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L-Fuzzy Fixed Point Theorems for L-Fuzzy Mappings Satisfying Rational Inequality

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Abstract We utilize the idea of Hausdorff metric on L-fuzzy sets to establish the existence of common L-fuzzy fixed point theorems for L-fuzzy mappings satisfying a rational expression. Our findings extend and improve many results in the literature, to support the validity of our results some examples and applications are also given.

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1. Introduction

Fuzzy set theory introduced by Russian mathematician L. A Zadeh [1] in 1965 plays an important role in solving real world problems, by making the description of vagueness and imprecision clear and more precise. Later in 1967, his student J. A Goguen [2] extended this idea by replacing the unit interval $[0,1]$ with a completely distributive lattice L to form L-fuzzy set theory.

In the late 1960s Nadler [3] established a multi-valued entension of Banach Contraction Principle [4]. Later in 1981, Heilpern [5] presented a fuzzy extension of Banach contraction principle [4] and Nadler's [3] fixed point theorems by introducing the concept of fuzzy contraction mappings and established a fixed point theorem for fuzzy contraction mappings in a complete metric linear spaces. Afterwards, several authors [6–14] among others studied and generalized the result in [5]. In 1975, Dass and Gupta [15] established an extension of the Banach Contraction Principle [4] satisfying a rational expression and derived some related results. Later, Fisher [16] presented a common fixed point result for single valued mappings satisfying rational expressions in complete metric spaces. Subsequently, many researchers ([17–22] and references therein) proposed different generalizations.

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On the other hand, Rashid et al. [23, 24] introduced the notions of d_L^∞-metric and Hausdorff distances for L-fuzzy sets, they presented some fixed point results for L-fuzzy set valued mappings and some coincidence theorems concerning a crisp mapping and a sequence of L-fuzzy mappings.

Recently, Azam [25] studied and deduced some common fuzzy fixed point results satisfying a rational inequality in a complete metric space. Following suit, in this manuscript we establish the existence of common L-fuzzy fixed point results for L-fuzzy mappings satisfying a rational expression via a Hausdorff metric on L-fuzzy sets. Our results improve and extend results in [25]. Some examples and applications are also given to support the validity of our results.

2. Preliminaries

In this section, some basic definitions and preliminary results which will used throughout this paper are recalled.

Let (X, d) be a metric space. We denote and define.

$$C(X) = \{A : A \text{ is nonempty and compact subsets of } X\}$$

$$CB(X) = \{A : A \text{ is nonempty closed and bounded subsets of } X\}.$$

Let $A, B \in CB(X)$ and define

$$d(x, A) = \inf_{y \in A} d(x, y)$$

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y).$$

The Hausdorff distance H on $CB(X)$ induced by d defined as:

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}.$$

Definition 2.1. (Zadeh [1]). A fuzzy set in X is a function with domain X and values in $[0, 1]$. i.e A is a fuzzy set if $A : X \longrightarrow [0, 1]$.

Let $\mathcal{F}(X)$ denotes the collection of all fuzzy subsets of X. If A is a fuzzy set and $x \in X$, then $A(x)$ is called the grade of membership of x in A. The α-level set of A is denoted by $[A]_\alpha$ and is defined as below:

$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}$, for $\alpha \in (0, 1]$,
$[A]_0 =$ closure of the set $\{x \in X : A(x) > 0\}$.

Definition 2.2. (Abdullahi and Azam [6]). A partially ordered set $(L, \preceq_L)$ is called

(i) a lattice; if $a \vee b \in L, a \wedge b \in L$ for any $a, b \in L$,

(ii) a complete lattice; if $\bigvee A \in L, \bigwedge A \in L$ for any $A \subseteq L$,

(iii) a distributive lattice; if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ for any } a, b, c \in L,$$

(iv) a complete distributive lattice; if $a \vee (\bigwedge b_i) = \bigwedge_i (a \wedge b_i)$,

$$a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i) \text{ for any } a, b_i \in L,$$

(v) a bounded lattice; if it is a lattice and additionally has a top element 1_L and a bottom element 0_L, which satisfy $0_L \preceq_L x \preceq_L 1_L$ for every $x \in L$.

Definition 2.3. (Abdullahi and Azam [6]). An L-fuzzy set A on a nonempty set X is a function $A : X \longrightarrow L$, where L is bounded complete distributive lattice with 1_L and 0_L.

Definition 2.4. (Goguen [2]).

Let L be a lattice, the top and bottom elements of L are 1_L and 0_L respectively, and if $a, b \in L, a \vee b = 1_L$ and $a \wedge b = 0_L$ then b is a unique complement of a denoted by $\acute{a}$.

