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Stability of Some Iteration Schemes in Cone Banach Spaces

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Abstract In this paper, we discuss some stability theorems for mappings in cone Banach Spaces. Let X be a cone Banach space with normal cone P and T be a self mapping on X satisfying some specific condition. Here we derive different conditions under which the Mann iteration and Picard-Mann hybrid iteration schemes are stable in cone Banach space X. The stability of another iteration process converging to a common fixed point of two self mappings S and T on X is also discussed here.

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1. Introduction

In 2007, Huang Long-Guang, Zhang Xian [1] introduced cone metric spaces, and derived some fixed point theorems of contractive mappings on such spaces. In 2008, Mujahid Abbas, B.E. Rhoades [2] derived common fixed point theorems for mappings without using commutativity conditions and periodic point results in cone metric spaces. In 2009, Asadi et al. [3] investigated the T-stability of Picard's iteration procedures in cone metric spaces with some applications. Since then a number of researchers viz., T. Abdeljawad et al. [4], Sh. Rezapour and R. Hamlbarani [5], Radonvic [6] etc., developed the fixed point theory in cone metric spaces. In 2009, Duran Turkoglu et al. [7] defined cone normed spaces, and Erdal Karapinar [8] proved some fixed point theorems in Cone Banach Spaces.

A subset P_1 of a real Banach space E_1 is called a cone if and only if

(i) P_1 is closed, nonempty and $P_1 \neq \{0\}$.
(ii) If $\alpha, \beta \in \mathbb{R}$; $\alpha, \beta \geqslant 0$ and $x, y \in P_1$, then $\alpha x + \beta y \in P_1$.
(iii) If both $x \in P_1$ and $-x \in P_1$ then $x = 0$.

For a given a cone P_1 in E_1, a partial ordering " $\leqslant$ " on E_1 with respect to P_1 is defined by $x \leqslant y$ if and only if $y - x \in P_1$. We write $x < y$ to indicate that $x \leqslant y$ but $x \neq y$, while $x << y$ will stand for $y - x \in int(P_1)$.

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Cone metric space: [1] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E_1$ satisfies:

(i) $d(x, y) \geqslant 0$ for all $x, y \in X$,
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$ and
(iv) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$

then the pair (X, d) is called a cone metric space.

Cone normed space: [7] Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|.\|_P : X \to E_1$ satisfies:

(i) $\|x\|_P > 0$ for all $x \in X$,
(ii) $\|x\|_P = 0$ if and only if $x = 0$,
(iii) $\|kx\|_P = |k|\|x\|_P$ for all $k \in \mathbb{R}$, $x \in X$ and
(iv) $\|x + y\|_P \leqslant \|x\|_P + \|y\|_P$ for all $x, y, z \in X$

then the pair $(X, \|.\|_P)$ is called a cone normed space.

Normal cone: [1] A cone P_1 is called a normal cone if there is a number $K > 0$ s.t. $\forall x, y \in E_1$

$$0 \leqslant x \leqslant y \Rightarrow \|x\| \leqslant K\|y\|.$$

Regular Cone: [1] The cone P_1 is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant y$$

for some $y \in E_1$, then there is $x \in E_1$ such that $\|x_n - x\| \to 0$ *as* $n \to \infty$.

Example 1.1. Let $X = \mathbb{R}$, $E = (\mathbb{R}, \|.\|)$, $P = \{x : x \geqslant 0\}$. Then $\|.\|_P : X \to E_1$ defined by $\|x\|_P = |x|$ is a cone norm on $\mathbb{R}$. Clearly, $(\mathbb{R}, \|.\|_P)$ is a cone Banach space.

In [9], Qing and Rhoades derived a result regarding the T-stability of Picard's iteration in metric spaces. Asadi et al. [3] generalized their results in cone metric spaces.

An iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable w.r.t. T, if $\{x_n\}$ converges to a fixed point q of T and whenever $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$, we have $\lim_{n\to\infty} y_n = q$.

Lemma 1.2 ([3])**.** *Let P be a normal cone with constant K, and let $\{a_n\}$ and $\{b_n\}$ be sequences in E satisfying the following inequality:*

$$a_{n+1} \leqslant h a_n + b_n,$$

where $h \in (0, 1)$ and $b_n \to 0$ as $n \to \infty$. Then $a_n \to 0$ as $n \to \infty$.

