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On Hankel Type Integral Transform Associated with Whittaker and Hypergeometric Functions

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Abstract In the present research note, we establish a Hankel type integral transform involving the product of Whittaker function $W_{k,m}$ and hypergeometric function ${}_1F_2$. By using the result of Erdelyi et al. [A. Erdelyi et al., Table of Integral Transfoms, Vol. 1, McGraw-Hill, New York, 1954], we express this transform into Srivastava triple hypergeometric series $F^{(3)}[x, y, z]$. Some special cases of our main transform are also indicated.

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1. Introduction

The theory of integral transforms introduced from time to time by many authors (see for example, [1–5] etc), play a very crucial role in the area of mathematical physics and engineering sciences (for example, astrophysics, plasma physics, neutron theory etc). Due to the great importance of such transforms, in this short note, we present a Hankel type integral transform involving the product of Whittaker function $W_{k,m}(z)$ and hypergeometric function ${}_1F_2$, which is given in terms of Srivastava triple hypergeometric series $F^{(3)}[x, y, z]$.

For purpose the our present research work, we begin by recalling here the following definitions of some well known functions:

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We have the generalized hypergeometric function represented is defined by (see [6]):

$$ {}_pF_q\left[\begin{array}{c}(\alpha_p);\\(\beta_q);\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}(\alpha_j)_n}{\Pi_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!} \tag{1.1}$$

provided $p\leq q$, $|z|<\infty$; $p=q+1$, $|z|<1$ and $(\alpha)_n$ is well known Pochhammer symbol, $\alpha\in\mathbb{C}$ (see [6]).

The Whittaker function of second kind $W_{k,\mu}(z)$ [7, 8] is defined as

$$W_{k,\mu}(z)=z^{\mu+\frac{1}{2}}\exp\left(-\frac{z}{2}\right)\Psi\left(\mu-k+\frac{1}{2},2\mu+1;z\right), \tag{1.2}$$

where Ψ denotes Humbert's confluent hypergeometric function of one-variable (see [9]).

Also, we have the Srivastava general triple hypergeometric series $F^{(3)}[x,y,z]$ (see [9, p. 69]) defined as

$$F^{(3)}\left[\begin{array}{ll}(a)::(h);(h');(h''):(g);(g');(g'') & ;\\ & \quad x,y,z\\ (b)::(f);(f')(f''):(e);(e');(e'') & ;\end{array}\right]$$

$$=\sum_{m,n,p=0}^{\infty}\frac{[(a)_{m+n+p}(h)_{m+n}(h')_{n+p}(h'')_{p+m}(g)_m(g')_n(g'')_p]}{[(b)_{m+n+p}(f)_{m+n}(f')_{n+p}(f'')_{p+m}(e)_m(e')_n(e'')_p]}\frac{x^my^nz^p}{m!n!p!} \tag{1.3}$$

and the Kampé de Fériet's function $F^{(2)}[x,y]$ (see [2, 10]) defined by

$$F^{(2)}\left[\begin{array}{ll}a:b;c & ;\\ & \quad x,y\\ e:f;g & ;\end{array}\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_nx^my^n}{(e)_{m+n}(f)_m(g)_nm!n!}. \tag{1.4}$$

The generalized Bessel function $\omega^b_{\nu,c}(z)$ of the first kind is defined for $z\in\mathbb{C}\backslash\{0\}$ and $b,c,\nu\in\mathbb{C}$ with $\Re(\nu)>-1$ by the following series (see [1, 3]):

$$\omega^b_{\nu,c}(z)=\sum_{k=0}^{\infty}\frac{(-1)^kc^k\ \left(\frac{z}{2}\right)^{\nu+2k}}{k!\ \Gamma(\nu+k+\frac{1+b}{2})}, \tag{1.5}$$

where $\mathbb{C}$ denotes the set of complex numbers, $\Gamma(z)$ is the familiar Gamma function and $\omega^b_{\nu,c}(0)=0$. It is well known that

$$\omega^1_{\nu,1}(z)=J_\nu(z), \tag{1.6}$$

where $J_\nu(z)$ is the Bessel function of first kind [6] and

$$\omega^1_{\nu,-1}(z)=I_\nu(z), \tag{1.7}$$

where $I_\nu(z)$ is the Modified Bessel function of first kind [6].