Remark 2.5. If $L = [0, 1]$, then the L-fuzzy set reduces to fuzzy set (in the original sense of Zadeh [1]), which shows that L-fuzzy set is larger.

Let $\mathcal{F}_L(X)$ denotes the class of all L-fuzzy subsets of X. The α_L-level set of an L-fuzzy set A is denoted by A_{α_L} and define as below:
$A_{\alpha_L} = \{x \in X : \alpha_L \preceq_L A(x)\}$ for $\alpha_L \in L \backslash \{0_L\}$,
$A_{0_L} =$ closure of the set $\{x \in X : 0_L \preceq_L A(x)\}$.

Definition 2.6. An L-fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if A_{α_L} is compact and convex in V for each $\alpha_L \in L$ and $\sup A(x) = 1$.

We suppose that $\hat{T}$ is the mapping induced by an L-fuzzy mapping T. That is, for $x, y, t \in X$

$$\hat{T}(x)(t) = \{y : T(x)(y) = \max_t T(x)(t)\}.$$

Now, we define some sub-collections of $\mathcal{F}_L(X)$ and $\mathcal{F}_L(V)$.

$$\mathcal{W}_L(V) = \{A \in \mathcal{F}_L(V) : A \text{ is an approximate quantity in } V\}$$

$$K(X) = \{A \in \mathcal{F}_L(X) : \hat{A} \in C(X)\}$$

$$D_L(X) = \{A \in \mathcal{F}_L(X) : A_{\alpha_L} \in C(X), \text{ for each } \alpha_L \in L\}$$

$$E_L(X) = \{A \in \mathcal{F}_L(X) : A_{\alpha_L} \in CB(X), \text{ for each } \alpha_L \in L\}$$

$$\mathcal{D}_L(X) = \{A \in \mathcal{F}_L(X) : A_{\alpha_L} \in C(X), \text{ for some } \alpha_L \in L\}$$

$$\mathcal{E}_L(X) = \{A \in \mathcal{F}_L(X) : A_{\alpha_L} \in CB(X), \text{ for some } \alpha_L \in L\}.$$

For $A, B \in \mathcal{F}_L(X)$, $A \subset B \iff A(x) \preceq_L B(x)$ for all $x \in X$. If there exists $\alpha_L \in L \backslash \{0_L\}$ such that $A_{\alpha_L}, B_{\alpha_L} \in CB(X)$. Then, we define

$$p_{\alpha_L}(x, A) = \inf_{y \in A_{\alpha_L}} d(x, y)$$

$$p_{\alpha_L}(A, B) = \inf_{x \in A_{\alpha_L}, y \in B_{\alpha_L}} d(x, y)$$

$$D_{\alpha_L}(A, B) = H(A_{\alpha_L}, B_{\alpha_L}).$$

If $A_{\alpha_L}, B_{\alpha_L} \in CB(X)$ for each $\alpha_L \in L \backslash \{0_L\}$. Then, we define

$$p(A, B) = \sup_{\alpha_L} p_{\alpha_L}(A, B)$$

$$d_L^\infty(A, B) = \sup_{\alpha_L} D_{\alpha_L}(A, B).$$

Note that, d_L^∞ is a metric on $\mathcal{F}_L(X)$ and the completeness of (X, d) implies that $(C(X), H)$ and $(\mathcal{F}_L(X), d_L^\infty)$ are complete.

Definition 2.7. (Rashid et al. [23]). Let X be an arbitrary set, Y be a metric space. A mapping T is called L-fuzzy mapping, if T is a mapping from X to $\mathcal{F}_L(Y)$(i.e class of L-fuzzy subsets of Y). An L-fuzzy mapping T is an L-fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

For convenience, we denote the α_L-level sets of $T(x)$ by $[Tx]_{\alpha_L}$ instead of $[T(x)]_{\alpha_L}$.

Definition 2.8. Let (X, d) be a metric space and $T : X \longrightarrow \mathcal{F}_L(X)$. A point $z \in X$ is said to be an L-fuzzy fixed point of T if $z \in [Tz]_{\alpha_L}$, for some $\alpha_L \in L\backslash\{0_L\}$ (see [23, 24]). The point z is known as a fixed point of T if $T(z)(z) \geq T(z)(x)$ for all $x \in X$ (see [7]). Moreover, we say z is an Heilpern fixed point of T if $\{z\} \subset Tz$ (see [5]). If $z \in [Sz]_{\alpha_L} \cap [Tz]_{\alpha_L}$, then $z \in X$ is a common L-fuzzy fixed point of S and T.

