Thai Journal of Mathematics
Volume 19 Number 2 (2021)
Pages 469–477

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

A Study of New Type of Minimal Open and Maximal Closed Sets

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Abstract In this paper, we show that the notions of maximal feebly open and minimal feebly closed sets are nothing except maximal α-open and minimal α-closed sets. Further, we show that the ideas of minimal feebly open and maximal feebly closed sets coincide with minimal open and maximal closed sets. Therefore, we introduce new notions of minimal open and maximal closed sets, namely f-minimal open and f-maximal closed sets, some of their properties, characterizations, and relationships established.

MSC: 54A05; 54B05; 54B10

Keywords: maximal and minimal open and closed sets; f-minimal open and f-maximal closed sets

Submission date: 08.04.2016 / Acceptance date: 15.06.2020

1. Introduction

In 1996, Rosen and Peters [1] used topology as a body of mathematics to unify diverse areas of computer-aided geometric design and engineering design, while El Naschie [2] derived quantum gravity from set theory, and so topology plays an important role in so many areas of applications, for example, in quantum physics [2–4], in Image analysis [5, 6], and in Biology (DNA) [7–9]. The notions of semi-open and α-open sets were introduced by Levine [10] in 1963, and Njasted [11] in 1965, respectively. The Study of feebly open sets was initiated by Maheshwari and Tapi [12] in 1978. However, Jankovic and Reilly [13] proved that the class of feebly open sets is exactly the class of α-open sets. Nakaoka and Oda [14, 15] introduced and studied the idea of minimal and maximal open sets. Thereafter, dualizing those concepts they introduced the concepts of minimal and maximal closed sets [16]. In 2013, Shakir [17] gave the concepts of minimal and maximal f-open (f-closed) sets. While, Hasan [18] in 2015, defined and studied the notions of maximal α-open and minimal α-closed sets. In the present paper, we prove that the concepts of maximal α-open and maximal f-open coincide as well as minimal α-closed and minimal f-closed sets. Also, we show that the concepts of minimal f-open and maximal f-closed sets coincide with the concepts of minimal open and maximal closed sets, respectively. Therefore, we introduce a new type of minimal open and maximal closed sets. Also, we state and prove some of their properties, characterizations, and

Published by The Mathematical Association of Thailand.

relationships. In what follows, by a space X we mean a topological space (X, τ) on which no separation axioms are assumed unless explicitly stated, and for a subset A of X , we denote by $ClA(Cl_X A)$ and $IntA(Int_X A)$ the closure and interior of A in X, respectively. A subset A of a space X is said to be semi-open [10] (resp., α-open [11]) if there exists an open set G (resp., open set G and a nowhere dense set N) such that $G \subseteq A \subseteq ClG$ (resp., $A = G - N$), equivalently if $A \subseteq IntClA$ (resp., $A \subseteq IntClIntA$). The complement of semi-open (resp., α-open) is said to be semi-closed (resp., α-closed). The intersection (union) of all semi-closed (semi-open) subsets of a space X contains (contained in) $A \subseteq X$ is the semi-closure (semi-interior) [19] of A which is denoted by $sCl_X A$ ($sInt_X A$), we may do not subscribe X when there is no possibility of confusion. A subset A of a space X is said to be feebly open [12], if there exists an open set G such that $G \subseteq A \subseteq sClG$.
The authors of [13] established:

Theorem 1.1. *For a subset A of a space X, the following statements are true:*

(1) A is feebly open if and only if it is α-open.
(2) $sClA = A \bigcup IntClA$ and $sIntA = A \bigcap ClIntA$.

Definition 1.2. A non-empty proper open (resp., feebly open) subset A of a space X is said to be a minimal open [14] (resp., minimal f-open [17]) set, if any open (resp., feebly open) subset of X contained in A is either $\emptyset$ or A.

The complement of a minimal open (resp., minimal f-open) set is called maximal closed [16] (resp., maximal f-closed [17]).