Theorem 1.3 ([3])**.** *Let (X, d) be a cone metric space. P a normal cone, and T be a self-mapping on X having a fixed point q. If there exist numbers $a \geqslant 0$ and $0 \leqslant b < 1$, s.t.*

$$d(Tx, q) \leqslant a(x, Tx) + bd(x, q)$$

for each $x \in X$, and in addition, whenever $\{y_n\}$ is a sequence with $d(y_n, Ty_n) \to 0$ as $n \to \infty$, then Picard's iteration is T-stable.

In [2], we have the following fixed point theorem in a complete cone metric space.

Theorem 1.4 ([2]). *Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose T be a self-mapping on X satisfying*

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)] \tag{1.1}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a unique fixed point q in X.

For $x_0 \in X$, we have the Mann iteration scheme ([10]) as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \ \forall\ n = 0, 1, 2, ...$$

where $\{\alpha_n\}_0^\infty$ is a real sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The Mann iteration scheme is said to be stable w.r.t T [11], if $\{x_n\}$ converges to the unique fixed point q of T, and whenever $\{y_n\}$ is a sequence in X with

$$\lim_{n\to\infty} (\|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ty_n\|_p = \delta_n) = 0 \text{ then } \lim_{n\to\infty} y_n = q \tag{1.2}$$

In 2013, Khan [12] introduced Picard-Mann hybrid iteration scheme as:
For $x_0 \in X$

$$x_{n+1} = Ty_n$$
$$y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n \ \forall\ n = 0, 1, 2, ...$$

where $\{\alpha_n\}_0^\infty$ is a real sequence in $[0, 1]$.

The Picard-Mann iteration scheme is said to be stable [12] w.r.t T, if $\{x_n\}$ converges to the unique fixed point q of T, and whenever $\{z_n\}$ and $\{y_n\}$ are real sequences in X with

$$\lim_{n\to\infty} (\|z_{n+1} - Ty_n\|_p = \psi_n) = 0 \text{ then } \lim_{n\to\infty} z_n = q \tag{1.3}$$

where

$$z_{n+1} = Ty_n$$
$$y_n = (1 - \alpha_n)z_n + \alpha_n Tz_n \ \forall\ n = 0, 1, 2, ...$$

For $x_0 \in X$, we have the Ishikawa iteration scheme ([13]) as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tx_n \ \forall\ n = 0, 1, 2, ...$$

satisfying the conditions $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1 \ \forall n$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n = 0$.

The Ishikawa iteration scheme is said to be stable [14] w.r.t T, if $\{x_n\}$ converges to the unique fixed point q of T, and whenever $\{z_n\}$ and $\{y_n\}$ are real sequences in X with

$$\lim_{n\to\infty} (\|z_{n+1} - (1 - \alpha_n)z_n - \alpha_n Ty_n\|_p = \phi_n) = 0 \text{ then } \lim_{n\to\infty} z_n = q \tag{1.4}$$

where

$$y_n = (1 - \beta_n)z_n + \beta_n Tz_n \ \forall\ n = 0, 1, 2, ...$$

Here, we discuss the stability of the above iteration schemes w.r.t T in a cone Banach space X with a normal cone P (with normal constant 1).
[We have, every cone normed spaces is a cone metric space with $d(x, y) = \|x - y\|_p$].

2. Main Results

In all the following results we take $d(x, 0) = \|x\|_p$, where $x \in X$(the cone Banach space).

Theorem 2.1. *Let $(X, \|.\|_p)$ be a cone Banach space with the normal cone P(normal constant 1). Let T be self mapping on X satisfying the condition* (1.1)*. Then the Mann iteration scheme is stable w.r.t. T if for the real sequence $\{\alpha_n\}_0^\infty$ in $[0, 1]$, $0 < k \leqslant \alpha_n$, $\forall\, n = 0, 1, 2, ...$ and $\alpha + 2\beta + 4\gamma < 1$.*

Proof. Let $\{y_n\}$ be the sequence in X satisfying

$$\lim_{n\to\infty}(\|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|_p = \delta_n) = 0.$$