2. Main Transform

In this section, we establish the following Hankel type integral transform:

$$\int_0^\infty t^{\sigma-1}e^{-zt}W_{k,m}(pt)w^b_{\nu,c}(\alpha t)\,{}_1F_2\left(\begin{matrix}\beta & ; \\ & & y^2t^2 \\ \gamma,\delta & ;\end{matrix}\right)dt$$

$$= \frac{p^{m+\frac{1}{2}}\alpha^\nu\Gamma(P)\Gamma(Q)}{2^\nu\Gamma(\nu+\frac{1+b}{2})(z+\frac{p}{2})^P\Gamma(E)}$$

$$\times\left\{F^{(3)}\left[\begin{matrix}\frac{P}{2},\frac{P+1}{2} :: \frac{Q}{2},\frac{Q+1}{2};-;-:\beta;-;\frac{D}{2},\frac{D+1}{2}; \\ \frac{E}{2},\frac{E+1}{2} :: -;-;-;\gamma,\delta;\nu+\frac{1+b}{2};\frac{1}{2};\end{matrix}\ \frac{4y^2}{(z+p/2)^2},\frac{-c\alpha^2}{(z+p/2)^2},\left(\frac{z-p/2}{z+p/2}\right)^2\right]\right.$$

$$+\frac{PD}{E}\left(\frac{z-p/2}{z+p/2}\right)$$

$$\left.\times F^{(3)}\left[\begin{matrix}\frac{P+1}{2},\frac{P+2}{2} :: \frac{Q}{2},\frac{Q+1}{2};-;-:\beta;-;\frac{D+1}{2},\frac{D+2}{2}; \\ \frac{E+1}{2},\frac{E+2}{2} :: -;-;-;\gamma,\delta;\nu+\frac{1+b}{2};\frac{3}{2};-;\end{matrix}\ \frac{4y^2}{(z+p/2)^2},\frac{-c\alpha^2}{(z+p/2)^2},\left(\frac{z-p/2}{z+p/2}\right)^2\right]\right\}, \tag{2.1}$$

where $P=\sigma+\nu+m+\frac{1}{2}$, $Q=\sigma+\nu-m+\frac{1}{2}$, $D=m-k+\frac{1}{2}$, $E=\sigma+\nu-k+1$, $\Re(P)>0$, $\Re(Q)>0$, $\Re(E)>0$, $\Re(\nu)>-\left(\frac{1+b}{2}\right)$, $\Re(z+\frac{p}{2})>0$ and $F^{(3)}[x,y,z]$ is the Srivastava triple hypergeometric series (see Eq.(1.3)).

Proof of Result (2.1):

In order to establish the result (2.1), expanding ${}_1F_2$ and $w^b_{\nu,c}$ in their respective series and integrating term by term with the help of the result [11, p. 216(16)], we get

$$I = \frac{p^{m+1/2}(\frac{\alpha}{2})^\nu}{\Gamma(\nu+\frac{1+b}{2})}\sum_{l,s=0}^{\infty}\frac{(\beta)_s(\frac{-c\alpha^2}{4})^l y^{2s}}{(\gamma)_s(\delta)_s(\nu+\frac{1+b}{2})_s l!s!}\frac{\Gamma(P+2l+2s)\Gamma(Q+2l+2s)}{\Gamma(E+2l+2s)(z+\frac{1}{2}p)^{P+2l+2s}}$$

$$\times{}_2F_1\left(\begin{matrix}P+2l+2s,D & ; \\ & & \frac{z-p/2}{z+p/2} \\ E+2l+2s & ;\end{matrix}\right),$$

where $P=\sigma+\nu+m+\frac{1}{2}$, $Q=\sigma+\nu-m+\frac{1}{2}$, $D=m-k+\frac{1}{2}$, $E=\sigma+\nu-k+1$, $\Re(P)>0$, $\Re(Q)>0$, $\Re(E)>0$, $\Re(\nu)>-\frac{(1+b)}{2}$, $\Re(z+\frac{p}{2})>0$.

