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Solving Split Monotone Variational Inclusion Problem and Fixed Point Problem for Certain Multivalued Maps in Hilbert Spaces

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Abstract In this paper, we introduce a new iterative algorithm for approximating the common solution of Split Monotone Variational Inclusion Problem (SMVIP) and Fixed Point Problem for multivalued strictly pseudocontractive-type mappings in real Hilbert spaces. Under standard and mild assumption of inverse strongly monotonicity and maximal and monotonicity of the SMVIP associated mappings, we establish the strong convergence of the sequence generated by our iterative algorithm. We further applied our result to solve Split Minimization Problem (SMP) and Split Variational Inequality Problem (SVIP).

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1. Introduction

Let H be a real Hilbert space and K a nonempty, closed and convex subset of H. We shall denote the family of all nonempty closed and bounded subsets of K by $CB(K)$, the family of all nonempty subsets of K by 2^K. A mapping $T : K \to K$ is said to be *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y|| \; \forall \; x, y \in K. \tag{1.1}$$

A point $x \in K$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$ and a Fixed Point Problem (FPP) for T is to find $x \in F(T)$. It is a common knowledge that if T is nonexpansive and $F(T) \neq \emptyset$ then $F(T)$ is a closed and convex subset of K. Let $T : K \to 2^K$ be a multivalued map, then $x \in D(T)$ is a fixed point of T if $x \in F(T)$. If T is multivalued then the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$

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is called the strict fixed point set of T. Let H be a Hilbert space. A subset K of H is called proximal, if for each $x \in H$ there exists $k \in K$ such that

$$||x-k|| = \inf\{||x-y|| : y \in K\} = d(x, K), \tag{1.2}$$

and the family of all proximal subsets of H will be denoted by $P(H)$. It is known that every closed convex subset of a Hilbert space is proximal. Let $H(.,.)$ denote the Hausdorff metric induced by the metric d on H, that is for $A, B \in CB(H)$,

$$H(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B}(b, A)\Big\}. \tag{1.3}$$

Let H be a Hilbert space and $T : D(T) \subseteq H \to 2^H$ be a multivalued mapping. T is said to be $L-$Lipschzian (see [1, 2]) if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L||x-y||. \tag{1.4}$$

In (1.4), if $L \in [0, 1)$, T is a contraction while T is nonexpansive if $L = 1$. T is called quasi-nonexpansive if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for $p \in F(T)$,

$$H(Tx, Tp) \leq ||x-p||. \tag{1.5}$$

T is said to be κ-strictly pseudocontractive-type in the sense of Isiogugu [3], if there exists $\kappa \in [0, 1)$ such that, given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u-v|| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq ||x-y||^2 + \kappa||x-u-(y-v)||^2. \tag{1.6}$$

$T : D(T) \subseteq H \to CB(H)$ is said to be κ-strictly pseudocontractive in the sense of Chidume et al. [4], if there exists $\kappa \in [0, 1)$ such that for all $x, y \in D(T)$

$$H^2(Tx, Ty) \leq ||x-y||^2 + \kappa||x-u-(y-v)||^2, \ \forall u \in Tx, v \in Ty. \tag{1.7}$$

It has been observed (see Isiogugu [5]) that every κ-strictly pseudocontractive mapping $T : D(T) \subseteq H \to P(H)$ is κ-strictly pseudocontractive-type.
The study of fixed points for multivalued contractions and nonexpansive mappings (see [6]) was initiated by Nadler [7] and Markin [8] respectively, and by now there exists an extensive literature on multivalued fixed point theory which has applications in convex optimization, differential inclusions, fractals, discontinuous differential equations, optimal control, computing homology of maps, computer-assisted proofs in dynamics, digital imaging and economics (e.g., [9–13] and references cited therein). There are many classical and well developed areas of applications, where a multivalued map is used as a generalization of a single valued map.

Let $f : H \to H$ be a single valued nonlinear mapping and let $M : H \to 2^H$ be a set valued mapping. The variational inclusion problem is to find $x \in H$ such that

$$0 \in f(x) + M(x), \tag{1.8}$$

where 0 is the zero vector in H. The set of solutions to the variational inclusion problem (1.8) is denoted by $I(f, M)$. For further studies on variational inclusion problem see for example [14–20] and some of the references therein.
A mapping $T : H \to H$ is said to be
(i) monotone, if

$$\langle Tx - Ty, x-y\rangle \geq 0, \ \forall x, y \in H;$$

(ii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha||x - y||^2, \ \forall x, y \in H;$$

(iii) β-inverse strongly monotone(β-ism), if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta||Tx - Ty||^2, \ \forall x, y \in H;$$