Definition 1.3. A non-empty proper open (resp., feebly open , α-open) subset A of a space X is said to be a maximal open [15] (resp., maximal f-open [17], maximal α-open [18]) set if any open (resp., feebly open, α-open) subset of X contains A is either A or X.

The complement of maximal open (resp., maximal f-open , maximal α-open) sets are called minimal closedb[16] (resp., minimal f-closed [17], minimal α-closed [18]) sets.

Proposition 1.4. *For a non-empty proper subset A of a space X, we have:*

(1) A is maximal f-open if and only if it is maximal α-open.
(2) A is minimal f-closed if and only if it is minimal α-closed.

Proof. Both followed by part (1) of Theorem 1.1. ∎

Because of Proposition 1.4, we notice that almost all results of section 3 of [18] are the same as the results in section 2 of [17]. However, in the next result, we show that the concepts of minimal open sets and minimal f-open sets coincide as well as the concepts of maximal closed sets and maximal f-closed sets.

Theorem 1.5. *For a non-empty proper subset A of a space X, we have:*

(1) A is minimal open if and only if it is minimal f-open.
(2) A is maximal closed if and only if it is maximal f-closed.

Proof. (1)
($\Rightarrow$) Let A be a minimal open subset of X. Then, A is feebly open. To show A is minimal f-open, we suppose that U is a feebly open subset of X such that $U \subsetneq A$. Then, either $IntU = \emptyset$ or $IntU \neq \emptyset$. If $IntU = \emptyset$, then $IntClIntU = \emptyset$, therefore, $U = \emptyset$. If $IntU \neq \emptyset$, then $IntU$ is a non-empty open subset of a minimal open set A, so $IntU = A$. Thus, $A = IntU \subseteq U \subseteq A$. Hence, $A = U$. This means that A is minimal f-open.
($\Leftarrow$) Let A be a minimal f-open subset of X. Then, A is feebly open. To show A is minimal open, Let G be any open subset of X such that $G \subsetneq A$. Since every open set is feebly open, so either $G = \emptyset$ or $G = A$. It remains only to show A is open. Since $IntA$ is an open set contained in A, so by what we have done for G, we have either $IntA = \emptyset$ or $IntA = A$, but, since A is feebly open and $A \neq \emptyset$, then $IntA = A$. Thus A is open, and hence it is a minimal open set.

(2) A is maximal f-closed if and only if $X - A$ is minimal f-open if and only if $X - A$ is minimal open { by part (1) of this theorem} if and only if A is maximal closed. ∎

2. f-Minimal Open and f-Maximal Closed Sets

In Theorem 1.5, we proved that the spaces (X, τ) and (X, τ^α) have the same class of minimal open sets and maximal closed sets. This equivalence between minimal f-open sets and minimal open sets leads us to introduce and study a new type of minimal open and maximal closed sets, namely f-minimal open and f-maximal closed sets.

Definition 2.1. A non-empty proper subset A of a space X is said to be an f-minimal open set, if there exists a minimal open sets G such that $G \subseteq A \subseteq sClG$. The complement of an f-minimal open set is called an f-maximal closed set.

Remark 2.2. It is evident that:

(1) Every minimal open set is an f-minimal open set.
(2) Every f-minimal open set is an α-open ($\approxeq feebly - open$) set.

But the converse of neither parts of the above remark is true, as we show by the following examples:

Example 2.3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the set $\{a, b\}$ is an f-minimal open set, but it is not minimal open.

Example 2.4. In the usual real line topological space $(\mathcal{R}, \Im_U)$, every open set is α-open ($\approxeq feebly - open$), but this space has no minimal open sets, so that, it has no f-minimal open sets.

