{y\} = \bigcup_{y \in L}\{x\} * \{y\} = \{x\} * L$, and hence $(K/L, \bullet) = (K/L, *)$. This implies that L is a lattice for $(K, \bullet)$.

In the following lemma we use the notation $W^n := W * \cdots * W$ (n times), where $n \in \mathbb{N}$ and $W \subseteq K$.

Lemma 2.4. *Let K be a locally compact hypergroup, and W be a compact symmetric neighborhood of e in K. Then $H := \bigcup_{n=1}^{\infty} W^n$, is an open, closed and σ-compact subhypergroup of K. In particular, if K is connected, then $K = H$ and every compact subset of K is in some W^n.*

Proof. The first and second parts are proved in [9, Lemma 1.5 and Theorem 1.9]. Let K be connected, E be a compact subset of K, and U is an open subset of K such that $e \in U \subseteq W$. For each $n \in \mathbb{N}$ we put $V_n := U * W^n$. Then $W^n \subseteq V_n \subseteq W^{n+1}$, and so $E \subseteq \bigcup_{n=1}^{\infty} V_n$. Since every V_n is open, then there exists $m \in \mathbb{N}$ such that $E \subseteq \bigcup_{n=1}^{m} V_n \subseteq \bigcup_{n=1}^{m} W^{n+1}$ and therefore $E \subseteq W^{m+1}$. ∎

Lemma 2.5. *Let K be a non-connected separable locally compact hypergroup. Then,*

(i) K/H is countable, if H is an open subhypergroup of K, and

(ii) there exists a countable family $W_1 \subseteq W_2 \subseteq W_3 \cdots$ of compact subsets of K such that every compact subset of K is contained in some W_j.

Proof. (i) By [2, 10.3A], the family of all cosets of H in K is a partition of K. Because of separablity of K, there is a set $A := \{t_1, t_2, ...\} \subseteq K$ such that $\overline{A} = K$. Thus for every $x \in K$ we have $(\{x\} * H) \cap A \neq \varnothing$, since $\{x\} * H$ is open and non-empty [2, 4.1D]. Put $m_x := \min\{k : t_k \in \{x\} * H\}$, and define $\varphi : K/H \to \mathbb{N}$ by $\varphi(\{x\} * H) := m_x$. Then φ is one-to-one, and K/H is countable.

(ii) Let V be an open relatively compact neighborhood of e, and U be an open symmetric neighborhood of e such that $W := \overline{U} \subseteq V$. Put $H := \bigcup_{n=1}^{\infty} W^n$. By Lemma 2.4, H is an open subhypergroup of K, and by part (i) we can suppose that $\{x_i\} * H$'s are distinct elements of K/H, where $i = 1, 2, \cdots$. Then for each compact subset E of K there is some m such that $E \subseteq \bigcup_{i=1}^{m}(\{x_i\} * H)$, and so $E = \bigcup_{i=1}^{m}(E \bigcap(\{x_i\} * H))$. Since H (and so $\{x_i\} * H$) is closed, $E_i := E \cap (\{x_i\} * H)$ is compact, where $1 \leq i \leq m$. We have $E_i \subseteq \bigcup_{j=1}^{\infty}(\{x_i\} * W^j) \subseteq \bigcup_{j=1}^{\infty}(\{x_i\} * V^j)$, and so there exists $p \in \mathbb{N}$ such that $E_i \subseteq \bigcup_{j=1}^{p}(\{x_i\} * V^j) \subseteq \bigcup_{j=1}^{p}(\{x_i\} * \overline{V}^j)$. Let $\Gamma := \{\{x_i\} * \overline{V}^j : i, j = 1, 2, \cdots\}$. Therefore, every element of Γ is a compact subset of K, and for each compact subset E of K there are $\Gamma_1, \Gamma_2, \cdots, \Gamma_j \in \Gamma$ such that $E \subseteq \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_j$. Now, if we put $W_j := \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_j$, then $\{W_j\}_{j=1}^{\infty}$ has the required property. ∎

Definition 2.6. Let H be a subhypergroup of K. A Borel subset V of K is called a *fundamental domain* for H if V intersects each coset of H in exactly one point.

Remark 2.7. If K is an abelian locally compact group and H is a subgroup of K, one can easily see that a Borel set $V \subseteq K$ is a fundamental domain for H if and only if $K = \bigcup_{x \in H} xV$ and for each distinct $x, y \in H$, xV and yV are disjoint. Therefore, the above concept is equivalent with the usual definition of fundamental domain in the group case.

Lemma 2.8. *Let K be a separable locally compact hypergroup, H be a subhypergroup of K and $q : K \to K/H$ be the quotient mapping. Then there exists a fundamental domain $B \subseteq K$ such that for each compact subset E of K, $q^{-1}(q(E)) \bigcap B$ is relatively compact.*