(iv) firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geq ||Tx - Ty||^2, \ \forall x, y \in H,$$

see [21]. A set valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$ with $u \in M(x)$ and $v \in M(y)$ then

$$\langle x - y, u - v\rangle \geq 0.$$

A monotone mapping M is said to be maximal if the graph of M denoted by $G(M)$ is not properly contained in the graph of any other monotone mapping. The graph of a multi valued mapping M is the set,

$$G(M) = \{(x, y) : y \in M(x)\}.$$

It is well known that M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$. The resolvent operator J_λ^M associated with M and λ is the mapping $J_\lambda^M : H \to H$ defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}x, \ x \in H, \lambda > 0. \tag{1.9}$$

It is known that the resolvent operator $J_\lambda^M(x)$ is single valued, nonexpansive and 1-inverse strongly monotone (for example see [22]) and the solution of (1.8) is a fixed point $J_\lambda^M(x)(I - \lambda f)$ $\forall \lambda > 0$ (see for example [23]). If f is α-inverse strongly monotone mapping with $0 < \lambda < 2\alpha$, then one can easily see that $J_\lambda^M(x)(I - \lambda f)$ is nonexpansive and $I(f, M)$ is closed and convex.

Let H_1 and H_2 be real Hilbert spaces. Let $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$ be inverse strongly monotone mappings and $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear mapping. The Split Monotone Variational Inclusion Problem (SMVIP) is to find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) \tag{1.10}$$

and

$$y^* = Ax^* \in H_2 \text{ such that, } 0 \in f_2(y^*) + B_2(y^*). \tag{1.11}$$

We shall denote by Ω the solution set of (1.10)–(1.11). That is,

$$\Omega = \{x^* \in H_1 : 0 \in f_1(x^*) + B_1(x^*) \text{ and } y^* = Ax^* \in H_2 \text{ such that } 0 \in f_2(y^*) + B_2(y^*)\}.$$

If we consider (1.10) and (1.11) seprately, we have that (1.10) is a variational inclusion problem with its solution set $I(f_1, B_1)$ and (1.11) is a variational inclusion problem with solution set $I(f_2, B_2)$.

Moudafi [24] introduced SMVIP (1.10)–(1.11) and proposed an iterative method for solving it. In Moudafi [24], it was noted that the SMVIP generalises the Split Fixed Point Problem (SFPP), Split Variational Inequality Problem (SVIP), Split Zero Problem (SZP) and Split Fasibility Problem (SFP) (see [24–31]), which have been studied extensively by many authors and applied to solving many real life problems such as in modelling intensity-modulated radiation therapy treatment planning, modelling of inverse problems arising from phase retrieval, and in sensor networks in computerised tomography and data compression [32, 33].

Suppose in SMVIP (1.10)–(1.11), $f_1 \equiv 0$ and $f_2 \equiv 0$, we obtain the following Split Variational Inclusion Problem (SVIP): Find $x^* \in H_1$ such that

$$0 \in B_1(x^*) \tag{1.12}$$

and

$$y^* = Ax^* \in H_2 \text{ such that, } 0 \in B_2(y^*). \tag{1.13}$$

Byrne *et al.* [34] using the following iterative scheme: for a given $x_0 \in H_1$, the sequence $\{x_n\}$ generated iteratively by;

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \ \lambda > 0,$$

obtained a weak and strong convergence theorem solving SVIP (1.12)–(1.13). Inspired by the work of Byrne *et al.*, Kazmi and Rizvi [35] proposed the following algorithm for approximating a solution of SVIP (1.12)–(1.13) which is a fixed point of a nonexpansive mapping S: for a given $x_0 \in H_1$ let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, n \geq 0, \end{cases} \tag{1.14}$$

and proved that both $\{u_n\}$ and $\{x_n\}$ converge strongly to $z \in F(S) \cap \Gamma$, where Γ is the solution set of SVIP (1.12)–(1.13). For more on variational inclusion problem see [36, 37].

Recently, Shehu and Ogbuisi [38] motivated by the works of Moudafi [24] and Kazmi and Rizvi [35] propose an iterative scheme for approximating a common solution of a fixed point problem and the SMVIP (1.10)–(1.11) without f_1 and f_2 being necessarily zero and obtained a strong convergence result.
In this paper, we introduce an iterative scheme and obtain a strong convergence result for approximating a solution of the SMVIP (1.10)–(1.11) (f_1 and f_2 not necessarily zero) which is also a common solution of two multivalued strictly pseudocontractive mappings in the sense of Isiogugu [3].

2. Preliminaries

In the sequel, we will need the following important definition and lemmas to establish our main results.