Let $\{x_n\}$ be the sequence in X converging to the unique fixed point q of T, so,

$$\lim_{n\to\infty}\|x_n - Tx_n\|_p = 0.$$

$$\begin{aligned}\|y_{n+1} - q\|_p &\leqslant \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|_p + \|(1-\alpha_n)y_n + \alpha_n Ty_n - Tq\|_p \\ &\leqslant \delta_n + \|(1-\alpha_n)(y_n - Tq) + \alpha_n(Ty_n - Tq)\|_p \\ &\leqslant \delta_n + (1-\alpha_n)\|y_n - q\|_p + \alpha_n(\alpha\|y_n - q\|_p + \beta[\|y_n - Ty_n\|_p + \|q - Tq\|_p] \\ &\qquad + \gamma[\|y_n - Tq\|_p + \|q - Ty_n\|_p]) \\ &\leqslant \delta_n + (1-\alpha_n)\|y_n - q\|_p + \alpha_n((\alpha + 2\gamma)\|y_n - q\|_p + (\beta+\gamma)\|y_n - Ty_n\|_p) \\ &= [(1-\alpha_n) + \alpha_n(\alpha + 2\gamma)]\|y_n - q\|_p + \delta_n + \alpha_n(\beta+\gamma)\|y_n - Ty_n\|_p \qquad (2.1)\end{aligned}$$

$$\begin{aligned}\|y_n - Ty_n\|_p &\leqslant \|y_n - x_{n+1}\|_p + \|x_{n+1} - Ty_n\|_p \\ &\leqslant \|y_n - x_{n+1}\|_p + \|(1-\alpha_n)x_n + \alpha_n Tx_n - Ty_n\|_p \\ &= \|y_n - x_{n+1}\|_p + \|(1-\alpha_n)(x_n - Ty_n) + \alpha_n(Tx_n - Ty_n)\|_p \\ &\leqslant \|y_n - x_{n+1}\|_p + (1-\alpha_n)[\|x_n - y_n\|_p + \|y_n - Ty_n\|_p] + \alpha_n(\alpha\|x_n - y_n\|_p \\ &\quad + \beta[\|x_n - Tx_n\|_p + \|y_n - Ty_n\|_p] + \gamma[\|x_n - Ty_n\|_p + \|y_n - Tx_n\|_p])\end{aligned}$$

$$\begin{aligned}\Rightarrow \|y_n - Ty_n\|_p \leqslant\ & \frac{1}{\alpha_n(1-\beta-\gamma)}[\|y_n - q\|_p + \|x_{n+1} - q\|_p] \\ &+ \frac{1-\alpha_n + (\alpha+2\gamma)\alpha_n}{\alpha_n(1-\beta-\gamma)}[\|y_n - q\|_p + \|x_n - q\|_p] \\ &+ \frac{\alpha_n(\beta+\gamma)}{\alpha_n(1-\beta-\gamma)}\|x_n - Tx_n\|_p. \qquad (2.2)\end{aligned}$$

From (2.1) and (2.2) we get,

$$\begin{aligned}
\|y_{n+1}-q\|_p &\leqslant \frac{[1-(\beta+\gamma)][1-\alpha_n+\alpha_n(\alpha+2\gamma)]+[\beta+\gamma][2-\alpha_n+\alpha_n(\alpha+2\gamma)]}{1-\beta-\gamma}\|y_n-q\|_p+\delta_n \\
&\quad + \frac{\beta+\gamma}{1-\beta-\gamma}\|x_{n+1}-q\|_p + \frac{[\beta+\gamma][1-\alpha_n+(\alpha+2\gamma)\alpha_n]}{1-\beta-\gamma}\|x_n-q\|_p \\
&\quad + \frac{\alpha_n(\beta+\gamma)^2}{1-\beta-\gamma}\|x_n-Tx_n\|_p \\
&= \frac{1-\alpha_n+\alpha_n(\alpha+2\gamma)+\beta+\gamma}{1-\beta-\gamma}\|y_n-q\|_p+\delta_n \\
&\quad + \frac{\beta+\gamma}{1-\beta-\gamma}\|x_{n+1}-q\|_p + \frac{[\beta+\gamma][1-\alpha_n+(\alpha+2\gamma)\alpha_n]}{1-\beta-\gamma}\|x_n-q\|_p \\
&\quad + \frac{\alpha_n(\beta+\gamma)^2}{1-\beta-\gamma}\|x_n-Tx_n\|_p.
\end{aligned}$$

We take $a_n = \|y_n-q\|_p$, $h = \dfrac{1-\alpha_n+\alpha_n(\alpha+2\gamma)+\beta+\gamma}{1-\beta-\gamma}$ and

$$\begin{aligned}
b_n &= \delta_n + \frac{\beta+\gamma}{1-\beta-\gamma}\|x_{n+1}-q\|_p + \frac{[\beta+\gamma][1-\alpha_n+(\alpha+2\gamma)\alpha_n]}{1-\beta-\gamma}\|x_n-q\|_p \\
&\quad + \frac{\alpha_n(\beta+\gamma)^2}{1-\beta-\gamma}\|x_n-Tx_n\|_p.
\end{aligned}$$

