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Common Fixed Point Theorem for Minimal Commutativity Type Mappings in Fuzzy Metric Spaces

(In the memory of his grand father late Pt. Bhim Singh)

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Abstract : In this paper, we introduce the concept of R-weak commutative of type (Ag) mappings in fuzzy metric spaces akin to to concept of R-weak commutative of type (Ag) mappings in metric spaces, introduced by Pathak *et al.* [16] and prove common fixed point theorems for weakly compatible mappings, R-weak commutative mappings and R-weak commutative of type (Ag) mappings (we call all these mappings as minimal commutativity type mappings) which generalizes the Results of various authors present in literature.

Keywords : Weakly compatible mappings, R-weak commutative mappings, R-weak commutative of type (Ag) mappings, Non compatible mappings, Lipschitz type mappings.
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1 Introduction

The notion of a probabilistic metric space corresponds to the situation when we do not know the distance between the points but know only probabilities of possible value of this distance. Since the 16th century, probability theory has been studying a kind of uncertainty randomness, that is, the uncertainty of the occurrence of an event; but in this case, the event itself is completely certain and the only uncertain thing is whether the event will occur or not and the causality is not clearly known.

Following the study on certainty and on randomness, the study of mathematics began to explore the restricted zone - fuzziness. Fuzziness is a kind of uncertainty i.e., for some events, it cannot be completely determined that in which cases these events should be subordinated to, (they have already occurred or not yet), they are in non-black or non-white state. We can say that the law of excluded middle in logic cannot be applied any more. Zadeh [26] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. A fuzzy set A in X is a function with domain X and values in $[0, 1]$. Since then, many authors regarding the theory of fuzzy sets and its applications have developed a lot of literatures. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable.

There are many viewpoints of the notion of the metric space in fuzzy topology. We can divide them into following two groups:

The first group involves those results in which a fuzzy metric on a set X is treated as a map $d : X \times X \to R^+$ where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.

Definition 1.1. *A t-norm is a 2-place function* $*: [0,1] \to [0,1]$ *satisfying the following*

(i) $*(0,0) = 0$,

(ii) $*(0,1) = 1$,

(iii) $*(a,b) = *(b,a)$,

(iv) if $a \le c$, $b \le d$, *then* $*(a,b) \le *(c,d)$,

(v) $*(*(a,b),c) = *(a,*(b,c))$ *for all* a,b,c *in* $[0,1]$.

Example 1.2. (i) $*(a,b) = ab$;

(ii) $*(a,b) = \max(a+b-1,0)$;

(iii) $*(a,b) = \min(a,b)$.

In 1975, Kramosil and Michalek [13] introduced the concept of fuzzy metric spaces as follows:

Definition 1.3. *The 3-tuple* $(X,M,*)$ *is called a fuzzy metric space (shortly, FM-space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set in* $X^2 \times [0,\infty)$ *satisfying the following conditions:*

(FM-1) $M(x,y,0) = 0$,

(FM-2) $M(x,y,t) = 1$, *for all* $t > 0$ *if and only if* $x = y$,

(FM-3) $M(x,y,t) = M(y,x,t)$,

(FM-4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$ *and*

(FM-5) $M(x,y,\cdot) : [0,1) \to [0,1]$ *is left continuous for all* $x,y,z \in X$ *and* $s,t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with $t = 0$. Since $*$ is a continuous t-norm, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. In 1994, George and Veermani [6] introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric space induces a fuzzy metric space.

We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way:

Let (X, d) be a metric space. Define $a * b = ab$ for all x, y in X and $t > 0$.

Define $M(x, y, t) = t/(t + d(x, y))$ for all x, y in X and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric.

We consider M be a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all x, y in X and $t > 0$.

Definition 1.4. *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be*

(i) a convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$),

$$\text{if } \lim_{n\to\infty} M(x_n, x, t) = 1, \text{ for all } t > 0,$$

(ii) a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \text{ for all } t > 0 \text{ and } p > 0.$$

(iii) a complete fuzzy metric space in which every Cauchy sequence converges to a point in it.

Especially, Erceg [5], Kaleva and Seikkala [12], Kramosil and Michalek [13] have introduced the concept of fuzzy metric spaces in different ways. Grabiec's [7] followed Kramosli and Michalek and obtained the fuzzy version of Banach contraction principle. Grabiec results [7] were further generalized by Subramanyam [23] for a pair of commuting mappings. In fact, Mishra [15], Chugh and Kumar [1], Chugh et al. *[2] introduced the notions of compatible,compatible of type (A) and compatible of type (P) mappings respectively in fuzzy metric spaces and proved some common fixed point theorems for these maps.*

Recently, several authors [5-7, 12, 22-25] proved fixed-point theorems for fuzzy metric spaces in different ways.