Definition 2.1. Let $T : H \to 2^H$ be a multivalued mapping; for each $x \in H, P_Tx$ is defined by

$$P_T(x) = \{y \in Tx : ||x - y|| = d(x, Tx)\}. \tag{2.1}$$

Lemma 2.2 ([5])**.** *Let K be a nonempty subset of a real Hilbert space H and let $T : K \to P(K)$ be a κ-strictly pseudocontractive-type mapping such that $F_s(T)$ is nonempty. Then $F_s(T)$ is closed and convex.*

Lemma 2.3 ([39, 40]). *Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping, then for all $x, y \in H$,*

$$\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2}||(Tx - x) - (Ty - y)||^2, \tag{2.2}$$

and consequently if $y \in F(T)$ then

$$\langle x - Tx, Ty - Tx \rangle \leq \frac{1}{2}||Tx - x||^2. \tag{2.3}$$

Lemma 2.4 ([41]). *Let H be a real Hilbert space. Then the following result holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle, \forall\ x, y \in H.$$

Lemma 2.5 ([41]). *Let H be a Hilbert space, then $\forall x, y \in H$ and $\alpha \in (0, 1)$, we have*

$$||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2.$$

Lemma 2.6. (Demiclosedness principle) *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to K$ a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x \in K$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.7 ([42]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n,\ n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i)$\Sigma_{n=0}^{\infty}\gamma_n = \infty$,
(ii)$\limsup_{n\to\infty} \delta_n \leq 0$, or $\Sigma_{n=0}^{\infty}||\delta_n\gamma_n|| < \infty$,
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 ([23]). *Let $M : H \to 2^H$ be a maximally monotone mapping and $f : H \to H$ be a Lipschitz continuous mapping. Then the mapping $G = M + f : H \to 2^H$ is a maximal monotone mapping.*

A mapping $T : H \to H$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1 - \beta)I + \beta S$$

where $\beta \in (0, 1)$ and $S : H \to H$ is a nonexpansive mapping and I is the identity mapping on H. Every averaged mapping is nonexpansive and every firmly nonexpansive mapping is averaged. Thus since the resolvent of maximal monotone operators are firmly nonexpansive, they are averaged and therefore nonexpansive. For details, see [24, 43].

3. Main Results

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω be a solution set of (1.10)–(1.11). Let $S, T : H_1 \to P(H_1)$ be two strictly pseudocontractive-type mappings with contractive* *coefficients* κ_1 *and* κ_2 *such that* $F_s(S)\cap F_s(T)\cap\Omega\neq\emptyset$. *Let* $\{x_n\}$ *be the sequence generated for* $x_0\in H_1$ *by*

$$\begin{cases} w_n=(1-\alpha_n)x_n\\ y_n=J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)\\ x_{n+1}=\beta_n y_n+(1-\beta_n)[\rho_n v_n+(1-\rho_n)u_n],\ \forall n\geq 0\end{cases} \tag{3.1}$$

where $v_n\in Ty_n$ *and* $u_n\in Sy_n$, $0<\lambda<2\mu,2\nu$ *and* $\gamma\in(0,\frac{1}{L})$, L *is the spectral radius of the operator* AA^* *and* A^* *is the adjoint of* A. *Suppose* $\{\alpha_n\}_{n=1}^\infty$, $\{\rho_n\}_{n=1}^\infty$ *and* $\{\beta_n\}_{n=1}^\infty$ *are real sequences in* $(0,1)$ *satisfying the following conditions*

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^\infty\alpha_n=\infty$,

(ii) $\beta_n\geq\max\{\kappa_1,\kappa_2\}$ $\forall n\geq 0$,

(iii) $\liminf_{n\to\infty}(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)>0$,

(iv) $\liminf_{n\to\infty}(1-\beta_n)(\beta_n-\kappa_2)\rho_n>0$.

Then $\{x_n\}_{n=1}^\infty$ *converges strongly to* $p\in F_s(S)\cap F_s(T)\cap\Omega$.

Proof. Let $p\in F_s(S)\cap F_s(T)\cap\Omega$ and let $z_n=\rho_n v_n+(1-\rho_n)u_n$, then

$$\begin{aligned}||x_{n+1}-p||^2 &= ||[\beta_n y_n+(1-\beta_n)[\rho_n v_n+(1-\beta_n)u_n]]-p||^2\\ &= ||[\beta_n y_n+(1-\beta_n)z_n]-p||^2\\ &= \beta_n||y_n-p||^2+(1-\beta_n)||z_n-p||^2-\beta_n(1-\beta_n)||y_n-z_n||^2, \end{aligned}\tag{3.2}$$

and

$$\begin{aligned}||z_n-p||^2 &= ||\rho_n v_n+(1-\rho_n)u_n-p||^2\\ &= \rho_n||v_n-p||^2+(1-\beta_n)||u_n-p||^2\\ &\quad -\rho_n(1-\rho_n)||v_n-u_n||^2. \end{aligned}\tag{3.3}$$