Hence $b_n \to 0$ as $n\to\infty$. Also,

$$\begin{aligned}
0 &< (1-\alpha_n)+\alpha_n(\alpha+2\gamma)+2\beta+2\gamma \\
&< 1-k(1-\alpha-2\gamma)+2\beta+2\gamma \\
&= 1-(k-(\alpha k+2\beta+2(1+k)\gamma)) \\
&< 1 \text{ [Since maximum value of } k \text{ is 1 and } \alpha+2\beta+4\gamma<1]
\end{aligned}$$

$$\Rightarrow h = \frac{1-\alpha_n+\alpha_n(\alpha+2\gamma)+\beta+\gamma}{1-\beta-\gamma} < 1$$

so, by the Lemma 1.2, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\|y_{n+1}-q\|_p = 0$. Hence the given iteration is stable w.r.t. T.

We can also show that if $y_n \to q$ then $\delta_n \to 0$ *as* $n \to \infty$.

$$\begin{aligned}
\delta_n &= \|y_{n+1}-(1-\alpha_n)y_n-\alpha_n Ty_n\|_p \\
&\leqslant \|y_{n+1}-q\|_p + \|q-(1-\alpha_n)y_n-\alpha_n Ty_n\|_p \\
&\leqslant \|y_{n+1}-q\|_p + (1-\alpha_n)\|y_n-q\|_p + \alpha_n\|Ty_n-Tq\|_p
\end{aligned}$$

$$\begin{aligned}
\|Ty_n-Tq\|_p &\leqslant \alpha\|y_n-q\|_p + \beta[\|y_n-Ty_n\|_p+\|q-Tq\|_p] + \gamma[\|y_n-Tq\|_p+\|Ty_n-q\|_p] \\
&\leqslant \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\|y_n-q\|_p.
\end{aligned}$$

Clearly, $\delta_n \to 0$ *as* $n \to \infty$. ∎

In 2006, Rhoades and Soltuz [15], derived the following result.

Theorem 2.2. *Let X be a normed space and $T : X \to X$ a mapping. If for all $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, $\lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ then following are equivalent:*

(i) for $\{\alpha_n\} \subset (0,1)$, Mann iteration scheme is T stable.
(ii) for $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, Ishikawa iteration scheme is T stable.

For the normal cone P (with normal constant 1) with the above stated conditions we can also derive the similar type of result:

Corollary 2.3. *For the self mapping T on the cone Banach space X satisfying* (1.1), *if the real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the above conditions (Theorem* 2.2*) together with $0 < k \leqslant \alpha_n$, $\forall\, n = 0, 1, 2, \ldots$ and $\alpha + 2\beta + 4\gamma < 1$, then the Ishikawa iteration scheme is also T stable.*

Example 2.4. Let $X = [0,1]$, $E = (\mathbb{R}, \|.\|)$, $P = \{x : x \geqslant 0\}$. We take the same cone norm as in Example 1.1. Let $T : [0,1] \to [0,1]$ be defined by $Tx = \frac{x}{8}$, where $[0,1]$ has the usual norm.

$$\begin{aligned}
\|Tx - Ty\|_p &= \|\frac{x}{8} - \frac{y}{8}\|_p = 2\|\frac{y}{16} - \frac{x}{16}\|_p \\
&= 2[\|\frac{x}{16} - \frac{y}{16} + \frac{x}{128} - \frac{x}{16} + \frac{y}{16} - \frac{y}{128} + \frac{y}{16} - \frac{x}{128} + \frac{y}{128} - \frac{x}{16}\|_p] \\
&\leqslant \frac{1}{8}\|x - y\|_p + \frac{1}{8}[\|x - \frac{x}{8}\|_p + \|y - \frac{y}{8}\|_p] + \frac{1}{8}[\|y - \frac{x}{8}\|_p + \|x - \frac{y}{8}\|_p] \\
&= \frac{1}{8}\|x - y\|_p + \frac{1}{8}[\|x - Tx\|_p + \|y - Ty\|_p] + \frac{1}{8}[\|y - Tx\|_p + \|x - Ty\|_p].
\end{aligned}$$