Theorem 2.5. *A subset A of a space X is f-minimal open if and only if there exists a minimal open set G in X such that $G \subseteq A \subseteq IntClG$.*

Proof. It follows from Definition 2.1 and part (2) of Theorem 1.1. ∎

Theorem 2.6. *A subset A of a Hausdorff space X is f-minimal open if and only if minimal open.*

Proof. Let A be an f-minimal open subset of a Hausdorff space X. Then, there exists a minimal open subset G of X such that $G \subseteq A \subseteq sClG$. Since $\emptyset \neq G$, so there exists a point $x \in G$. Now, if y is any point of $X \setminus G$, there exist disjoint open sets U_x and V_y such that $x \in U_x$ and $y \in V_y$. By [14, part 2 of Lemma 2.2], we have $G \subseteq U_x$. Therefore, $G \bigcap V_y = \emptyset$.This implies that $y \notin ClG$, so $y \notin sClG$. Hence, $A = G$ which is minimal open.
Conversely; it follows from part (1) of Remark 2.2. ■

Theorem 2.7. *A subset A of a space X is f-maximal closed if and only if there exists a maximal closed sets H in X such that $sIntH \subseteq A \subseteq H$.*

Proof. (⇒) Let A be an f-maximal closed subset of X. Then, $X - A$ is f-minimal open, so there is a minimal open set G such that $G \subseteq (X - A) \subseteq sClG$. Therefore, $sInt(X - G) = X - sClG \subseteq A \subseteq X - G$. By Definition 1.2, the set $H = X - G$ is the required maximal closed set.
(⇐) Let F be a maximal closed subset of X such that $sIntH \subseteq A \subseteq H$. By Definition 1.2, $X - H$ is a minimal open subset of X such that $X - H \subseteq X - A \subseteq X - sIntH = sCl(X - H)$. Therefore, $X - A$ is f-minimal open, and hence A is f-maximal closed. ■

Proposition 2.8. *If A is an f-minimal open subset of X and B a subset of X such that $A \subseteq B \subseteq sClA$, then B is f-minimal open.*

Proof. Let G be a minimal open subset of X such that $G \subseteq A \subseteq sClG$. Then, $sClA = sClB$, So $G \subseteq A \subseteq B \subseteq sClA \subseteq sClG$. Hence, B is an f-minimal open subset of X. ■

Corollary 2.9. *The semi-closure of any minimal open (resp, f-minimal open) is f-minimal open.*

The following example shows that Proposition 2.8 and Corollary 2.9 may fail if we replace $sClA$ with ClA.

Example 2.10. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Then, the set $A = \{b\}$ is minimal open (whence, f-minimal open) and $A \subseteq B \subseteq ClA$, where $B = ClA = \{b, c, d\}$ is not f-minimal open.

It is easy to see that the only f-minimal open subsets of X in Example 2.10 are the sets $A = \{a\}$, $B = \{b\}$ and $C = \{b, c\}$, but neither $A \bigcup B = \{a, b\}$ nor $A \bigcap B = \emptyset$ is f-minimal open. So, it is natural to ask (under what condition the intersection (union) of two f-minimal open sets is also f-minimal open?) The answer is in the following results:

Theorem 2.11. *The intersection of two f-minimal open subsets A and B of a space X is not f-minimal open if and only if $A \bigcap B = \emptyset$.*

Proof. (⇒) Let A and B be two f-minimal open subsets of X such that $A \bigcap B$ is not f-minimal open. Then, there exist two minimal open sets G and U such that $G \subseteq A \subseteq sClG$ and $U \subseteq B \subseteq sClU$. By [14, part 2 of Lemma 2.2], either $G \bigcap U = \emptyset$ or $G = U$. If $G = U$, then $sClG = sClU$. Thus, $G = G \bigcap U \subseteq A \bigcap B \subseteq sClG \bigcap sClU = sClG$. This means that $A \bigcap B$ is f-minimal open which is impossible. Hence, $G \bigcap U = \emptyset$. Then $IntClG \bigcap IntClU = \emptyset$, so by part 2 of Theorem 1.1 $sClG \bigcap sClU = \emptyset$. Therefore, $\emptyset = G \bigcap U \subseteq A \bigcap B \subseteq sClG \bigcap sClU = \emptyset$. Hence, $A \bigcap B = \emptyset$.
(⇐) It is obvious. ■

Corollary 2.12. *The intersection of two f-minimal open sets A and B of a space X is f-minimal open if and only if $A \bigcap B \neq \emptyset$.*

Corollary 2.13. *Let A and B be two f-minimal open sets. Then:*

(1) $A \bigcap B$ is f-minimal open if and only if there exists a minimal open set G such that $G \subseteq A, B \subseteq sClG$.
(2) $A \bigcap B$ is not f-minimal open if and only if there exist disjoint minimal open sets G and U such that $G \subseteq A \subseteq sClG$ and $U \subseteq B \subseteq sClU$.