Two mappings f *and* g *of a fuzzy metric space* $(X, M, *)$ *into itself are said to be weakly commuting if*

$$M(fgx, gfx, t) \geq M(fx, gx, t) \quad \text{for each } \; x \text{ in } X.$$

Further, Mishra [15] introduced more generalized commutativity, so called compatibility. Let f *and* g *be self-mappings of a fuzzy metric space* $(X, M, \cdot)$.

The mappings f *and* g *are said to be compatible if* $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, *whenever* $\{x_n\}_{n=1}^{\infty}$ *is a sequence in* X *such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ *for some* p *in* X *and for each* $t > 0$, *which is more general than that of weak commutativity. Commuting, weakly commuting mappings are compatible but neither implication is reversible, see Jungck [9].*

Remark 1.5. *Compatible maps are not necessarily weakly commuting mappings.*

Let X be the set $\{1/n : n \in N\} \cup \{0\}$ with the metric d defined by

$$d(x, y) = |x - y|.$$

For each $t \in (0, \infty)$ define

$$M(x, y, t) = \frac{t}{t + |x - y|} \quad \text{for } x, y \in X \text{ and } \; M(x, y, 0) = 0\,.$$

Clearly, $(X, M, *)$ is a complete fuzzy metric on X where $*$ is defined by $a * b = ab$.

Define $Ax = x^3$ and $Bx = 2 - x$ for all $x \in X$ and for some $t > 0$.

Since

$$M(Ax_n, Bx_n, t) = \frac{t}{t + |Ax_n - Bx_n|} = \frac{t}{t + |(x_n - 1)|\,|(x_n^2 + x_n + 2)|} \to 1 \text{ iff } \; x_n \to 1\,,$$
$$M(BAx_n, ABx_n, t) = \frac{t}{t + 6|(x_n - 1)^2|} \to 1 \quad \text{iff } \; x_n \to 1$$

Thus, A and B are compatible. But they are not weakly commuting mappings since

$$M(ABx, BAx, t) = \frac{t}{t + |ABx - BAx|} = \frac{t}{t + 6} \quad \text{for } x = 0 \text{ in } \; X\,,$$

and

$$M(Ax, Bx, t) = \frac{t}{t + |Ax - Bx|} = \frac{t}{t + 2} \quad \text{for } x = 0 \text{ in } \; X.$$

In 1998, Jungck and Rhoades [10] introduced the notion of weakly compatible maps as followed:

Definition 1.6. *Two maps f and g are said to be weakly compatible if they commute at coincidence points.*

Example 1.7. *Let $X = R$ and define $f, g : R \to R$ by $fx = x/3$, $x \in R$ and $gx = x^2$, $x \in R$. Here 0 and 1/3 are two coincidence points for the maps f and g. Note that f and g commute at 0, i.e., $fg(0) = gf(0) = 0$, but $fg(1/3) = f(1/9) = 1/27$ and $gf(1/3) = g(1/9) = 1/81$ and so f and g are not weakly compatible maps on R.*

Remark 1.8. *(i) Weakly compatible maps need not be compatible.*

Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = 2 \;\; \text{if} \;\; x = 2 \;\; \text{or} \;\; x > 5, \quad fx = 4 \;\; \text{if} \;\; 2 < x \leq 5,$$

$$g2 = 2, \; gx = 12 \;\; \text{if} \;\; 2 < x \leq 5, \quad gx = (x+1)/3 \;\; \text{if} \;\; x > 5.$$

Also define $M(x, y, t) = t/(t + d(x, y))$. Clearly $(X, M, *)$ is a complete fuzzy metric on X, where $*$ is defined by $a * b = ab$.

Moreover, $fX = \{2\} \cup \{4\}$, $gX = [2, 7] \cup \{9\}$. Clearly $fX \subset gX$.