Remark 2.9. If $\alpha_L = 1_L$, then it is called a fixed point of the L-fuzzy mapping T.

Lemma 2.10. (Nadler [3]). *Let (X, d) be a metric space and $A, B \in CB(X)$. If $a \in A$ then $d(a, B) \leq H(A, B)$.*

Lemma 2.11. (Nadler [3]). *Let (X, d) be a metric space and $A, B \in CB(X)$ and $\psi > 0$. Then, for any $a \in A$ there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \psi$.*

In the following, we give an L-fuzzy versions of lemmas due to Abu-Donia [7] and Arora and Sharma [8] respectively.

Lemma 2.12. *Let (X, d) be a metric space, $z \in X$ and $T : X \longrightarrow \mathcal{F}_L(X)$ be an L-fuzzy mapping such that $\hat{T}x \in C(X)$ for all $x \in X$. Then $z \in \hat{T}(z) \Longleftrightarrow T(z)(z) \geq T(z)(x)$ for all $x \in X$.*

Lemma 2.13. *Let (V, d) be a complete metric linear space, $x_0 \in V$ and $T : V \longrightarrow \mathcal{W}_L(V)$ be an L-fuzzy mapping. Then there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$.*

3. Main Results

In the following, the existence of a common L-fuzzy fixed point result satisfying a rational type inequality is presented.

3.1. L-Fuzzy Fixed Points of L-Fuzzy Mappings

Theorem 3.1. *Let (X, d) be a complete metric space, $S, T : X \longrightarrow \mathcal{F}_L(X)$ be L-fuzzy mappings and for $x \in X$, there exists $\alpha_{L_{S(x)}}, \alpha_{L_{T(x)}} \in L\backslash\{0_L\}$ such that $[Sx]_{\alpha_{L_{S(x)}}}$, $[Tx]_{\alpha_{L_{T(x)}}} \in CB(X)$. If for all $x, y \in X$*

$$H([Sx]_{\alpha_{L_{S(x)}}}, [Ty]_{\alpha_{L_{T(y)}}}) \leq \lambda_1 d(x, y) + \lambda_2 d(x, [Sx]_{\alpha_{L_{S(x)}}}) + \lambda_3 d(y, [Ty]_{\alpha_{L_{T(y)}}}) + \frac{\lambda_4 d(x, [Sx]_{\alpha_{L_{S(x)}}}) d(y, [Ty]_{\alpha_{L_{T(y)}}})}{1 + d(x, y)}, \tag{3.1}$$

and

$$\lambda_3 + \frac{\lambda_4 d(x, [Sx]_{\alpha_{L_{S(x)}}})}{1 + d(x, y)} < 1, \qquad \lambda_2 + \frac{\lambda_4 d(y, [Ty]_{\alpha_{L_{T(y)}}})}{1 + d(x, y)} < 1, \tag{3.2}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are non-negative real numbers with $\lambda_1+\lambda_2+\lambda_3+\lambda_4<1$. Then, there exists $x^* \in X$ such that $x^* \in [Sx^*]_{\alpha_{L_S(x^*)}} \cap [Tx^*]_{\alpha_{L_T(x^*)}}$.

Proof. We will consider the following 3 possible cases:

(i) $\lambda_1+\lambda_2=0$;
(ii) $\lambda_1+\lambda_3=0$;
(iii) $\lambda_1+\lambda_2\neq 0, \lambda_1+\lambda_3\neq 0$.

Case i: For $x \in X$, there exists $\alpha_{L_S(x)} \in L\backslash\{0_L\}$ such that $[Sx]_{\alpha_{L_S(x)}}$ is nonempty, closed and bounded subset of X. Let $y \in [Sx]_{\alpha_{L_S(x)}}$. Then from Lemma 2.10, we have

$$d(y,[Ty]_{\alpha_{L_T(y)}}) \leq H([Sx]_{\alpha_{L_S(x)}}, [Tx]_{\alpha_{L_T(x)}}) \tag{3.3}$$

(3.1) and (3.3) implies

$$d(y,[Ty]_{\alpha_{L_T(y)}}) \leq \lambda_1 d(x,y)+\lambda_2 d(x,[Sx]_{\alpha_{L_S(x)}})+\lambda_3 d(y,[Ty]_{\alpha_{L_T(y)}}) + \frac{\lambda_4 d(x,[Sx]_{\alpha_{L_S(x)}})d(y,[Ty]_{\alpha_{L_T(y)}})}{1+d(x,y)}.$$

Since $\lambda_1+\lambda_2=0$, we get

$$\left(1-\lambda_3-\frac{\lambda_4 d(x,[Sx]_{\alpha_{L_S(x)}})}{1+d(x,y)}\right)d(y,[Ty]_{\alpha_{L_T(y)}}) \leq 0.$$

By (3.2), we have

$$d(y,[Ty]_{\alpha_{L_T(y)}}) \leq 0.$$

Thus, $y \in [Ty]_{\alpha_{L_T(y)}}$. Applying (3.1) again, yields

$$(1-\lambda_2)d(y,[Sy]_{\alpha_{L_S(y)}}) \leq 0.$$

which implies $y \in [Sy]_{\alpha_{L_S(y)}}$. Hence, $y \in [Sy]_{\alpha_{L_S(y)}} \cap [Ty]_{\alpha_{L_T(y)}}$.

