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Subclasses of Bi-Sakaguchi Function Associated with $q-$Difference Operator

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Abstract In this work, the authors introduce a subclass $\zeta\Sigma_q^\lambda(b,\phi)$ in the open unit disk which associated with q-difference operator and satisfy some subordination conditions. The estimates for $|a_2|$ and $|a_3|$ are obtained by making use of Taylor-Maclaurin series. Our results serve as a new generalization in this direction and still pave way for researchers to study this class by means of the Chebyshev polynomials and Sigmoid functions.

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1. Introduction

Let A denotes a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disk $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. For two analytic functions f, g such that $f(0) = g(0)$, we say that f is *subordinate* to g in u and write $f(z) \prec g(z)$, $z \in \Delta$ if there exist a Schwartz function $w(z)$ (analytic in Δ with $w(0) = 0$, and $|w(z)| \le |z|$, $z \in \Delta$) such that $f(z) = g(w(z))$ $(z \in \Delta)$. Furthermore if the function g is univalent in Δ, then we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

See details in [1].

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Ma and Minda [2] unified various subclasses of starlike and convex functions for which either of the quantities

$$\frac{zf'(z)}{f(z)} \quad and \quad \frac{zf''(z)}{f'(z)}+1, \tag{1.2}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk Δ, $\phi(0)=1$, $\phi'(0)>0$ and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. In the light of this, it is assumed that such a function has a series expansion of the form

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+... \quad (B_1>0), \tag{1.3}$$

for details check [3].

Koebe theorem in [4] shows that the image of Δ under every univalent function $f\in A$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z))=z$, $(z\in\Delta)$ and $f^{-1}(f(w))=w$ $(|w|<r_0(f), r_0(f)\geq\frac{1}{4})$. A function $f\in A$ is said to be *bi-univalent* in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions defined in the unit disk Δ. Since $f\in\Sigma$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g=f^{-1}$ has the expansion.

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3+... \tag{1.4}$$

Several researchers have introduced and investigated subclasses of bi-univalent functions and obtained the first few bounds (see [5–9]).

The q-analysis plays a vital role in complex analysis and geometric function theory. It has been used to construct and investigate several subclasses of analytic functions and their interesting results are too voluminous to discuss. Just to mention but few, it has an application in dynamical system, quantum groups, q-deformed superalgebras and so on. Jackson [10, 11] initiated the application of q-calculus and later, the geometrical interpretation of q-analysis has been recognized through the studies on quantum groups. This has led many researchers in the field of q-theory for extending all the vital results involving the classical analysis to their q-analogs. For recent work on q-calculus, see [12–16].

In this paper, we provide some basic definitions and concept details for q-calculus by the following notation and terminology.

Definition 1.1. Let $0<q<1$ and define

$$[n]_q=\frac{1-q^n}{1-q} \tag{1.5}$$

for $n\in N=\{1,2,...\}$.

Definition 1.2. The *q-derivative* of a function f, defined on a subset of $\mathbb{C}$ is given by

$$(D_qf)(z)=\begin{cases}\dfrac{f(z)-f(qz)}{(1-q)z}, & \text{for } z\neq 0\\ f'(0), & \text{for } z=0.\end{cases} \tag{1.6}$$

It is noted that $\lim_{q\to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z. Additionally, in view of (1.6), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=1}^{\infty}[n]_q a_n z^{n-1}. \tag{1.7}$$

For functions g of the form (1.4), we define

$$(D_q g)(w) = 1 - a_2[2]_q w + (2a_2^2 - a_3)[3]_q w^2 + ... \tag{1.8}$$

and introduce new subclass of bi-sakaguchi functions to obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ by Ma-Minda subordination.

2. Bi-Sakaguchi Function Class $\zeta\Sigma_q^{\lambda}(b, \phi)$

In this section, due to Sharma and Riana [17], we introduce a subclass $\zeta\Sigma_q^{\lambda}(b, \phi)$ of Σ and find coefficients $|a_2|$ and $|a_3|$ for the functions in this new subclass by subordination. Throughout our study, we let

$$\lambda \geq 0; \quad 0 < q < 1; \quad n \in N_0.$$

Definition 2.1. For $\lambda \geq 0$, a function $f \in \Sigma$ of the form (1.1) is said to be *in the class* $\zeta\Sigma_q^{\lambda}(b, \phi)$ if the following subordination hold:

$$(D_q f)(z)\left(\frac{(1-b)z}{f(z) - f(bz)}\right)^{\lambda} \prec \phi(z) \tag{2.1}$$

and

$$(D_q g)(w)\left(\frac{(1-b)w}{g(w) - g(bw)}\right)^{\lambda} \prec \phi(w) \tag{2.2}$$

where $1 \neq b \in C, |b| \leq 1, z, w \in \Delta$ and g is given by (1.8).