Proof. By Lemmas 2.4 and 2.5, there exists a countable family $W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots$ of compact subsets of K such that every compact subset E of K is contained in some W_j. By [10, Theorem 5.1], for each j there exists a Borel subset $A_j \subseteq W_j$ such that $q(A_j) = q(W_j)$ and q is one-to-one on A_j. Let some Borel sets $B_1, B_2, ..., B_j$ have been chosen such that $B_1 := A_1$, $B_1 \subseteq B_2 \subseteq ... \subseteq B_j \subseteq K$, $q(B_i) = q(W_i)$ and q be one-to-one on B_i, for all $1 \le i \le j$. Then by [11, Theorem 15.1], $B_{j+1} := (A_{j+1} - q^{-1}(q(A_j))) \bigcup B_j$ is a Borel set, and $B := \bigcup_{j=1}^{\infty} B_j$ satisfies the required properties. To see this, let $x \in K$. Then for some j we have $\{x\} \subseteq W_j$, and since $q(W_j) = q(B_j)$, there is $b \in B_j$ such that $q(x) = q(b)$ i.e. $\{x\} * H = \{b\} * H$. Therefore $b \in B \cap (\{x\} * H)$. If in addition $a \in B \cap (\{x\} * H)$, then $q(a) = q(b)$. For some j we have $a, b \in B_j$ because $\{B_j\}_{j=1}^{\infty}$ is increasing. Since q is one-to-one on B_j, we have $a = b$. Thus B intersects each coset of H in exactly one point. For every $j \in \{1, 2, \ldots\}$, we have $B_j \subseteq (W_1 \cup \ldots \cup W_j)$. So, every B_j is relatively compact, and this completes the proof. ∎

Remark 2.9. In the above lemma, if H is an open subhypergroup of K (for example one can consider the open subhypergroup introduced in Lemma 2.4), then every B_j is finite. For this, let $j \in \{1, 2, \ldots\}$, and for each $x \in B_j$ suppose that $U_x := \{x\} * H$. Then every U_x (and so $q(U_x)$) is open, $q(B_j) \subseteq \bigcup_{x \in B_j} q(U_x)$ and by continuity of q, $q(B_j) = q(W_j)$ is compact. So, for some $x_1, \ldots, x_n \in B_j$ we have

$$q(B_j) \subseteq \bigcup_{i=1}^{n} q(U_{x_i}) = \{q(x_1), \ldots, q(x_n)\} \subseteq q(B_j),$$

and then

$$B_j = q^{-1}(q(B_j)) \cap B = q^{-1}(\{q(x_1), \ldots, q(x_n)\}) \cap B = \{x_1, \ldots, x_n\}.$$

Theorem 2.10. *Let K be a separable locally compact hypergroup and L be a uniform lattice of K. Then, there exists a relatively compact fundamental domain for L.*

Proof. By Lemma 2.8, there exists a Borel subset V of K such that V intersects each coset of L in exactly one point and for each compact subset E of K, $q^{-1}(q(E)) \bigcap V$ is relatively compact. We show that V is relatively compact. For each $x \in K$, we suppose that V_x is an open relatively compact neighborhood of x. Thus $V \subseteq \bigcup_{x\in V} V_x$ and $K/L = q(V) \subseteq \bigcup_{x\in V} q(V_x)$. Since K/L is compact, for some $x_1, \ldots, x_n \in V$ we have $K/L = q(V) = q(V_{x_1}) \cup q(V_{x_2}) \cup \ldots \cup q(V_{x_n})$. Thus

$$V \subseteq q^{-1}(q(V)) \subseteq \bigcup_{i=1}^{n} q^{-1}(q(V_{x_i})) \subseteq \bigcup_{i=1}^{n} q^{-1}(q(\overline{V_{x_i}})),$$

and so

$$\overline{V} \subseteq \overline{\bigcup_{i=1}^{n} (V \cap q^{-1}(q(\overline{V_{x_i}}))} = \bigcup_{i=1}^{n} \overline{(V \cap q^{-1}(q(\overline{V_{x_i}})))}.$$

Therefore, V is relatively compact. ■

Corollary 2.11. *Every uniform lattice of a separable locally compact group has a Borel relatively compact fundamental domain.*

Definition 2.12. If H is a subhypergroup of K, then

$$H^{\perp} := \{\xi \in \hat{K} : \xi(x) = 1 \text{ for all } x \in H\}$$

is called the *annihilator* of H in $\hat{K}$.

If K is a commutative hypergroup, then $H^{\perp}$ is closed in $\hat{K}$, and if K is a strong hypergroup, then $H^{\perp}$ is a subhypergroup of $\hat{K}$ (see [1, 2.2.45]).

Corollary 2.13. *Let K be a second countable locally compact commutative strong hypergroup and L be a lattice of K. Then there exists a relatively compact Borel set $U \subseteq \hat{K}$ such that U intersects each coset of $L^{\perp}$ in exactly one point*

Proof. Since K/L is compact and L is a discrete, $\widehat{K/L} \cong L^{\perp}$ is discrete and $\hat{L} \cong \hat{K}/L^{\perp}$ is compact. Therefore, $L^{\perp}$ is a lattice of $\hat{K}$. Finally, $\hat{K}$ is separable and existence of the requested set U follows from Theorem 2.10. ■

By [6], if $m_{K/H}$ is a (left) Haar measure on K/H, we have a (left) Haar measure m on K such that the Weil's formula holds:

$$\int_K f(z)dm(z) = \int_{K/H} \int_H f(x * t)dm_H(t)dm_{K/H}(x * H).$$

See [6, Proposition 1.8 and Theorem 2.3], for some sufficient conditions that H has Weil's property. As an application, by Lemma 2.10 and Weil's formula we have the following.

Corollary 2.14. *Let K be a second countable locally compact hypergroup that has Weil's property and L be a uniform lattice of K. Then*

$$\int_{K/L} f d\sigma = \frac{1}{\nu(L)} \sum_{t\in V} f(t * L)\mu(t * L) \qquad f \in C_c(K/L),$$

where V is a fundemental domain for L, and σ, μ and ν are Haar measures of K/L, K and L, respectively.

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