Case ii: For $x \in X$, let $y \in [Sx]_{\alpha_{L_S(x)}}$. Then, similar to case i, there exists $\alpha_{L_T(y)} \in L\backslash\{0_L\}$ such that $[Ty]_{\alpha_{L_T(y)}}$ is nonempty, closed and bounded subset of X. Let $z \in [Ty]_{\alpha_{L_T(y)}}$. Then from Lemma 2.10, (3.1) and (3.3), we have

$$\begin{aligned} d(z,[Sz]_{\alpha_{L_S(z)}}) &\leq H([Ty]_{\alpha_{L_T(y)}}, [Sz]_{\alpha_{L_S(z)}}) \\ &\leq \lambda_1 d(z,y)+\lambda_2 d(z,[Sz]_{\alpha_{L_S(z)}})+\lambda_3 d(y,[Ty]_{\alpha_{L_T(y)}}) \\ &\quad + \frac{\lambda_4 d(z,[Sz]_{\alpha_{L_S(z)}})d(y,[Ty]_{\alpha_{L_T(y)}})}{1+d(z,y)}. \end{aligned}$$

Using $\lambda_1+\lambda_3=0$, we have

$$\left(1-\lambda_2-\frac{\lambda_4 d(y,[Ty]_{\alpha_{L_T(y)}})}{1+d(z,y)}\right)d(z,[Sz]_{\alpha_{L_S(z)}}) \leq 0.$$

Therefore $z \in [Sz]_{\alpha_{L_S(z)}}$. Using (3.1) again, will yield

$$(1-\lambda_3)d(z,[Tz]_{\alpha_{L_T(z)}}) \leq 0.$$

Thus, implying $z \in [Tz]_{\alpha_{L_T(z)}}$. Hence, $z \in [Sz]_{\alpha_{L_S(z)}} \cap [Tz]_{\alpha_{L_T(z)}}$.

Case iii: Let

$$q = \max\left\{\left(\frac{\lambda_1+\lambda_3}{1-\lambda_2-\lambda_4}\right), \left(\frac{\lambda_1+\lambda_2}{1-\lambda_3-\lambda_4}\right)\right\}.$$

Then, since $\lambda_1+\lambda_2 \neq 0, \lambda_1+\lambda_3 \neq 0$ and $\lambda_1+\lambda_2+\lambda_3+\lambda_4 < 1$, it implies that $q < 1$ and non-zero. Choose $x_0 \in X$, then there exists $\alpha_{L_S(x_0)} \in L\backslash\{0_L\}$ such that $[Sx_0]_{\alpha_{L_S(x_0)}}$ is nonempty, closed and bounded subset of X. Take $x_1 \in [Sx_0]_{\alpha_{L_S(x_0)}}$. It follows that there exists $\alpha_{L_T(x_1)} \in L\backslash\{0_L\}$ such that $[Tx_1]_{\alpha_{L_T(x_1)}}$ is nonempty, closed and bounded subset of X. Then by Lemma 2.11, there exists $x_2 \in [Tx_1]_{\alpha_{L_T(x_1)}}$ such that

$$d(x_1,x_2) \leq H([Sx_0]_{\alpha_{L_S(x_0)}}, [Tx_1]_{\alpha_{L_T(x_1)}}) + q(1-\lambda_3-\lambda_4). \tag{3.4}$$

Similarly, one can get $\alpha_{L_S(x_2)} \in L\backslash\{0_L\}$ and $x_3 \in [Sx_2]_{\alpha_{L_S(x_2)}}$ such that

$$d(x_2,x_3) \leq H([Sx_2]_{\alpha_{L_S(x_2)}}, [Tx_1]_{\alpha_{L_T(x_1)}}) + q^2(1-\lambda_2-\lambda_4). \tag{3.5}$$

Continuing in this fashion, we can construct a sequence $\{x_n\}$ in X such that, for each $n = 0,1,2,\dots$ we have