Since

$$M(fx_n, gx_n, t) = \frac{t}{t + |fx_n - gx_n|} = \frac{t}{t + |(x_n - 1)|\,|(x_n^2 + x_n + 2)|} \to 1 \;\; \text{iff} \;\; x_n \to 1\,,$$
$$M(fgx_n, gfx_n, t) = \frac{t}{t + 6|(x_n - 1)^2|} \to 1 \;\; \text{iff} \;\; x_n \to 1\,.$$

To see that f and g are noncompatible maps, consider the sequence $\{x_n = 5 + 1/n; n \geq 1\}$ in X. Then $fx_{n\to\infty} = 2$, $\lim_{n\to\infty} gx_n = 2$, $\lim_{n\to\infty} fgx_n = 6$ and $\lim_{n\to\infty} gfx_n = 2$. It is easy to check that f and g are noncompatible maps. Also, f and g are weakly compatible maps since they commute at their coincidence point at $x = 2$. Moreover, both f and g are discontinuous at the common fixed point $x = 2$.

(ii) Let $X = [0, 2]$ and $a * b = \min\{a, b\}$. Let $M(x, y, t) = t/(t + d(x, y))$ be the standard fuzzy metric space induced by d, where $d(x, y) = |x - y|$ for all x, y in X.

Define $f, g : X \to X$ by

$$fx = \begin{cases} 2 - x & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2 \end{cases} \qquad gx = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2 \end{cases}$$

Consider the sequence $\{x_n\}$ defined by $\{x_n = 1 - 1/n; n \ge 1\}$. Clearly f and g are weakly compatible but not compatible.

In 2003, Chugh and Kumar [1] defined the concept of compatible mapping of type (A) or (Sharma [22] call it compatible of type (α)) as follows:

Two mappings f and g are said to be compatible of type (A) if

$$\lim_{n\to\infty} M(fgx_n, ggx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(gfx_n, ffx_n, t) = 1,$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some $p \in X$ and for each $t > 0$.

Recently, Chugh et al. *[2] introduced the concept of compatible mappings of type (P) and compared with compatible mappings of type (A) and compatible mappings.*

The mappings f and g are said to be compatible of type (P) if $\lim_{n\to\infty} M(ffx_n, ggx_n, t) = 1$ whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X and for each $t > 0$, see for more details [2].

Remark 1.9. *Weakly compatible maps need not be compatible of type (A) and compatible of type (P).*

Let $X = [2, 20]$ and d be the usual metric on X. Define mappings $B, T : X \to$

X by

$$Bx = x \ \ if \ \ x = 2 \ \ or \ \ > 5, \ Bx = 8 \ \ if \ \ 2 < x \leq 5, \ Tx = x \ \ if \ \ x = 2,$$

$$Tx = 12 + x \ \ if \ \ 2 < x \leq 5, \ Tx = x - 3 \ \ if \ \ x > 5$$

Also define $M(x, y, t) = t/(t + d(x, y))$. Clearly $(X, M, *)$ is a complete fuzzy metric on X, where $*$ is defined by $a * b = ab$.

To see that B and T are not compatible of type (A) and compatible of type (P). Let us consider a sequence $\{x_n\}$ defined by $\{x_n = 5 + (1/n), \ n \geq 1\}$. Then $Tx_n \to 2$, $Bx_n = 2$,$TTx_n \to 14$, $BTx_n = 8$, $BBx_n = 2$, $\lim_{n\to\infty} d(TTx_n, \ BBx_n) = 12 \neq 0$ and $\lim_{n\to\infty} d(BTx_n, \ TTx_n) = 6 \neq 0$, therefore, B and T are not compatible of type (A) and compatible of type (P), but B and T commute at the coincidence point at $x = 2$, therefore, they are weakly compatible maps.

In 1994, Pant [19] introduced the concept of R-weakly commuting of mappings as follows:

Two self maps f and g of a metric space (X, d) are called R-weakly commuting at a point $x \in X$ if

$$d(fgx, gfx) \leq Rd(fx, gx) \ \textit{for some} \ R > 0.$$

Also, f and g are called point wise R-weakly commuting on X if given x in X, there exists $R > 0$ such that

$$d(fgx, gfx) \leq Rd(fx, gx).$$

Later on Pathak et al. *[16] introduced an interesting generalization of R-weak commutativity of maps by defining R-weak commutativity of type (Ag) as follows:*

Two self maps f and g of a metric space (X, d) are called R-weakly commuting of type (Ag) if there exists some positive real number $R > 0$ such that

$$d(ffx, gfx) \leq Rd(fx, gx) \text{ for all } x \text{ in } X.$$

Moreover, such mappings commute at their coincidence points.