In order to prove our main results, we shall make use of the lemma below:

Lemma 2.1 ([18]). *If a function $p \in P$ is given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \quad (z \in \Delta) \tag{2.3}$$

then $|p_i| \leq 2 \quad (i \in N)$.

Here P is the family of functions p, analytic in Δ , for which

$$p(0) = 1; \quad \text{and} \quad Re(p(z)) > 0 \quad (z \in \Delta).$$

3. Main Results

Theorem 3.1. *Let f be given by* (1.1) *be in the class $\zeta\Sigma_q^{\lambda}(b, \phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left(\lambda[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b) - (1+b+b^2)] + [3]_q\right)B_1^2 + (B_1 - B_2)([2]_q - \lambda(1+b))^2}} \tag{3.1}$$

and

$$|a_3| \leq \frac{B_1}{[3]_q - \lambda(1+b+b^2)} + \left(\frac{B_1}{[2]_q - \lambda(1+b)}\right)^2 \tag{3.2}$$

where $\lambda \geq 0$, *for* $1 \neq b \in \mathbb{C}$ *and* $|b| \leq 1$.

Proof. Let $f \in \zeta\Sigma_q^\lambda(b,\phi)$ and $g = f^{-1}$. Then there are two analytic functions $u, v : \Delta \longrightarrow \Delta$ with $u(0) = 0 = v(0)$ satisfying

$$(D_q f)(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^\lambda = \phi(u(z)) \tag{3.3}$$

and

$$(D_q g)(w)\left(\frac{(1-b)w}{g(w)-g(bw)}\right)^\lambda = \phi(v(w)). \tag{3.4}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + ... \tag{3.5}$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + ... \tag{3.6}$$

or equivalently,

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + ...\right] \tag{3.7}$$

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + ...\right] \tag{3.8}$$

Then $p(z)$ and $q(w)$ are analytic in Δ with $p(0) = 1 = q(0)$. Since $u, v : \Delta \longrightarrow \Delta$, the functions $p(z)$ and $q(w)$ have a positive real part in Δ, $|p_i| \leq 2$ and $|q_i| \leq 2$.
Using (3.7) and (3.8) in (3.3) and (3.4) respectively, we have

$$(D_q f)(z)\left(\frac{(1-b)z}{f(z)-f(bz)}\right)^\lambda = \phi\left[\frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + ...\right)\right] \tag{3.9}$$

and

$$(D_q g)(w)\left(\frac{(1-b)w}{g(w)-g(bw)}\right)^\lambda = \phi\left[\frac{1}{2}\left(q_1 w + \left(q_2 - \frac{q_1^2}{2}\right)w^2 + ...\right)\right]. \tag{3.10}$$

In light of (1.1) - (1.8) and from (3.9) and (3.10), we have

$$1 + ([2]_q - \lambda(1+b))a_2 z + \left(\lambda\left[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b)\right]a_2^2 + ([3]_q - \lambda(1+b+b^2))a_3\right)z^2 + ...$$
$$= 1 + \frac{1}{2}B_1 p_1 z + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]z^2 + ...$$

and

$$1 + (\lambda(1+b) - [2]_q)a_2 w + \left[\left(\lambda\left[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b)\right] + 2([3]_q - \lambda(1+b+b^2))\right)a_2^2 - ([3]_q - \lambda(1+b+b^2))a_3\right]w^2 + ...$$

$$= 1 + \frac{1}{2}B_1 q_1 w + \left[\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right]w^2 + ...$$

which yields the following relations

$$([2]_q - \lambda(1+b))a_2 = \frac{1}{2}B_1 p_1 \tag{3.11}$$

$$\lambda\left[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b)\right]a_2^2 + ([3]_q - \lambda(1+b+b^2))a_3 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2 \tag{3.12}$$

$$(\lambda(1+b) - [2]_q)a_2 = \frac{1}{2}B_1 q_1 \tag{3.13}$$

and

$$\left(\lambda\left[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b)\right] + 2([3]_q - \lambda(1+b+b^2))\right)a_2^2 - ([3]_q - \lambda(1+b+b^2))a_3 = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2. \tag{3.14}$$

From (3.11) and (3.13) it follows that

$$p_1 = -q_1 \tag{3.15}$$

and

$$8([2]_q - \lambda(1+b))^2 a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{3.16}$$

From (3.12),(3.14) and (3.16), we obtain

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{4\left[\left(\lambda[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b) - (1+b+b^2)] + [3]_q\right)B_1^2 + (B_1 - B_2)([2]_q - \lambda(1+b))^2\right]} \tag{3.17}$$

Applying Lemma 2.1 to the coefficients p_2 and q_2, we have

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left(\lambda[\frac{1+\lambda}{2}(1+b)^2 - [2]_q(1+b) - (1+b+b^2)] + [3]_q\right)B_1^2 + (B_1 - B_2)([2]_q - \lambda(1+b))^2}} \tag{3.18}$$