From (3.16) and (3.3), we have

$$\begin{aligned}||x_{n+1}-p||^2 &= \beta_n||y_n-p||^2+(1-\beta_n)\rho_n||v_n-p||^2+(1-\beta_n)(1-\rho_n)||u_n-p||^2\\ &\quad -(1-\beta_n)\rho_n(1-\rho_n)||v_n-u_n||^2-\beta_n(1-\beta_n)||y_n-z_n||^2\\ &\leq \beta_n||y_n-p||^2+(1-\beta_n)\rho_n H^2(Ty_n,Tp)+(1-\beta_n)(1-\rho_n)H^2(Sy_n,Sp)\\ &\quad -(1-\beta_n)\rho_n(1-\beta_n)||v_n-u_n||^2-\beta_n(1-\beta_n)||y_n-z_n||^2\\ &\leq \beta_n||y_n-p||^2+(1-\beta_n)\rho_n[||y_n-p||^2+\kappa_2||y_n-v_n||^2]\\ &\quad +(1-\beta_n)(1-\rho_n)[||y_n-p||^2+\kappa_1||y_n-u_n||^2]\\ &\quad -(1-\beta_n)\rho_n(1-\beta_n)||v_n-u_n||^2-\beta_n(1-\beta_n)||y_n-z_n||^2. \end{aligned}\tag{3.4}$$

Again

$$\begin{aligned}||y_n-z_n||^2 &= ||y_n-[\rho_n v_n+(1-\rho_n)u_n]||^2\\ &= \rho_n||y_n-v_n||^2+(1-\rho_n)||y_n-u_n||^2-\rho_n(1-\rho_n)||v_n-u_n||^2. \end{aligned}\tag{3.5}$$

Inserting (3.5) into (3.4), we obtain

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq [\beta_n+(1-\beta_n)\rho_n+(1-\beta_n)(1-\rho_n)||y_n-p||^2\\
&\quad+[(1-\beta_n)\rho_n\kappa_2-\beta_n(1-\beta_n)\rho_n]||y_n-v_n||^2\\
&\quad+[(1-\beta_n)(1-\rho_n\kappa_1-\beta_n(1-\beta_n)(1-\rho_n)]||y_n-v_n||^2\\
&\quad+[(1-\beta_n)(1-\rho_n)\rho_n\beta_n-(1-\beta_n)(1-\rho_n)\rho_n||v_n-u_n||^2\\
&= ||y_n-p||^2-\rho_n(1-\beta_n)(\beta_n-\kappa_2)||y_n-v_n||^2\\
&\quad-(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2\\
&\quad-(1-\beta_n)^2(1-\rho_n)\rho_n||v_n-u_n||^2\\
&\leq ||y_n-p||^2. \qquad (3.6)
\end{aligned}$$

But

$$\begin{aligned}
||y_n-p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2\\
&\leq ||w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p||^2\\
&= ||w_n-p||^2+\gamma^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
&\quad+2\gamma\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle, \qquad (3.7)
\end{aligned}$$

and

$$\begin{aligned}
\gamma^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 &= \gamma^2\langle(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n, AA^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle\\
&\leq L\gamma^2\langle(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n,(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle\\
&= L\gamma^2||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2. \qquad (3.8)
\end{aligned}$$

Let $\Upsilon_n = 2\gamma\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle$ then from (2.3), we have

$$\begin{aligned}
\Upsilon_n &= 2\gamma\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle\\
&= 2\gamma\langle A(w_n-p)+(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n,(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle\\
&\quad+2\gamma\langle-(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n,(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle\\
&= 2\gamma[\langle J_\lambda^{B_2}(I-\lambda f_2)Aw_n-Ap, J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle\\
&\quad-||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2] \qquad (3.9)\\
&\leq 2\gamma\Big[\frac{1}{2}||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\Big]\\
&= -\gamma||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2.
\end{aligned}$$

From (3.7), (3.8) and (3.9), we have

$$\begin{aligned}
||y_n-p||^2 &\leq ||w_n-p||^2+L\gamma^2||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-\gamma||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
&= ||w_n-p||^2+\gamma(L\gamma-1)||J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \qquad (3.10)\\
&\leq ||w_n-p||^2.
\end{aligned}$$

By (3.6) and (3.10)

$$\begin{aligned}
||x_{n+1}-p|| &\leq ||w_n - p|| \\
&= ||(1-\alpha_n)x_n - p|| \\
&= ||(1-\alpha_n)(x_n - p) - \alpha_n p|| \\
&\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||p|| \\
&\leq \max\{||x_n - p||, ||p||\} \\
&\vdots \\
&\leq \max\{||x_0 - p||, ||p||\}.
\end{aligned}$$

Therefore $\{x_n\}$ is bounded and consequently $\{y_n\}$,$\{Sy_n\}$ and $\{w_n\}$ are bounded. We divide into two cases to establish the strong convergence of $\{x_n\}$ to p.