The study of fixed points of contractive type mappings in fuzzy metric spaces has been concerned around compatible mapping around a decade. However, the study of common fixed points of noncompatible mappings is also equally interesting in fuzzy metric spaces, moreover, interestingly enough, the best example of noncompatible mapping are found among pair of mappings which are discontinuous at their common fixed point.

The notion of point wise R-weakly commuting is useful in studying common fixed of non-compatible maps.

However, noncompatibility of f and g implies that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X but $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or nonexistent.

In 1999, Vasuki [25] introduced the notion of R-weakly commuting in fuzzy metric space as follows:

The mappings f and g of a fuzzy metric space $(X, M, *)$ into itself are R-weakly commuting, provided there exists some positive real number R such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R) \text{ for all } x \text{ in} X.$$

R-weak commutativity implies weak commutativity only when $R \leq 1$.

Two self maps f and g of a fuzzy metric space $(X, M, *)$ are called point wise R-weakly commuting if given x in X, there exists some real number $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R) \text{ for each } t > 0.$$

Clearly point wise R-weak commutativity is

(i) *Equivalent to commutativity at coincidence points(weakly compatible)*

(ii) *a necessary, hence minimal condition for existence of common fixed points of contractive type mappings.*

The notion of point wise R-weakly commuting is useful in studying common fixed of non-compatible maps. However, non compatibility of f and g implies that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X but $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonunit or non existent.

Remark 1.10. *R-weakly commuting need not be weakly commuting Let $X = R$, the set of all real numbers. Define $a * b = ab$ and*

$$M(x, y, t) = \left(\exp\left(\frac{|x-y|}{t}\right)\right)^{-1}, \text{ for all } x, y \text{ in } X \text{ and } t > 0 \text{ and } M(x, y, 0) = 0.$$

Then $(X, M, *)$ is a fuzzy metric space. Define $f(x) = 2x - 1$ and $g(x) = x^2$. Now

$$\begin{aligned} M(fgx, gfx, t) &= \left(\exp\left(2\frac{|x-1|^2}{t}\right)\right)^{-1}, \\ M(fx, gx, t/2) &= \left(\exp\left(2\frac{|x-1|^2}{t}\right)\right)^{-1}. \end{aligned}$$

Therefore, for $R = 2$, f and g are R-weakly commuting. But f and g are not weakly commuting since exponential function is strictly increasing

Now we introduce the notion of R-weakly commuting of type (Ag) in fuzzy metric space as follows:

Two self maps f and g of a fuzzy metric space $(X, M, *)$ are called R-weakly commuting of type (Ag) if there exists some positive real number R such that

$$M(ffx, gfx, t/R) \geq M(fx, gx, t) \text{ for each } t > 0 \text{ and for all } x \text{ in } X.$$

We note that such mappings commute at their coincidence points. The converse of this is also true.

For, if f and g commute at their coincidence points, we can define

$$R = \{M(gfx, ffx, t)/M(fx, gx, t)\}, \text{ when } fx \neq gx \text{ and } R = 1 \text{ when } fx = gx.$$

Therefore, f and g can fail to be point wise R- weakly commuting of type (Ag) only if there exists some x in X such that $fx = gx$ but $gfx \neq ffx$. Therefore point wise R-weakly commuting mappings commute at their coincidence points

Remark 1.11. *Point wise R-weakly commuting mappings need not be compatible (resp. compatible of type (A) and compatible of type (P)).*

Let $X = [2, 20]$ and d be the usual metric on X.

For each $t \in [0, \infty)$, define

$$M(x, y, t) = \begin{cases} 0, & \text{if } t = 0 \\ \dfrac{t}{(t + |x - y|)}, & \text{if } t > 0, x, y \in X. \end{cases}$$

Clearly $(X, M, *)$ is a complete fuzzy metric on X, where $*$ is defined by $a * b = ab$.

Define mappings B, $T : X \to X$ by

$$Bx = x \text{ if } x = 2 \text{ or } x > 5,\ Bx = 8 \text{ if } 2 < x \leq 5,\ Tx = x \text{ if } x = 2,$$

$$Tx = 12 + x \text{ if } 2 < x \leq 5,\ Tx = x - 3 \text{ if } x > 5.$$

To see that B and T are not compatible (resp.compatible of type (A) and compatible of type (P)). Let us consider a sequence $\{x_n\}$ defined by $\{x_n = 5 + (1/n), n \geq 1\}$. Then $Tx_n \to 2, Bx_n = 2, TTx_n \to 14$ and $BTx_n = 8,\ BBx_n = 2$. Clearly B and T are not compatible of type (A) and compatible of type (P).