Thus T satisfies (1.1) with $\alpha = \beta = \gamma = \frac{1}{8}$. Let $\alpha_n = \frac{1}{4}$, $y_n = \frac{1}{n+1}$, $\forall n = 1, 2, \ldots$ For $k \in (0, \frac{1}{4}]$ we have $k \leqslant \alpha_n$, $\forall n = 0, 1, 2, \ldots$, and $\alpha + 2\beta + 4\gamma < 1$.
Now, $q = 0$ is the unique fixed point of T and $\lim_{n\to\infty} y_n = 0$. Also we have,

$$\begin{aligned}
\delta_n &= \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T y_n\|_p \\
&= \|\frac{1}{n+2} - \frac{3}{4(n+1)} - \frac{1}{32(n+1)}\|_p \\
&\to 0 \ as\ n \to \infty.
\end{aligned}$$

Hence the Mann iteration scheme is T-stable for the cone Banach space X.

As the above theorem we can prove similar type of result for Picard-Mann hybrid iteration scheme:

Theorem 2.5. *Let $(X, \|.\|_p)$ be a cone Banach space with the normal cone P(normal constant 1). Let T be self mapping on X satisfying the condition* (1.1). *Then the Picard-Mann iteration scheme is stable w.r.t. T if $\alpha + 2\beta + 2\gamma < 1$ and*

$$\lim_{n\to\infty} \|z_n - Tz_n\|_p = 0.$$

Proof. Let $\{y_n\}$ and $\{z_n\}$ be real sequences in X with

$$\lim_{n\to\infty} (\|z_{n+1} - Ty_n\|_p = \psi_n) = 0.$$

We have, for the unique fixed point q of T, $\lim_{n\to\infty} x_n = q$. Now,

$$\|z_{n+1} - q\|_p \leqslant \|z_{n+1} - Ty_n\|_p + \|Ty_n - q\|_p \tag{2.3}$$

$$\begin{aligned}
\|Ty_n - Tq\|_p &\leqslant \alpha\|y_n - q\|_p + \beta[\|y_n - Ty_n\|_p + \|q - Tq\|_p] \\
&\quad + \gamma[\|y_n - Tq\|_p + \|q - Ty_n\|_p] \\
\Rightarrow \|Ty_n - Tq\|_p &\leqslant \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\|y_n - q\|_p \\
&= \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\|(1-\alpha_n)z_n + \alpha_n Tz_n - q\|_p \\
&\leqslant \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}[(1-\alpha_n)\|z_n - q\|_p + \alpha_n\|Tz_n - q\|_p] \\
&\leqslant \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}[(1-\alpha_n)\|z_n - q\|_p + \alpha_n\|z_n - q\|_p + \alpha_n\|Tz_n - z_n\|_p].
\end{aligned} \tag{2.4}$$

From (2.3) and (2.4),

$$\|z_{n+1} - q\|_p \leqslant \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\|z_n - q\|_p + \psi_n + \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\alpha_n\|Tz_n - z_n\|_p. \tag{2.5}$$

We take $a_n = \|z_n - q\|_p$, $h = \dfrac{\alpha+\beta+\gamma}{1-\beta-\gamma}$ and $b_n = \psi_n + \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\alpha_n\|Tz_n - z_n\|_p$. Now, $b_n \to 0$ as $n \to \infty$. Also, $0 < h < 1$. By the Lemma 1.2,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \|z_{n+1} - q\|_p = 0.$$

Hence the given iteration is stable w.r.t. T. ■

In [2], M. Abbas, B.E. Rhoades et al. established the following common fixed point theorem in cone metric spaces.

Theorem 2.6 ([2]). *Let (X, d) be a complete cone metric space, and P be a normal cone with normal constant K. Suppose that S and T are two self-mappings on X satisfying*

$$d(Sx, Ty) \leqslant \alpha d(x, y) + \beta[d(x, Sx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Sx)] \tag{2.6}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then S and T have a unique common fixed point q in X . Moreover, any fixed point of S is a fixed point of T, and conversely.

For $x_0 \in X$, we consider the following iteration scheme ([2]) converging the above common fixed point of S and T:

$$x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ n = 0, 1, 2, ... \tag{2.7}$$

Here, we discuss the stability of the above iteration scheme w.r.t. S and T in a cone Banach space X with a normal cone P (with normal constant 1). The above iteration scheme is said to be stable w.r.t. S and T, if $\{x_n\}$ converges to the unique fixed point q of S and T, and whenever $\{y_n\}$ is a sequence in X with

$$\lim_{n\to\infty} \|y_{2n+1} - Sy_{2n}\|_p = 0,\ \textit{and} \lim_{n\to\infty} \|y_{2n+2} - Ty_{2n+1}\|_p = 0$$

we have $\lim_{n\to\infty} y_n = q$.