Case 1. Assume that $\{||x_n - p||\}$ is a monotonically decreasing sequence. Then $\{||x_n - p||\}$ is convergent and clearly

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{n\to\infty} ||x_{n+1} - p||.$$

Now,

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq ||y_n - p||^2 - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2 - (1-\beta_n)^2(1-\rho_n)\rho_n||v_n-u_n||^2 \\
&\leq ||w_n - p||^2 - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2 - (1-\beta_n)^2(1-\rho_n)\rho_n||v_n-u_n||^2 \\
&\leq ||(1-\alpha_n)x_n - p||^2 - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2 - (1-\beta_n)^2(1-\rho_n)\rho_n||v_n-u_n||^2 \\
&\leq ||x_n - p||^2 + \alpha_n^2||x_n||^2 - 2\alpha_n\langle x_n - p, x_n\rangle - \rho_n(1-\beta_n)(\beta_n - \kappa_2)||y_n - v_n||^2 \\
&\quad -(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2 \\
&\quad -(1-\beta_n)^2(1-\rho_n)\rho_n||v_n-u_n||^2. \qquad (3.11)
\end{aligned}$$

Let

$$\begin{aligned}
D_n &= \rho_n(1-\beta_n)(\beta_n-\kappa_2)||y_n - v_n||^2 \\
&\quad +(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)||y_n-u_n||^2.
\end{aligned}$$

Thus, from (3.11) we have

$$D_n \leq ||x_n - p||^2 - ||x_{n+1}-p||^2 + \alpha_n^2||x_n||^2 - 2\alpha_n\langle x_n - p, x_n\rangle \to 0, \qquad (3.12)$$

as $n \to \infty$. Thus, by conditions (iii) and (iv) and (3.12), we have

$$||y_n - v_n|| \to 0, \ as\ n \to \infty, \qquad (3.13)$$

and

$$||y_n - u_n|| \to 0, \ as\ n \to \infty. \qquad (3.14)$$

From (3.1), we have

$$||w_n - x_n|| = \alpha_n||x_n|| \to 0\ as\ n \to \infty. \qquad (3.15)$$

Again from (3.6)

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq ||y_n-p||^2\\
&= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2\\
&\leq ||w_n-p||^2+\gamma(L\gamma-1)||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
&\leq (1-\alpha_n)^2||x_n-p||^2+\alpha_n^2||p||^2-2\alpha_n(1-\alpha_n)\langle x_n-p,p\rangle\\
&\quad +\gamma(L\gamma-1)||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2. \qquad (3.16)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(1-L\gamma)||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||p||^2\\
&\quad -2\alpha_n(1-\alpha_n)\langle x_n-p,p\rangle\to 0, \qquad (3.17)
\end{aligned}$$

as $n\to\infty$. Hence,

$$||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||\to 0 \ as\ n\to\infty. \qquad (3.18)$$

From (3.10), we have

$$\begin{aligned}
||y_n-p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2\\
&\leq \langle y_n-p, w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p\rangle\\
&= \frac{1}{2}[||y_n-p||^2+||w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p||^2\\
&\quad -||y_n-p-(w_n+\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p)||^2]\\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2+\gamma(L\gamma-I)||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
&\quad -||y_n-w_n-\gamma A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p)||^2]\\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-(||y_n-w_n||^2+\gamma^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\\
&\quad -2\gamma\langle y_n-w_n, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle)]\\
&\leq \frac{1}{2}[||y_n-p||^2+||w_n-p||^2-||y_n-w_n||^2\\
&\quad +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||]. \qquad (3.19)
\end{aligned}$$

That is,

$$\begin{aligned}
||y_n-p||^2 &\leq ||w_n-p||^2-||y_n-w_n||^2\\
&\quad +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||. \qquad (3.20)
\end{aligned}$$

It then follows from (3.6) and (3.20) that

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq ||w_n-p||^2-||y_n-w_n||^2\\
&\quad +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||, \qquad (3.21)
\end{aligned}$$

which implies that

$$\begin{aligned}
||y_n-w_n||^2 &\leq ||w_n-p||^2-||x_{n+1}-p||^2+2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||\\
&= ||(1-\alpha_n)x_n-p||^2-||x_{n+1}-p||^2+2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||\\
&\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||p||^2+2\alpha_n(1-\alpha_n)\langle x_n-p,p\rangle\\
&\quad +2\gamma||A(y_n-w_n)||||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||\to 0 \ as\ n\to\infty. \qquad (3.22)
\end{aligned}$$

Therefore,

$$||y_n - w_n|| \to 0 \ as \ n \to \infty. \tag{3.23}$$

From (3.15),

$$||x_n - y_n|| \leq ||x_n - w_n|| + ||w_n - y_n|| \to 0. \tag{3.24}$$

Let $\theta_n = w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n$, then

$$||\theta_n - w_n||^2 = L\gamma^2||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2 \to 0. \tag{3.25}$$

Combining (3.23) and (3.25), we have

$$||y_n - \theta_n|| \leq ||y_n - w_n|| + ||w_n - \theta_n|| \to 0. \tag{3.26}$$

It follows from (3.13) and (3.14) that $\{y_n\}$ converges weakly to a point $p \in F(S) \cap F(T)$ and so do $\{x_n\}$ and $\{w_n\}$ converge weakly to p.