But they are Point wise R-weakly commuting mappings since they commute at coincidence point at $x = 2$.

2 Fixed Points for Weakly Compatible Maps

In this section, we prove some common fixed-point theorem for weakly compatible maps, which are not necessarily continuous. Before proving main results we prove two lemmas that are helpful in proving results in this section and further sections.

Lemma 2.1. *If for all $x, y \in X,\ t > 0$ and for a number $k \in (0,1)$, then $M(x,y,kt) \geq M(x,y,t)$ then $x = y$.*

Proof. Suppose that there exists $k \in (0,1)$ such that

$$M(x,y,kt) \geq M(x,y,t) \text{ for all } x,y \text{ in } X \text{ and } t > 0$$

Then $M(x,y,t) \geq M(x,y,t/k)$ and after n-th iteration $M(x,y,t) \geq M(x,y,t/k^n)$ for some positive integer n. Taking limit as $n \to \infty$, we have

$M(x,y,t) \geq 1$ and hence $x = y$. □

Lemma 2.2. *Let $\{y_n\}$ be a cauchy sequence in a fuzzy metric space $(X, M, *)$ with $*$ continuous t-norm with the condition* (FM-6). *If there exists a number $q \in (0,1)$*

such that

$$M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t) \text{ for all } t > 0$$

and $n = 1, 2, \ldots$ *then* $\{y_n\}$ *is cauchy in* X.

Proof. For $t > 0$, and $q \in (0, 1)$, we have

$$M(y_2, y_3, qt) \geq M(y_1, y_2, t) \geq M(y_0, y_1, t/q).$$

By induction, we have for all $t > 0$ and $n = 1, 2, 3 \ldots$

$$M(y_{n+1}, y_{n+2}, t) \geq M(y_1, y_2, t/q^n)$$

Thus for any positive integer p, and real number $t > 0$, we have

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * \cdots p\text{-times} \cdots * M(y_{n+p-1}, y_{n+p}, t/p)$$
$$\geq M(y_1, y_2, t/pq^{n-1}) * \cdots p\text{-times} \cdots * M(y_1, y_2, t/pq^{n+p-2})$$

Therefore, by (FM-6), we have $\lim_{n \to \infty} M(y_n, y_{n+p}, t) \geq 1 * \cdots p\text{-times} \cdots * 1 \geq 1$, which implies $\{y_n\}$ is a Cauchy sequence in X. □

In 2002 Sharma [22] proved the following theorem.

Theorem 2.3. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$. *Let* A, B, S, T, P *and* Q *be mappings from* X *into itself satisfying the following conditions:*

(2.1) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$,

(2.2) $AB = BA$, $ST = TS$, $PB = BP$, $SQ = QS$, $QT = TQ$, (2.3) *Pairs* (P, AB) *and* (Q, ST) *are compatible of type* (α) (*or compatible of type* (A)),

(2.4) A, B, S *and* T *are continuous,*

(2.5) *There exists a number* $k \in (0, 1)$ *such that*

$$M(Px, Qy, kt) \geq M(ABx, Px, t) * M(STy, Qy, t) * M(STy, Px, \beta t)$$
$$* M(ABx, Qy, (2-\beta)t) * M(ABx, STy, t),$$

for all $x, y \in X, \beta \in (0, 2)$ *and* $t > 0$.

Then A, B, S, T, P *and* Q *have a unique common fixed point in* X.

Now we prove the following results:

Theorem 2.4. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$. *Let* A, B, S, T, P *and* Q *be mappings from* X *into itself satisfying* (2.1), (2.2), (2.5) *and the following condition:*

(2.6) *Pairs* (P, AB) *and* (Q, ST) *are weakly compatible.*

If the range one of the subspaces $P(X)$ *or* $AB(X)$ *or* $Q(X)$ *or* $ST(X)$ *is complete, then* A, B, S, T, P *and* Q *have a unique common fixed point in* X.

Proof. By [22], $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, so $\{y_n\}$ converges to a point $z \in X$. Since $\{Px_{2n}\}$, $\{Qx_{2n+1}\}$, $\{ABx_{2n+1}\}$ and $\{STx_{2n+2}\}$ are subsequences of $\{y_n\}$, they also converge to the same point z.