Theorem 2.14. *The union of two f-minimal open sets is f-minimal open if and only if their intersection is f-minimal open.*

Proof. Let A and B be two f-minimal open subsets of a space X.
($\Rightarrow$) Let $A \bigcup B$ be an f-minimal open set. Then, there exists a minimal open sets V such that $V \subseteq A \bigcup B \subseteq sClV$. To show $A \bigcap B$ is f-minimal open. If, we suppose that $A \bigcap B$ is not f-minimal open, then by Theorem 2.11 $A \bigcap B = \emptyset$ and by Corollary 2.13, there exists two disjoint minimal open sets G and U such that $G \subseteq A \subseteq sClG$ and $U \subseteq B \subseteq sClU$. Since $G \neq U$, then either $V = G$ or $V = U$. If $V = G$, then $G \subseteq A \subseteq A \bigcup B \subseteq sClG$. This implies that $G \bigcap U \neq \emptyset$ which is impossible. Again, if $V = U$, then we obtain $G \bigcap U \neq \emptyset$ which is against to $G \bigcap U = \emptyset$. Thus, $A \bigcap B$ is f-minimal open.
($\Leftarrow$) Let $A \bigcap B$ be an f-minimal open set. By Corollary 2.13, there exists a minimal open set G such that $G \subseteq A, B \subseteq sClG$. Therefore, $A \bigcup B$ is f-minimal open. ∎

Corollary 2.15. *For any two f-minimal open sets A and B of a space X, the following statements are equivalent:*

(1) $A \bigcap B$ is f-minimal open.
(2) $A \bigcup B$ is f-minimal open.
(3) $A \bigcap B \neq \emptyset$.
(4) There exists a minimal open set G such that $G \subseteq A \bigcap B \subseteq A \bigcup B \subseteq sClG$.

Since in Example 2.10, the intersection of the sets $A = \{a, b\}$ and $B = \{b, c\}$ is f-minimal open but neither A nor B is f-minimal open. This means that the f-minimal openness of $A \bigcap B$ does not imply that A or B is f-minimal open.

Proposition 2.16. *Let G and U be two minimal open subsets of a space X such that $G \subseteq A \subseteq sClG$. If $G \bigcap U = \emptyset$, then $A \bigcap U = \emptyset$*

Proof. Since G and U are minimal open sets, then A and U are f-minimal open sets. Since G and U are disjoint, then by Corollary 2.13 $A \bigcap U$ is not f-minimal open. So by Theorem 2.11, $A \bigcap U = \emptyset$. ∎

It is easy to prove the following lemma:

Lemma 2.17. *Let G be an open subset of a subspace Y of a space X, If $x \notin sCl_Y G$, then $x \notin sCl_X G$.*

Theorem 2.18. *Let A be a non-empty proper subset of a subspace Y of a space X. If A is f-minimal open in X, then it is f-minimal open in Y.*

Proof. Let A be f-minimal open in X. Then, there exists a minimal open set G in X such that $G \subseteq A \subseteq sCl_X G$. Since $G \subseteq A \subseteq Y$, then G is minimal open in Y. To show A is f-minimal open in Y, we have to show $A \subseteq sCl_Y G$. If $x \notin sCl_Y G$, then by Lemma 2.17 $x \notin sCl_X G$. Thus $x \notin A$, this means that $A \subseteq sCl_Y G$. Hence, A is f-minimal open in Y. ∎

By the following example, we show that if A is f-minimal open in Y, where Y is a subspace of X, it isn't necessary to be f-minimal open in X.