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) \leq H([Sx_{2n}]_{\alpha_{L_S(x_{2n})}}, [Tx_{2n+1}]_{\alpha_{L_T(x_{2n+1})}}) \\ + q^{2n+1}(1-\lambda_3-\lambda_4), \end{aligned}$$

and

$$\begin{aligned} d(x_{2n+2},x_{2n+3}) \leq H([Sx_{2n+2}]_{\alpha_{L_S(x_{2n+2})}}, [Tx_{2n+1}]_{\alpha_{L_T(x_{2n+1})}}) \\ + q^{2n+2}(1-\lambda_2-\lambda_4), \end{aligned}$$

where

$$x_{2n+1} \in [Sx_{2n}]_{\alpha_{L_S(x_{2n})}} \text{ and } x_{2n+2} \in [Tx_{2n+1}]_{\alpha_{L_T(x_{2n+1})}}.$$

By (3.1) and (3.4), we have

$$\begin{aligned} d(x_1,x_2) \leq \lambda_1 d(x_0,x_1) + \lambda_2 d(x_0,[Sx_0]_{\alpha_{L_S(x_0)}}) + \lambda_3 d(x_1,[Tx_1]_{\alpha_{L_T(x_1)}}) \\ + \frac{\lambda_4 d(x_0,[Sx_0]_{\alpha_{L_S(x_0)}}) d(x_1,[Tx_1]_{\alpha_{L_T(x_1)}})}{1+d(x_0,x_1)} + q(1-\lambda_3-\lambda_4). \end{aligned}$$

Which implies

$$d(x_1,x_2) \leq \left(\frac{\lambda_1+\lambda_2}{1-\lambda_3-\lambda_4}\right) d(x_0,x_1) + q.$$

Similarly by (3.1) and (3.5), we have

$$\begin{aligned} d(x_2,x_3) \leq \lambda_1 d(x_2,x_1) + \lambda_2 d(x_2,[Sx_2]_{\alpha_{L_S(x_2)}}) + \lambda_3 d(x_1,[Tx_1]_{\alpha_{L_T(x_1)}}) \\ + \frac{\lambda_4 d(x_2,[Sx_2]_{\alpha_{L_S(x_2)}}) d(x_1,[Tx_1]_{\alpha_{L_T(x_1)}})}{1+d(x_2,x_1)} + q^2(1-\lambda_2-\lambda_4). \end{aligned}$$

Thus

$$\begin{aligned} d(x_2,x_3) &\leq \left(\frac{\lambda_1+\lambda_3}{1-\lambda_2-\lambda_4}\right) d(x_1,x_2) + q^2 \\ &\leq q d(x_1,x_2) + q. \end{aligned}$$

Which further implies

$$\begin{aligned} d(x_n, x_{n+1}) &\leq q d(x_{n-1}, x_n) + q^n \\ &\leq q^2 d(x_{n-2}, x_{n-1})) + 2q^n \\ &\leq q^3 d(x_{n-3}, x_{n-2}) + 3q^n \\ &\ \ \vdots \\ &\leq q^n d(x_0, x_1) + nq^n. \end{aligned}$$

Consequently, for each $n = 0, 1, 2, \dots$ we obtain

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1) + nq^n.$$

Next, we show that $\{x_n\}$ is a Cauchy in X. Let $m > n > 0$. By triangular inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} (q^k d(x_0, x_1) + kq^k) \\ &\leq \frac{q^m}{1-q} d(x_0, x_1) + S_{n-1} - S_{m-1}, \text{ where } S_n = \sum_{k=1}^{n} kq^k. \end{aligned}$$

Since $q < 1$, by Cauchy's root test it implies that $\sum nq^n$ is convergent. Hence, it follows that $\{x_n\}$ is Cauchy sequence, and since X is complete there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.
Now, consider

$$\begin{aligned} d(x^*, [Sx^*]_{\alpha_{L_S(x^*)}}) &\leq d(x^*, x_{2n}) + d(x_{2n}, [Sx^*]_{\alpha_{L_S(x^*)}}) \\ &\leq d(x^*, x_{2n}) + H([Sx_{2n-1}]_{\alpha_{L_S(x_{n-1})}}, [Sx^*]_{\alpha_{L_S(x^*)}}) \\ &\leq \left[1 - \lambda_2 - \lambda_4\left(\frac{d(x_{2n-1}, x_{2n})}{1 + d(x^*, x_{2n-1})}\right)\right]^{-1} \\ &\qquad \left(d(x^*, x_{2n}) + \lambda_1 d(x^*, x_{2n-1}) + \lambda_3 d(x_{2n-1}, x_{2n})\right). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we have

$$d(x^*, [Sx^*]_{\alpha_{L_S(x^*)}}) \leq 0.$$

Thus, $x^* \in [Sx^*]_{\alpha_{L_S(x^*)}}$. Similarly one can show that $x^* \in [Tx^*]_{\alpha_{L_T(x^*)}}$ by using

$$d(x^*, [Tx^*]_{\alpha_{L_T(x^*)}}) \leq d(x^*, x_{2n+1}) + d(x_{2n+1}, [Tx^*]_{\alpha_{L_T(x^*)}}).$$