We now show that $p \in I(f_1, B_1)$. Since f_1 is $\frac{1}{\mu}$-Lipschitz monotone mapping and the domain of f_1 is H_1 then by Lemma 2.8, we conclude that $B_1 + f_1$ is maximally monotone. Let $(v, z) \in G(B_1 + f_1)$, that is $z - f_1 v \in B_1(v)$.

Since $y_n = J_\lambda^{B_1}(I - \lambda f_1)\theta_n$, we obtain

$$(I - \lambda f_1)\theta_n \in (I + \lambda B_1)y_n.$$

That is,

$$\frac{1}{\lambda}(\theta_n - \lambda f_1\theta_n - y_n) \in B_1(y_n).$$

Using the maximal monotonicity of $(B_1 + f_1)$, we have

$$\langle v - y_n, z - f_1 v - \frac{1}{\lambda}(\theta_n - \lambda f_1\theta_n - y_n)\rangle \geq 0.$$

Therefore,

$$\begin{aligned} \langle v - y_n, z\rangle &\geq \langle v - y_n, f_1 v + \frac{1}{\lambda}(\theta_n - \lambda f_1\theta_n - y_n)\rangle \\ &= \langle v - y_n, f_1 v - f_1 y_n + f_1 y_n - f_1\theta_n + \frac{1}{\lambda}(\theta_n - y_n)\rangle \\ &\geq 0 + \langle v - y_n, f_1 y_n - f_1\theta_n\rangle + \langle v - y_n, \frac{1}{\lambda}(\theta_n - y_n)\rangle. \end{aligned} \tag{3.27}$$

By (3.26), we obtain

$$\lim_{n\to\infty} ||f_1 y_n - f_1\theta_n|| = 0.$$

Also, since $y_n \rightharpoonup p$, we have

$$\lim_{n\to\infty} \langle v - y_n, z\rangle = \langle v - p, z\rangle.$$

Thus from (3.27)

$$\langle v - p, z\rangle \geq 0.$$

Since $B_1 + f_1$ is maximally monotone, we have $0 \in (B_1 + f_1)p$ which implies that

$$p \in I(f_1, B_1).$$

Moreover, since $||w_n - y_n|| \to 0$, we have that Aw_n converges weakly to Ap and by (3.18) and the fact that $J_\lambda^{B_2}(I - \lambda f_2)$ is nonexpansive, then by Lemma 2.6, we have

$$0 \in f_2 Ap + B_2(Ap).$$

That is $Ap \in I(f_2, B_2)$. Hence $p \in F_s(S) \cap F_s(T) \cap \Omega$.
We now show that $\{x_n\}$ converges strongly to p.

$$\begin{aligned} ||x_{n+1} - p||^2 &\leq ||y_n - p||^2 \\ &\leq ||w_n - p||^2 \\ &= ||(1-\alpha_n)x_n - p||^2 \\ &= ||(1-\alpha_n)(x_n - p) - \alpha_n p||^2 \\ &= (1-\alpha_n)^2||x_n - p||^2 + \alpha_n^2||p||^2 - 2\alpha_n(1-\alpha_n)\langle x_n - p, p\rangle \\ &\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n[\alpha_n||p||^2 - 2(1-\alpha_n)\langle x_n - p, p\rangle]. \end{aligned}$$

Therefore, by Lemma 2.7, we obtain $x_n \to p$, $n \to \infty$.

Case 2. Assume that $\{||x_n - p||\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$(for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \geq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$.
It follows from (3.11) that

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\leq \alpha_{\tau(n)}^2||x_{\tau(n)}||^2 - 2\alpha_{\tau(n)}\langle x_{\tau(n)} - p, x_{\tau(n)}\rangle - \rho_{\tau(n)}(1-\beta_{\tau(n)})(\beta_{\tau(n)} - \kappa_2)||y_{\tau(n)} - v_{\tau(n)}||^2 \\ &\quad -(1-\beta_{\tau(n)})(1-\rho_{\tau(n)})(\beta_{\tau(n)} - \kappa_1)||y_{\tau(n)} - u_{\tau(n)}||^2 \\ &\quad -(1-\beta_{\tau(n)})^2(1-\rho_{\tau(n)})\rho_{\tau(n)}||v_{\tau(n)} - u_{\tau(n)}||^2. \end{aligned}$$

Let

$$\begin{aligned} D_{\tau(n)} &= \rho_{\tau(n)}(1-\beta_{\tau(n)})(\beta_{\tau(n)} - \kappa_2)||y_{\tau(n)} - v_{\tau(n)}||^2 \\ &\quad +(1-\beta_{\tau(n)})(1-\rho_{\tau(n)})(\beta_{\tau(n)} - \kappa_1)||y_{\tau(n)} - u_{\tau(n)}||^2. \end{aligned}$$