Example 2.19. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. The only f-minimal open subsets of X are $\{a\}, \{a, b\}, \{a, c\}$ and X. If $Y = \{b, c\}$, then $\tau_Y = \{\emptyset, \{b\}, \{c\}, Y\} = P(Y)$. So the set $A = \{b\}$ is f-minimal open in Y but it is not f-minimal open in X.

Theorem 2.20. *Let A be an f-minimal open subset of a space X. Then:*

(1) $A \subseteq sClS \subseteq ClS$, for each non-empty semi-open subset S of X contained in A.

(2) $ClA = ClS$ and $sClA = sClS$, for each non-empty semi-open subset S of X contained in A.

(3) $sClA = IntClA$.

Proof. (1) Let S be a non-empty semi-open subset of X contained in A. Then, $IntS \neq \emptyset$ and $IntS \subseteq A$. Since A is f-minimal open, then there exists a minimal open set G such that $G \subseteq A \subseteq sClG$. Since $IntS \bigcap G \subseteq G$, so by [14, part 1 of Lemma 2.2], either $IntS \bigcap G = \emptyset$ or $G \subseteq IntS$. If $IntS \bigcap G = \emptyset$, then $IntS \bigcap sClG = \emptyset$. Therefore, $IntS \bigcap A = \emptyset$, that is, $IntS = \emptyset$ which is impossible. Hence, $G \subseteq IntS$. Thus, $A \subseteq sClIntS \subseteq sClS \subseteq ClS$.
(2) It follows from the fact that $S \subseteq A$, $sClsClS = sClS$ and part 1 of this theorem.
(3) Since the minimal open set G which is contained in A, is a non-empty semi-open subset of X contained in A, then by part 2, we have $sClA = sClG = IntClG = IntClA$. ∎

Remark 2.21. (1) Since every open and every α-open ($\approxeq$ feebly open) sets are semi-open, so we can replace the semi-open in both parts of Theorem 2.20 by an open and α-open ($\approxeq$ feebly open) set.

(2) Since in Example 2.3, the set $A = \{a, b\}$ is f-minimal open and $\{b\} \subseteq A$ but $A \nsubseteq sCl\{b\}$. Therefore, we cannot replace the semi-open set with any non-empty subset in Theorem 2.20.

(3) Since each non-empty open subset of the cofinite topological space $(\mathcal{R}, \Im_{cof})$ satisfies each part of Theorem 2.20 but no subset of $(\mathcal{R}, \Im_{cof})$ is f-minimal open. This means that, neither part of Theorem 2.20 is equivalent to the notion of the f-minimal open set.

(4) Since the indiscrete space $(X, \Im_{ind})$, where $\Im_{ind} = \{\emptyset, X\}$ and X is finite set has no minimal open set, therefore, [14, Theorem 3.1] fails to hold. To correct it, it must be adding the condition that X has at least one non-empty proper open subset.

Proposition 2.22. *A non-empty proper subset A of a finite space X is f-minimal open if and only if $IntA \neq \emptyset$ and $A \subseteq sClO$, for each non-empty open subset O of X contained in A.*

Proof. (⇒) Let A be an f-minimal open set. Then, there exists a minimal open subset G of X such that $G \subseteq A \subseteq sClG$. Therefore, $\emptyset \neq G \subseteq IntA$. If O is any non-empty open subset of X contained in A, then by [14, part 1 of Lemma 2.2], we have $G \subseteq O$. Hence, $A \subseteq sClG \subseteq sClO$.
(⇐) Since $A \neq X$ and $IntA \neq \emptyset$, then $\emptyset \neq A \neq X$, that is, X has a non-empty proper open subset. Since X is finite, so by [14, Theorem 3.1] and part 4 of Remark 2.21, $IntA$ contains a minimal open subset of X, say G. By our hypothesis, $A \subseteq sClG$. Hence, A is f-minimal open. ■

It is easy to see that Theorem 2.22 is true if we replace the open set with $semi-open$ or α-open ($\approxeq$ feebly open) and $IntA \neq \emptyset$ with $sIntA \neq \emptyset$ or $\alpha IntA \neq \emptyset$.