Now by making use of the following result of Carlson [12, p. 234(10)]:

$${}_2F_1\left(\begin{matrix}a,b & ; \\ & & x \\ c & ;\end{matrix}\right) = {}_4F_3\left(\begin{matrix}\frac{a}{2},\frac{a+1}{2},\frac{b}{2},\frac{b+1}{2} & ; \\ & & x^2 \\ \frac{1}{2},\frac{c}{2},\frac{c+1}{2} & ;\end{matrix}\right)+\frac{abx}{c}\,{}_4F_3\left(\begin{matrix}\frac{a+1}{2},\frac{a+2}{2},\frac{b+1}{2},\frac{b+2}{2} & ; \\ & & x^2 \\ \frac{3}{2},\frac{c+1}{2},\frac{c+2}{2} & ;\end{matrix}\right), \tag{2.2}$$

in the above equation and on further expanding ${}_4F_3$'s in series form and interpreting the resulting series in the form of $F^{(3)}$, we arrive at the required result (2.1).

Remark. If we consider $b = c = 1$ in (2.1), then it reduces to the known result of Khan and Kashmin [4].

3. Special Cases

In this section, we derive some potentially useful integral transforms as special cases our main result. The following special cases of the main transformation (2.1) are given below:

(1) On setting $k = 0$ in (2.1) and using the result $W_{0,m}(z) = \sqrt{\frac{z}{\pi}}K_m(z/2)$, we get

$$\int_0^\infty t^{\sigma-\frac{1}{2}}e^{-zt}K_m(pt/2)w^b_{\nu,c}(\alpha t)\,{}_1F_2\left(\begin{array}{ccc}\beta & ; & \\ & & y^2t^2 \\ \gamma,\delta & ; & \end{array}\right)dt$$

$$= \frac{p^m\alpha^\nu\Gamma(P)\Gamma(Q)}{2^\nu\Gamma(\nu+\frac{1+b}{2})\Gamma(E')(z+\frac{p}{2})^P}$$

$$\times\Bigg\{F^{(3)}\left[\begin{array}{c}\frac{P}{2},\frac{P+1}{2}::\frac{Q}{2},\frac{Q+1}{2};-;-:\beta;-;\frac{D'}{2},\frac{D'+1}{2}; \\ \frac{E'}{2},\frac{E'+1}{2}::-;-;-;\gamma,\delta;\nu+\frac{1+b}{2};\frac{1}{2};\end{array}\frac{4y^2}{(z+p/2)^2},\frac{-c\alpha^2}{(z+p/2)^2},\left(\frac{z-p/2}{z+p/2}\right)^2\right]$$

$$+\frac{PD'}{E'}\left(\frac{z-p/2}{z+p/2}\right)$$

$$\times F^{(3)}\left[\begin{array}{c}\frac{P+1}{2},\frac{P+2}{2}::\frac{Q}{2},\frac{Q+1}{2};-;-:\beta;-;\frac{D'+1}{2},\frac{D'+2}{2}; \\ \frac{E'+1}{2},\frac{E'+2}{2}::-;-;-;\gamma,\delta;\nu+\frac{1+b}{2};\frac{3}{2};-;\end{array}\frac{4y^2}{(z+p/2)^2},\frac{-c\alpha^2}{(z+p/2)^2},\left(\frac{z-p/2}{z+p/2}\right)^2\right]\Bigg\}, \tag{3.1}$$

where $P = \sigma+\nu+m+\frac{1}{2}$, $Q = \sigma+\nu-m+\frac{1}{2}$, $D' = m+\frac{1}{2}$, $E' = \sigma+\nu+1$, $\Re(P) > 0$, $\Re(Q) > 0$, $\Re(E') > 0$, $\Re(\nu) > -\frac{(1+b)}{2}$, $\Re(z+\frac{p}{2}) > 0$ and $K_m(z)$ is the modified Bessel function (see [9]).