Then,

$$D_{\tau(n)} \leq \alpha_{\tau(n)}^2||x_{\tau(n)}||^2 - 2\alpha_{\tau(n)}\langle x_{\tau(n)} - p, x_{\tau(n)}\rangle \to 0, \ as\ n \to \infty.$$

Thus, by conditions (iii) and (iv) and (3.12), we have

$$||y_{\tau(n)} - v_{\tau(n)}|| \to 0, \ as\ n \to \infty,$$

and

$$||y_{\tau(n)} - u_{\tau(n)}|| \to 0, \ as\ n \to \infty.$$

By the same argument as in case 1, we conclude that $\{x_{\tau(n)}\}$, $\{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ converge weakly to $p \in F_s(S) \cap F_s(T) \cap \Omega$. Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\leq (1-\alpha_{\tau(n)})||x_{\tau(n)} - p||^2 + \alpha_{\tau(n)}^2||p||^2 - 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)}\langle x_{\tau(n)} - p, p\rangle - ||x_{\tau(n)} - p||^2 \\ &= \alpha_{\tau(n)}[\alpha_{\tau(n)}||p||^2 - 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, p\rangle - ||x_{\tau(n)} - p||^2]. \end{aligned}$$

Therefore,

$$||x_{\tau(n)} - p||^2 \leq \alpha_{\tau(n)}||p||^2 - 2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)} - p, p\rangle \to 0.$$

Thus,

$$\lim_{n\to\infty} ||x_{\tau(n)} - p||^2 = 0.$$

And hence

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$(that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)} + 1\} = \Gamma_{\tau(n)} + 1.$$

So $\lim_{n\to\infty} \Gamma_n = 0$, that is $\{x_n\}$,$\{y_n\}$ and $\{w_n\}$ converge strongly to $p \in F_s(S) \cap F_s(T) \cap \Omega$.
■

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω be a solution set of (1.10)–(1.11). Let $S, T : H_1 \to P(H_1)$ be two multivalued nonexpansive mappings such that $F_s(S) \cap F_s(T) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0 \in H_1$ by*

$$\begin{cases} w_n = (1 - \alpha_n)x_n \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n) \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)[\rho_n v_n + (1 - \rho_n)u_n], \ \forall n \geq 0 \end{cases} \tag{3.28}$$

where $v_n \in Ty_n$ and $u_n \in Sy_n$, $0 < \lambda < 2\mu, 2\nu$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}_{n=1}^\infty$, $\{\rho_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0, 1)$ satisfying the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(iii) $\liminf_{n\to\infty}(1 - \beta_n)(1 - \rho_n)\beta_n > 0$,

(iv) $\liminf_{n\to\infty}(1 - \beta_n)\beta_n\rho_n > 0$.

Then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F_s(S) \cap F_s(T) \cap \Omega$.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω be a solution set of (1.10)–(1.11). Let $S, T : H_1 \to P(H_1)$ be two strictly pseudocontractive mappings with contractive coefficients κ_1 and κ_2 such that $F_s(S) \cap F_s(T) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0 \in H_1$ by*

$$\begin{cases} w_n = (1 - \alpha_n)x_n \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n) \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)[\rho_n v_n + (1 - \rho_n)u_n], \ \forall n \geq 0 \end{cases} \tag{3.29}$$

where $v_n \in P_T y_n$ and $u_n \in P_S y_n$, $0 < \lambda < 2\mu, 2\nu$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}_{n=1}^\infty$, $\{\rho_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $[0, 1]$ satisfying the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,

(ii) $\beta_n \geq \max\{\kappa_1, \kappa_2\}$ $\forall n \geq 0$,

(iii) $\liminf_{n\to\infty}(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1)>0$,

(iv) $\liminf_{n\to\infty}(1-\beta_n)(\beta_n-\kappa_2)\rho_n>0$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p\in F_s(S)\cap F_s(T)\cap\Omega$.

4. Applications

4.1. Split Minimization Problem

Consider the following Split Minimization Problem (SMP): Find $x^*\in H_1$ such that

$$x^*=\min_{x\in H_1}(\varphi_1(x)+\psi_1(x)), \tag{4.1}$$

and $y^*=Ax^*\in H_2$ is such that

$$y^*=\min_{y\in H_2}(\varphi_2(y)+\psi_2(y)), \tag{4.2}$$

where $\varphi_1,\psi_1:H_1\to\mathbb{R}$ and $\varphi_2,\psi_2:H_2\to\mathbb{R}$. Moreover φ_1 and φ_2 are assumed to be differentiable. $A:H_1\to H_2$ is a bounded linear operator. Denote the solution set of (4.1)–(4.2) by Λ.
Recall that the subdifferentials of a function $h:H\to\mathbb{R}$ at x is the set-valued operator on H defined by

$$\partial h(x):=\{z\in H: h(\bar{x})\geq h(x)+\langle z,\bar{x}-x\rangle\ \forall\ \bar{x}\in H\}.$$