Theorem 2.23. *If A_i is an f-minimal open set in a space X_i $(i = 1, 2)$, then $A_1 \times A_2$ is f-minimal open in $X = X_1 \times X_2$.*

Proof. Since A_i is f-minimal open in X_i $(i = 1, 2)$, then there is a minimal open set G_i in X_i such that $G_i \subseteq A_i \subseteq sCl_{X_i}G_i$ $(i = 1, 2)$. Therefore, $G_1 \times G_2 \subseteq A_1 \times A_2 \subseteq sCl_{X_1}G_1 \times sCl_{X_2}G_2 = sCl_X(G_1 \times G_2)$. Since $G_1 \times G_2$ is minimal open in X, then $A_1 \times A_2$ is f-minimal open in X. ■

It is easy to prove the following two Lemmas, so we omitted their proofs:

Lemma 2.24. *If $G = G_1 \times G_2$ is a minimal open subset of $X = X_1 \times X_2$, then G_1 is minimal open in X_1 or G_2 is minimal open in X_2*

Lemma 2.25. *If $X = X_1 \times X_2$ has a minimal open subset, then X_1 or X_2 has a minimal open subset.*

Theorem 2.26. *If $A = A_1 \times A_2$ is an f-minimal open subset of $X = X_1 \times X_2$, then A_1 is f-minimal open in X_1 or A_2 is f-minimal open in X_2.*

Proof. Let $A = A_1 \times A_2$ be an f-minimal open subset of X. Then, there exists a minimal open subset G of X such that $G \subseteq A \subseteq sCl_XG$. Since $G \neq \emptyset$, then there exists a point $(x_1, x_2) \in G$, and so there are open subsets V_i of X_i $(i = 1, 2)$ such that $(x_1, x_2) \in V_1 \times V_2 \subseteq G$. Since G is minimal open, then by [14, part 2 of Lemma 2.2] $G = V_1 \times V_2$, by Lemma 2.24 V_1 is minimal open in X_1 or V_2 is minimal open in X_2. Since $V_1 \times V_2 = G \subseteq A = A_1 \times A_2 \subseteq sCl_XG = sCl_XV_1 \times V_2 = sCl_{X_1}V_1 \times sCl_{X_2}V_2$. Its easy to see that $V_i \subseteq A_i \subseteq sCl_{X_i}V_i$ $(i = 1, 2)$. Thus, A_1 is f-minimal open in X_1 or A_2 is f-minimal open in X_2. ■

Theorem 2.27. *If $X = X_1 \times X_2$ has an f-minimal open subset, then X_1 or X_2 has an f-minimal open subset.*

Proof. Let A be an f-minimal open subset of X. Then, there exists a minimal open subset G of X such that $G \subseteq A \subseteq sClG$. So by Lemma 2.25, X_1 or X_2 has a minimal open subset. This implies that X_1 or X_2 has an f-minimal open subset. ■

The following example shows that the converse of Lemma 2.24, Lemma 2.25, Theorem 2.26 and Theorem 2.27 are not true in general:

Example 2.28. Let X_1 and X_2 be the spaces of Example 2.3 and Example 2.4, respectively. The set $A = \{a\}$ is minimal open (hence f-minimal open) in X_1, but $X_1 \times X_2$ has neither f-minimal open nor minimal open subset.

Remark 2.29. Since the non-empty proper subset $\{c\}$ of the space X in Example 2.3 is f-maximal closed but it is not f-maximal closed in the subspace $Y = \{b, c\}$ of X.This means that, Theorem 2.18 is not true for f-maximal closed sets. However, analogous to the other results, one can easily establish the dual results to f-maximal closed sets by using Theorem 2.7 and the facts that $X - sClA = sInt(X - A)$, $sCl(X - A) = X - sIntA$ and ($A \subseteq B$ if and only if $X - B \subseteq X - A$).

Acknowledgements

The author would like to thank the University of Sulaimani for its financial support. Also, I would like to thank the referees for their comments and suggestions on the manuscript to provide my results.

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