Hence,

$$x^* \in [Sx^*]_{\alpha_{L_S(x^*)}} \cap [Tx^*]_{\alpha_{L_T(x^*)}}.$$

■

Next, an example is given to help validate our result.

Example 3.2. Let $X = [0,1]$ and $L = \{\sigma, \beta, \gamma, \delta\}$ where $\sigma \preceq_L \beta \preceq_L \delta$, and $\sigma \preceq_L \gamma \preceq_L \delta$, such that β and γ are not comparable. Then, (X, d) is a complete metric space with the usual metric d and $(L, \preceq_L)$ is a complete distributive lattice. Let $S, T : X \longrightarrow \mathcal{F}_L(X)$ be L-fuzzy mappings such that $Sx, Tx \in \mathcal{F}_L(X)$ and be define as:

$$S(x)(t) = \begin{cases} \gamma, & \text{if } 0 \le t \le \frac{x}{30}; \\ \beta, & \text{if } \frac{x}{30} < t \le \frac{x}{25}; \\ \delta, & \text{if } \frac{x}{25} < t < \frac{x}{20}; \\ \sigma, & \text{if } \frac{x}{20} \le t \le 1. \end{cases}$$

and

$$T(x)(t) = \begin{cases} \delta, & \text{if } 0 \le t < \frac{x}{15}; \\ \gamma, & \text{if } \frac{x}{15} \le t \le \frac{x}{10}; \\ \sigma, & \text{if } \frac{x}{10} < t \le \frac{x}{5}; \\ \beta, & \text{if } \frac{x}{5} < t \le 1. \end{cases}$$

Observe that,

$$\hat{S}x = [Sx]_\delta = \left(\frac{x}{25}, \frac{x}{20}\right), \qquad \hat{T}x = [Tx]_\delta = \left[0, \frac{x}{15}\right).$$

Hence, $Sx, Tx \notin K(X)$. But, for $x \in X$ if $\alpha_{L_{S(x)}} = \alpha_{L_{T(x)}} = \gamma \in L \backslash \{0_L\}$.

$$[Sx]_\gamma = \left[0, \frac{x}{30}\right], \qquad [Tx]_\gamma = \left[0, \frac{x}{10}\right].$$

Thus, $Sx, Tx \in \mathcal{D}_L(X) \subseteq \mathcal{E}_L(X)$. Furthermore,

$$\lambda_3 + \frac{\lambda_4 d(x, [Sx]_{\alpha_{L_{S(x)}}})}{1 + d(x,y)} \le \frac{1}{15} + \frac{1}{20}\left(\frac{|x - \frac{x}{30}||y - \frac{y}{10}|}{1 + |x - y|}\right) < 1.$$

Similarly,

$$\lambda_2 + \frac{\lambda_4 d(y, [Ty]_{\alpha_{L_{T(y)}}})}{1 + d(z,y)} < 1.$$

So, whenever $x = y$ we have

$$H([Sx]_{\alpha_{L_{S(x)}}}, [Ty]_{\alpha_{L_{T(y)}}}) = 0,$$

but whenever $x \neq y$, we have

$$H([Sx]_{\alpha_{L_{S(x)}}}, [Ty]_{\alpha_{L_{T(y)}}}) \le \frac{1}{5}|x - y| + \frac{1}{10}|x - \frac{x}{30}| + \frac{1}{15}|y - \frac{y}{10}| + \frac{1}{20}\left(\frac{|x - \frac{x}{30}||y - \frac{y}{10}|}{1 + |x - y|}\right).$$

Since $[Sx]_{\alpha_{L_{S(x)}}}, [Tx]_{\alpha_{L_{T(x)}}} \notin C(X)$ for each $\alpha_L \in L$ and X is not linear, one can not apply many known results of the literature (see; [7–10]) even if $\lambda_4 = 0$. But, S and T satisfy all the hypothesis of Theorem 3.1 for $\lambda_1 = \frac{1}{5}, \lambda_2 = \frac{1}{10}$, $\lambda_3 = \frac{1}{15}$ and $\lambda_4 = \frac{1}{20}$. Hence, a common L-fuzzy fixed point for S and T exists.