(2) On setting $k = \frac{n}{2}+\frac{1}{4}$, $m = \frac{1}{4}$ in (2.1), and using the result $W_{\frac{n}{2}+\frac{1}{4},\frac{1}{4}}(z^2) = 2^{-n}e^{\frac{-z^2}{2}}\sqrt{z}H_n(z)$, we get another transform

$$\int_0^\infty t^{\sigma-\frac{3}{4}}e^{-(z+p/2)t}H_n(\sqrt{(}pt))w^b_{\nu,c}(\alpha t)\,{}_1F_2\left(\begin{array}{ccc}\beta & ; & \\ & & y^2t^2 \\ \gamma,\delta & ; & \end{array}\right)dt$$

$$= \frac{p^{m+\frac{1}{4}}\alpha^\nu\Gamma(P')\Gamma(Q')}{2^{\nu-n}\Gamma(\nu+\frac{1+b}{2})\Gamma(E'')(z+\frac{p}{2})^{P'}}$$

$$\times\left\{F^{(3)}\left[\begin{array}{l}\frac{P'}{2},\frac{P'+1}{2}::\frac{Q'}{2},\frac{Q'+1}{2};-;-:\beta;-;\frac{D''}{2},\frac{D''+1}{2};\\ \frac{E''}{2},\frac{E''+1}{2}::-;-;-;\gamma,\delta;\nu+\frac{1+b}{2};\frac{1}{2};\end{array}\frac{4y^2}{(z+p/2)^2},\frac{-c\alpha^2}{(z+p/2)^2},\left(\frac{z-p/2}{z+p/2}\right)^2\right]\right.$$

$$+\frac{P'D''}{E''}\left(\frac{z-p/2}{z+p/2}\right)$$

$$\left.\times F^{(3)}\left[\begin{array}{l}\frac{P'+1}{2},\frac{P'+2}{2}::\frac{Q'}{2},\frac{Q'+1}{2};-;-:\beta;-;\frac{D''+1}{2},\frac{D''+2}{2};\\ \frac{E''+1}{2},\frac{E''+2}{2}::-;-;-;\gamma,\delta;\nu+\frac{1+b}{2};\frac{3}{2};-;\end{array}\frac{4y^2}{(z+p/2)^2},\frac{-c\alpha^2}{(z+p/2)^2},\left(\frac{z-p/2}{z+p/2}\right)^2\right]\right\},\qquad(3.2)$$

where $P' = \sigma+\nu+\frac{3}{2}$, $Q' = \sigma+\nu+\frac{1}{4}$, $D'' = \frac{-n}{2}+\frac{1}{2}$ and $E'' = \sigma+\nu-\frac{n}{2}+\frac{3}{4}$, $\Re(P')>0$, $\Re(Q')>0$, $\Re(E'')>0$, $\Re(\nu)>-\frac{(1+b)}{2}$.