By subtracting (3.14) from (3.12) and using (3.15) and (3.16), we get

$$a_3 = \frac{B_1^2(p_1^2+q_1^2)}{8([2]_q - \lambda(1+b))^2} + \frac{B_1(p_2 - q_2)}{4([3]_q - \lambda(1+b+b^2))}$$

Applying Lemma 2.1 once again to the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{B_1}{[3]_q - \lambda(1+b+b^2)} + \left(\frac{B_1}{[2]_q - \lambda(1+b)}\right)^2 \tag{3.19}$$

Setting $b = 0$ we have ■

Corollary 3.2. *Let f be given by* (1.1) *be in the class $\zeta\Sigma_q^\lambda(b,\phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left[\lambda\left(\frac{1+\lambda}{2} - [2]_q - 1\right) + [3]_q\right]B_1^2 + (B_1 - B_2)([2]_q - \lambda)^2}} \tag{3.20}$$

and

$$|a_3| \leq \frac{B_1}{[3]_q - \lambda} + \left(\frac{B_1}{[2]_q - \lambda}\right)^2. \tag{3.21}$$

Putting $\lambda = 0$ in corollary 1, we obtain

Corollary 3.3. *Let f be given by* (1.1) *be in the class $\zeta\Sigma_q^{\lambda}(b,\phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{[3]_q B_1^2 + (B_1 - B_2)[2]_q^2} \tag{3.22}$$

and

$$|a_3| \leq \frac{B_1}{[3]_q} + \frac{B_1^2}{[2]_q^2}. \tag{3.23}$$

Also taking $\lambda = 1$ in corollary 1, we get

Corollary 3.4. *Let f be given by* (1.1) *be in the class $S\Sigma_q^{\lambda}(b,\phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{([3]_q - [2]_q - \frac{1}{2})B_1^2 + (B_1 - B_2)([2]_q - 1)^2}} \tag{3.24}$$

and

$$|a_3| \leq \frac{B_1}{[3]_q - 1} + \left(\frac{B_1}{[2]_q - 1}\right)^2. \tag{3.25}$$

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References

[1] Z. Nehari, Conformal Mappings, McGraw-Hill, New York, 1952.

[2] W.C. Ma, D. Minda, A unified treatment of some special classes of functions, in: Proceedings of the Conference on Complex Analysis, Tianjin (1992), 157–169, Conf. Proc. Lecture Notes Anal. I. Int. Press, Cambridge, MA, 1994.

[3] G. Murugusundaramoorthy, K. Vijaya, Subclasses of Bi-univalent functions defined by Salagean type q-difference operator, (2017) arXiv:1710.00143v1.

[4] P.L. Duren, Univalent Functions, Springer-Verlag, New York, 1983.

[5] D.A. Branman, J.G. Clunie, W.E. Kirwan, Coefficient estimates for a class of starlike functions, Canad J. Math. 22 (1970) 476–485.

[6] D.A. Branman, T.S. Taha, On some classes of bi-univalent functions. Studia Univ. Babes. Bolyai Math. 31 (2) (1986) 70–77.

[7] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Anal. 2 (1) (2013) 49–60.

[8] T. Hayanu, S. Owa, Coefficient bounds for bi-univalent functions, Pan. Amer. Math. J. 22 (4) (2012) 15–26.

[9] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18 (1967) 63–68.

[10] F.H. Jackson, On q-functions and a certain difference operator, Trans. Royal soc. Edinburgh. 46 (1909) 253–281.

[11] F.H. Jackson, On q-definite integrals, Quaterly J. Pure. Appl. Math. 41 (1910) 193–203.

[12] S. Altinkaya, S. Yalcin, Estimates on coefficients of a general subclass of Bi-univalent functions associated with symmetric q-derivative operator by means of the chebyshev polynomials, Asia Pacific J. Math. 4 (2) (2017) 90–99.

[13] A. Aral, V. Gupta, R.P. Agarwal, Applications of q-Calculus in Operator Theory, Springer-Verlag, New York, 2013.

[14] B. Kamel, S. Yosr, On some symmetric q-special functions, Le Matematiche 68 (2) (2013) 107–122.

[15] N. Magesh, S. Altinkaya, S. Yalcin, Construction of Toeplitz matrices whose elements are the coefficients of univalent functions associated with q-derivative operator, (2017) arXiv:1708.03600v1.

[16] S.D. Purohit, R.K. Riana, Fractional q-calculus and certain subclasses of univalent analytic functions, Mathematica 55 (78) (2013) 62–74.

[17] P. Sharma, R.K. Riana, On a Sakaguchi type class of analytic functions associated with quasi-subordination, Commentarii Mathematici Universitatis Sancti Pauli 64 (1) (2015) 59–70.

[18] C. Pommerenke, Univalent Functions, Vandenhoeck and Rupercht, Gottingen, 1975.