Below, we obtain some common fixed points results for L-fuzzy mappings and multi-valued mappings as an application of the above L-fuzzy fixed point result (see [3, 12, 13]).

Theorem 3.3. *Let (X, d) be a complete metric space and $S, T : X \longrightarrow \mathcal{F}_L(X)$ be L-fuzzy mappings such that $\hat{S}x, \hat{T}x \in CB(X)$. If for all $x, y \in X$*

$$H(\hat{S}x, \hat{T}y) \leq \lambda_1 d(x, y) + \lambda_2 d(x, \hat{S}x) + \lambda_3 d(y, \hat{T}y) + \frac{\lambda_4 d(x, \hat{S}x) d(y, \hat{T}y)}{1 + d(x, y)},$$

and

$$\lambda_3 + \frac{\lambda_4 d(x, \hat{S}x)}{1 + d(x, y)} < 1, \qquad \lambda_2 + \frac{\lambda_4 d(y, \hat{T}y)}{1 + d(x, y)} < 1,$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$. Then, there exists $z \in X$ such that $S(z)(z) \geq S(z)(x)$ and $T(z)(z) \geq T(z)(x)$ for all $x \in X$.

Proof. For $x, t \in X$. Let

$$\max_t S(x)(t) = \lambda \text{ and } \max_t T(x)(t) = \kappa.$$

Then for every $x, y \in X$, we have

$$\hat{S}x = [Sx]_\mu \text{ and } \hat{T}x = [Tx]_\kappa.$$

Thus

$$\begin{aligned} H(\hat{S}x, \hat{T}y) &= H([Sx]_\mu, [Tx]_\kappa) \\ &\leq \lambda_1 d(x, y) + \lambda_2 d(x, [Sx]_\mu) + \lambda_3 d(y, [Ty]_\kappa) \\ &\qquad + \frac{\lambda_4 d(x, [Sx]_\mu) d(y, [Ty]_\kappa)}{1 + d(x, y)}. \end{aligned}$$

Then, from Theorem 3.1 we obtain $z \in X$ such that

$$z \in [Sz]_\mu \cap [Tz]_\kappa = \hat{S}z \cap \hat{T}z.$$

Therefore, using Lemma 2.12 we have

$$S(z)(z) \geq S(z)(x),$$

and

$$T(z)(z) \geq T(z)(x),$$

for all $x \in X$ as required. ∎

Theorem 3.4. *Let (X, d) be a complete metric space and $K, L : X \longrightarrow CB(X)$ be multi-valued mappings. If for all $x, y \in X$*

$$H(Kx, Ly) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Kx) + \lambda_3 d(y, Ly) + \frac{\lambda_4 d(x, Kx) d(y, Ly)}{1 + d(x, y)},$$

and

$$\lambda_3 + \frac{\lambda_4 d(x, Kx)}{1 + d(x, y)} < 1, \qquad \lambda_2 + \frac{\lambda_4 d(y, Ly)}{1 + d(x, y)} < 1,$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$. Then, there exists $x^* \in X$ such that $x^* \in Kx^* \cap Lx^*$.

Proof. Take a pair (arbitrary) of mappings $A, B : X \longrightarrow L\backslash\{0_L\}$ and let $S, T : X \longrightarrow \mathcal{F}_L(X)$ be L-fuzzy mappings defined as follows:

$$S(x)(t) = \begin{cases} Ax, & \text{if } t \in Kx; \\ 0_L, & \text{if } t \notin Kx. \end{cases}$$

and

$$T(x)(t) = \begin{cases} Bx, & \text{if } t \in Lx; \\ 0_L, & \text{if } t \notin Lx. \end{cases}$$

For each $x \in X$.

$$[Sx]_{\alpha_{L_{S(x)}}} = \{t \in X : \alpha_{L_{S(x)}} \preceq_L S(x)(t)\} = Kx.$$

Similarly

$$[Tx]_{\alpha_{L_{T(x)}}} = Lx.$$

Now, from Theorem 3.1 we obtain $x^* \in X$ such that

$$x^* \in [Sx^*]_{\alpha_{L_{S(x^*)}}} \cap [Tx^*]_{\alpha_{L_{T(x^*)}}} = Kx^* \cap Lx^*,$$

completing the proof. ∎

Moreover, as an application of Theorem 3.1, we establish the existence of a common fixed point for L-fuzzy mappings under a rational contractive condition on a metric space with the d_L^∞-metric for L-fuzzy sets.