(3) On setting $z = \frac{p}{2}$ in (2.1), we get

$$\int_0^\infty t^{\sigma-1}e^{-pt/2}W_{k,m}(pt)\,{}_1F_2\left(\begin{array}{ll}\beta & ;\\ & y^2t^2\\ \gamma,\delta & ;\end{array}\right)w^b_{\nu,c}(\alpha t)dt$$

$$=\frac{(\frac{\alpha}{2})^\nu p^{m+\frac{1}{2}}\Gamma(P)\Gamma(Q)}{\Gamma(\nu+\frac{1+b}{2})\Gamma(E)p^P}F^{(2)}\left[\begin{array}{ll}\frac{P}{2},\frac{P+1}{2},\frac{Q}{2},\frac{Q+1}{2}:\beta;- & ;\\ & \frac{4y^2}{p^2},\frac{-c\alpha^2}{p^2}\\ \frac{E}{2},\frac{E+1}{2}:\gamma,\delta;\nu+\frac{b+1}{2} & ;\end{array}\right],\qquad(3.3)$$

where $P = \sigma+\nu+m+\frac{1}{2}$, $Q = \sigma+\nu-m+\frac{1}{2}$, $D = m-k+\frac{1}{2}$, $E = \sigma+\nu-k+1$, $\Re(P)>0$, $\Re(Q)>0$, $\Re(E)>0$, $\Re(\nu)>-\frac{(1+b)}{2}$, $\Re(z+\frac{p}{2})>0$ and $F^{(2)}$ is the Kampé de Fériet's function defined by (see Eq.(1.4)).

(4) On setting $k=m+1/2$ in (2.1) and using the result $W_{m+1/2,m}(z)=z^{m+1/2}e^{-\frac{z}{2}}$, we get

$$\int_0^\infty t^{\sigma+m-\frac{1}{2}-1}e^{-(z+\frac{p}{2})t}{}_1F_2\left(\begin{array}{ll}\beta & ;\\ & y^2t^2\\ \gamma,\delta & ;\end{array}\right)w^b_{\nu,c}(\alpha t)dt$$

$$=\frac{(\frac{\alpha}{2})^\nu\Gamma(P)}{\Gamma(\nu+\frac{1+b}{2})(z+\frac{p}{2})^P}F^{(2)}\left[\begin{array}{ll}\frac{P}{2},\frac{P+1}{2}:\beta;- & ;\\ & \frac{4y^2}{(z+p/2)},\frac{-c\alpha^2}{(z+p/2)}\\ -:\gamma,\delta;\nu+\frac{b+1}{2} & ;\end{array}\right],\qquad(3.4)$$

where $P = \sigma+\nu+m+\frac{1}{2}$, $\Re(P)>0$, $\Re(\nu)>-\frac{(1+b)}{2}$, $\Re(z+\frac{p}{2})>0$.

4. Concluding Remarks

In the present article, we have derived a Hankel type integral transform involving the product of Whittaker function $W_{k,m}$ and the hypergeometric function ${}_1F_2$, which is expressed in terms of Srivastava triple hypergeometric series. We have also considered some special cases of our main transform.
As we know that the hypergeometric function ${}_1F_2$ have the following relations with the Lommel and Struve functions

$$ {}_1F_2\left[1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; \frac{-z^2}{4}\right] = (\mu-\nu+1)(\mu+\nu+1)z^{-(\mu+1)}S_{\mu,\nu}(z); $$

$$ {}_1F_2\left[1; \frac{3}{2}, \frac{\nu+3}{2}; \frac{-z^2}{4}\right] = \Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{-(\nu+1)} H_\mu(z); $$

$$ {}_1F_2\left[1; \frac{3}{2}, \frac{\nu+3}{2}; \frac{z^2}{4}\right] = \Gamma\left(\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{-(\nu+1)} L_\mu(z). $$

Therefore, by using the above relations of ${}_1F_2$, we can obtained some other integral transform (involving Lommel and Struve functions) as special cases of our main result.
Furthermore, we have the following interesting relation of Hermite polynomials with Laguerre polynomials

$$ H_{2k}(x) = (-1)^k 2^{2k} k! L_k^{(-1/2)}(x^2),\ H_{2k+1}(x) = (-1)^k 2^{2k+1} k! L_k^{(1/2)}(x^2). $$

So, by using the above relation of $H_n(x)$ in (3.2), we can obtained a new integral transform involving Laguerre polynomials.

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