Now, we derive a condition for stability of the iteration scheme (2.7).

Theorem 2.7. *Let $(X, \|.\|_p)$ be a cone Banach space with the normal cone P(normal constant 1) and S and T be self mappings on X satisfying the condition* (2.6)*. Then the iteration scheme* (2.7) *is stable w.r.t. S and T, if $\alpha + 2\beta + 2\gamma < 1$ and*

$$\lim_{n\to\infty} \|y_{2n} - Sy_{2n}\|_p = 0 \text{ and } \lim_{n\to\infty} \|y_{2n+1} - Ty_{2n+1}\|_p = 0.$$

Proof. We have,

$$\begin{aligned}
\|y_{2n+1} - q\|_p &\leqslant \|y_{2n+1} - Sy_{2n}\|_p + \|Sy_{2n} - Tq\|_p \\
&\leqslant \|y_{2n+1} - Sy_{2n}\|_p + \alpha\|y_{2n} - q\|_p + \beta[\|y_{2n} - Sy_{2n}\|_p + \|q - q\|_p] \\
&\qquad + \gamma[\|y_{2n} - Tq\|_p + \|q - Sy_{2n}\|_p] \\
&\leqslant \|y_{2n+1} - Sy_{2n}\|_p + \alpha\|y_{2n} - q\|_p + \beta\|y_{2n} - Sy_{2n}\|_p \\
&\qquad + \gamma[\|y_{2n} - q\|_p + \|y_{2n} - q\|_p + \|y_{2n} - Sy_{2n}\|_p] \\
&= (\alpha + 2\gamma)\|y_{2n} - q\|_p + \|y_{2n+1} - Sy_{2n}\|_p + (\beta + \gamma)\|y_{2n} - Sy_{2n}\|_p
\end{aligned}$$

we take $a_n = \|y_n - q\|_p$ and $b_{2n} = \|y_{2n+1} - Sy_{2n}\|_p + (\beta + \gamma)\|y_{2n} - Sy_{2n}\|_p$.

Now, $\|y_{2n+1} - Sy_{2n}\|_p \to 0$ *as* $n \to \infty$ and $\|y_{2n} - Sy_{2n}\|_p \to 0$ (given condition). Hence $b_n \to 0$ as $n \to \infty$. Also, $\alpha + 2\gamma < 1$ so, by the Lemma 1.2, $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \|y_{2n+1} - q\|_p = 0$. Again,

$$\begin{aligned}
\|y_{2n+2} - q\|_p &\leqslant \|y_{2n+2} - Ty_{2n+1}\|_p + \|Ty_{2n+1} - Sq\|_p \\
&\leqslant \|y_{2n+2} - Ty_{2n+1}\|_p + \alpha\|y_{2n+1} - q\|_p + \beta\|y_{2n+1} - Ty_{2n+1}\|_p \\
&\qquad + \gamma[\|q - y_{2n+1}\|_p + \|y_{2n+1} - Ty_{2n+1}\|_p + \|y_{2n+1} - q\|_p] \\
&= (\alpha + 2\gamma)\|y_{2n+1} - q\|_p + \|y_{2n+2} - Ty_{2n+1}\|_p + (\beta + \gamma)\|y_{2n+1} - Ty_{2n+1}\|_p.
\end{aligned}$$

Let $b'_{2n} = \|y_{2n+2} - Ty_{2n+1}\|_p + (\beta + \gamma)\|y_{2n+1} - Ty_{2n+1}\|_p$.

Now, $\|y_{2n+2} - Ty_{2n+1}\|_p \to 0$ as $n \to \infty$ and $\|y_{2n+1} - Ty_{2n+1}\|_p \to 0$ (given condition). Hence, $\lim_{n\to\infty} a_{2n+2} = \lim_{n\to\infty} \|y_{2n+2} - q\|_p = 0$.
Therefore, $\lim_{n\to\infty} \|y_n - q\|_p = 0$. Hence the given iteration is stable w.r.t. S and T. ∎

Remark: In all the above results we have considered the normal cones with normal constant 1. It can be further investigated whether the similar result holds without assuming this condition. In view of the recent development of research, the analogous study can be done considering cone Banach spaces over Banach algebras with some possible practical applications.

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