Theorem 3.5. *Let (X, d) be a complete metric space and $S, T : X \longrightarrow E_L(X)$. If for all $x, y \in X$*

$$d_L^\infty(Sx, Ty) \le \lambda_1 d(x, y) + \lambda_2 p(x, Sx) + \lambda_3 p(y, Ty) + \frac{\lambda_4 p(x, Sx) p(y, Ty)}{1 + d(x, y)}.$$

and

$$\lambda_3 + \frac{\lambda_4 p(x, Sx)}{1 + d(x, y)} < 1, \qquad \lambda_2 + \frac{\lambda_4 p(y, Ty)}{1 + d(x, y)} < 1.$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$. Then, there exists $z \in X$ such that $\{z\} \subset Sz$ and $\{z\} \subset Tz$.

Proof. Choose $x \in X$. By hypothesis $[Sx]_{\alpha_L}, [Tx]_{\alpha_L} \in CB(X)$ for all $\alpha_L \in L$. So, for every $x, y \in X$, we have

$$\begin{aligned} D_{\alpha_L}(Sx, Ty) &\le d_L^\infty(Sx, Ty) \\ &\le \lambda_1 d(x, y) + \lambda_2 p(x, Sx) + \lambda_3 p(y, Ty) + \frac{\lambda_4 p(x, Sx) p(y, Ty)}{1 + d(x, y)}. \end{aligned}$$

But

$$p(x, Sx) \le d(x, [Sx]_{\alpha_L}).$$

Which implies

$$\begin{aligned} H([Sx]_{\alpha_L}, [Ty]_{\alpha_L}) \le{}& \lambda_1 d(x, y) + \lambda_2 d(x, [Sx]_{\alpha_L}) + \lambda_3 d(y, [Ty]_{\alpha_L}) \\ &+ \frac{\lambda_4 d(x, [Sx]_{\alpha_L}) d(y, [Ty]_{\alpha_L})}{1 + d(x, y)}. \end{aligned}$$

Therefore, Theorem 3.1 implies the existence of $z \in X$ such that

$$\{z\} \subset Sz \text{ and } \{z\} \subset Tz. \qquad \blacksquare$$

Theorem 3.6. *Let (V, d) be a complete metric linear space and $S, T : V \longrightarrow \mathcal{W}_L(V)$. If for all $x, y \in V$*

$$d_L^{\infty}(Sx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 p(x, Sx) + \lambda_3 p(y, Ty) + \frac{\lambda_4 p(x, Sx) p(y, Ty)}{1 + d(x, y)}.$$

and

$$\lambda_3 + \frac{\lambda_4 p(x, Sx)}{1 + d(x, y)} < 1, \qquad \lambda_2 + \frac{\lambda_4 p(y, Ty)}{1 + d(x, y)} < 1.$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are non-negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$. Then, there exists $z \in V$ such that $\{z\} \subset Sz$ and $\{z\} \subset Tz$.

Proof. Choose $z_0 \in V$. By applying Lemma 2.13 there exists $z_1 \in V$ such that $\{z_1\} \subset Sz_0$. Which implies that $p_{\alpha_L}(z_1, Sz_0) = 0 \Longleftrightarrow z_1 \in [Sz_0]_{\alpha_L}$, for all $\alpha_L \in L$. Similarly, one can find $z_2 \in V$ so that $z_2 \in [Tz_0]_{\alpha_L}$.
Thus, for each $z \in V$, $[Sz]_{\alpha_L}, [Tz]_{\alpha_L} \in C(X)$. To complete the proof, one can employ a similar approach to the proof of Theorem 3.5 above. $\blacksquare$

Corollary 3.7. *Let (V, d) be a complete metric linear space and $T : V \longrightarrow \mathcal{W}_L(V)$ be an L-fuzzy mapping such that for all $x, y \in V$*

$$d_L^{\infty}(Tx, Ty) \leq \beta d(x, y),$$

where $0 \leq \beta < 1$. Then, there exists $z \in V$ such that $\{z\} \subset Tz$.

Remark 3.8.

(i) If we consider $L = [0, 1]$ in Theorems 3.1 3.3, 3.4, 3.5, 3.6 and Corollary 3.7 above, we get Theorems 2.1, 3.1, 3.2, 4.1, 4.2 and Corollary 4.3 of [25] respectively;

(ii) If $L = [0, 1]$ in Corollary 3.7, then the result reduces to Theorem 3.1 of [5];

(iii) If $\alpha_L = 1_L$ in Theorems 3.1, 3.3, 3.4, 3.5 and 3.6, then by Remark 2.9 the L-fuzzy mappings S and T have a common fixed point;

(iv) If $\alpha_L = 1_L$ in Corollary 3.7, then by Remark 2.9 the L-fuzzy mapping T has a fixed point.

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