It is well known that $\partial\psi_1$ and $\partial\psi_2$ are maximal monotone operators. Also we know that $J_\lambda^{\partial\psi_i}=prox_{\lambda\psi_i}(i=1,2)$. The proximal operators $prox_{\lambda\psi_i}(i=1,2)$ of ψ_i with parameter $\lambda>0$ is defined by

$$prox_{\lambda\psi_i}(x)=\arg\min_{u\in H_i}\{\psi_i(u)+\frac{1}{2\lambda}||x-u||\}.$$

Lemma 4.1. ([44] Lemma 1.5, [45] Corollary 10) *Let $\varphi:H\to\mathbb{R}$ be a differentiable convex function and let $L>0$. Suppose that $\nabla\varphi$ is L-Lipschitz continuous. Then $\nabla\varphi$ is L^{-1}-inverse strongly monotone. In [44], the word cocoercive is used for inverse strongly monotone.*

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert spaces and $A:H_1\to H_2$ be a bounded linear operator. Let $\varphi_1:H_1\to H_1$ be a differentiable convex function with $\frac{1}{\mu}$-Lipschitz continuous gradient and $\varphi_2:H_2\to H_2$ be a differentiable convex function with $\frac{1}{\nu}$-Lipschitz continuous gradient. Let $\psi_1:H_1\to 2^{H_1}$ and $\psi_2:H_2\to 2^{H_2}$ be convex lower semicontinuous functions. Let $S,T:H_1\to P(H_1)$ be two strictly pseudocontractive-type mappings with contractive coefficients κ_1 and κ_2 such that $F_s(S)\cap F_s(T)\cap\Lambda\neq\emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0\in H_1$ by*

$$\begin{cases} w_n=(1-\alpha_n)x_n\\ y_n=prox_{\lambda\psi_1}(I-\lambda\nabla\varphi_1)(w_n+\gamma A^*(prox_{\lambda\psi_2}(I-\lambda\nabla\varphi_2)-I)Aw_n)\\ x_{n+1}=\beta_n y_n+(1-\beta_n)[\rho_n v_n+(1-\rho_n)u_n],\ \forall n\geq 0\end{cases} \tag{4.3}$$

where $v_n\in Ty_n$ and $u_n\in Sy_n$, $0<\lambda<2\mu,2\nu$ and $\gamma\in(0,\frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}_{n=1}^{\infty}$, $\{\rho_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying the following conditions

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\beta_n \geq \max\{\kappa_1, \kappa_2\}$ $\forall n \geq 0$,

(iii) $\liminf_{n\to\infty}(1-\beta_n)(1-\rho_n)(\beta_n-\kappa_1) > 0$,

(iv) $\liminf_{n\to\infty}(1-\beta_n)(\beta_n-\kappa_2)\rho_n > 0$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F_s(S) \cap F_s(T) \cap \Omega$.

Proof. Let $f_1 = \nabla\varphi_1$, $f_2 = \nabla\varphi_2$, $g_1 = \partial\psi_1$ and $g_2 = \partial\psi_2$. Then the conclusion follows from Theorem 3.1. ∎

4.2. Split Variational Inequality Problem

Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two inverse strongly monotone operators and $A : H_1 \to H_2$ a bounded linear operator. Suppose C and Q are nonempty, closed and convex subsets of H_1 and H_2 respectively. We consider the following Split Variational Inequality Problem (SVIP):

$$\text{Find a point } x^* \in C \text{ such that } \langle f_1(x^*), x - x^*\rangle \geq 0 \ \forall\, x \in C, \tag{4.4}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } \langle f_2(y^*), y - y^*\rangle \geq 0 \ \forall\, y \in Q. \tag{4.5}$$

Let Θ denote the solution set of (4.4)–(4.5).
If considered alone, (4.4) is the classical variational inequality problem with solution set $VI(C, f_1)$, (see [46–49]) for details and recent results.
Let D be a nonempty, closed and convex subset of a real Hilbert space H. The normal cone of D at the point $x \in D$ is defined by

$$N_D(x) := \{d \in H : \langle d, y - x\rangle \leq 0, \forall y \in D\}. \tag{4.6}$$

By means of normal cones, (4.4)–(4.5) can be written as

$$\text{find a point } x^* \in C \text{ such that } 0 \in B_1(x^*) + N_C(x^*), \tag{4.7}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } 0 \in B_2(x^*) + N_Q(x^*). \tag{4.8}$$

It is well-known that the normal cone of a nonempty closed convex set is a maximal monotone operator (since, it is equal to the subdifferential of its indication function), then by applying Theorem 3.1 with $B_1 = N_C$ and $B_2 = N_Q$, we obtain a strong convergence result for approximating a point of $F_s(S) \cap F_s(T) \cap \Theta$.

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