Since $P(X) \subset AB(X)$, there exists a point $u \in X$ such that $ABu = z$. Then, using (2.5)

$$\begin{aligned} M(Pu,z,kt) &\geq M(Pu,Qx_{2n+1},kt) \\ &\geq M(ABu,Pu,t)*M(STx_{2n+1},Qx_{2n+1},t)*M(STx_{2n+1},Pu,\beta t) \\ &\quad *M(ABu,Qx_{2n+1},(2-\beta)t)*M(ABu,STx_{2n+1},t)\,. \end{aligned}$$

Proceeding limit as $n \to \infty$ and setting $\beta = 1$,

$$\begin{aligned} M(Pu,z,kt) &\geq M(Pu,z,t)*M(z,z,t)*M(z,Pu,\beta t)*M(z,z,t)*M(z,z,t), \\ &= M(Pu,z,t)*1*M(Pu,z,t)*1*1, \\ &\geq M(Pu,z,t)\,. \end{aligned}$$

By Lemma (2.1), $Pu = z$. Therefore, $ABu = Pu = z$.

Since $Q(X) \subset ST(X)$, there exists a point $v \in X$ such that $z = STv$. Then, again using (2.5)

$$\begin{aligned} M(Pu,Qv,kt) &\geq M(ABu,Pu,t)*M(STv,Qv,t)*M(STv,Pu,Bt) \\ &\quad *M(ABu,Qv,(2-\beta)t)*M(ABu,STv,t) \end{aligned}$$

Proceeding limit as $n \to \infty$, we have for $\beta = 1$, $Qv = z$.

Therefore, $ABu = Pu = STv = Qv = z$.

Since pair (P, AB) is weakly compatible, therefore, $Pu = ABu$ implies that $PABu = ABPu$ i.e., $Pz = ABz$. Now we show that z is a fixed point of P. For

$\beta = 1$, we have

$$\begin{aligned} M(Pz, Qv, kt) &\geq M(ABz, Pz, t) * M(STv, Qv, t) * M(STv, Pz, \beta t) \\ &\quad * M(ABz, Qv, (2-\beta)t) * M(ABz, STv, t) \\ &= 1 * 1 * M(z, Pz, t) * M(Pz, z, t) * M(Pz, z, t) \end{aligned}$$

Therefore, we have by Lemma 2.1, $Pz = z$. Hence

$$Pz = z = ABz.$$

Similarly, pair of map $\{Q, ST\}$ is weakly compatible, we have

$$Qz = STz = z.$$

Now we show that $Bz = z$, by putting $x = Bz$ and $y = x_{2n+1}$ with $\beta = 1$ in for (2.5) we have

$$\begin{aligned} M(PBz, Qx_{2n+1}, kt) &\geq M(AB(Bz), P(Bz), t) * M(STx_{2n+1}, Qx_{2n+1}, t) \\ &\quad * M(STx_{2n+1}, PBz, t) * M(AB(Bz), Qx_{2n+1}, t) \\ &\quad * M(AB(Bz), STx_{2n+1}, t) \,. \end{aligned}$$

Proceeding limits as $n \to \infty$, and using Lemma 2.1, we have $Bz = z$. Since $ABz = z$, therefore, $Pz = ABz = Bz = z = Qz = STz$. Finally, we show that $Tz = z$, by putting $x = z$ and $y = Tz$ with $\beta = 1$ in (2.5).

$$\begin{aligned} M(Pz, Q(Tz), kt) &\geq M(ABz, Pz, t) * M(ST(Tz), Q(Tz), t) \\ &\quad * M(ST(Tz), Pz, t) * M(ABz, Q(Tz), t) \\ &\quad * M(ABz, ST(Tz), t) \,. \end{aligned}$$

Therefore, $Tz = z$.

Hence $ABz = Bz = STz = Tz = Pz = Qz = z$.

Uniqueness follows easily. □

If we put $B = T = I$, the identity map on X, in Theorem 2.2, we have the following:

Corollary 2.5. *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in (0, 1)$ and the condition:*

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$,

Let A, S, P and Q be the mapping from X into itself such that

(2.7) $P(X) \subset A(X)$, $Q(X) \subset S(X)$.

The pairs (A, S) and (Q, S) are weakly compatible. There exists a number $k \in (0, 1)$ such that

(2.8) $M(Px, Qy, kt) \geq M(Ax, Px, t) * M(Sy, Qy, t) * M(Sy, Px, \beta t)*$

$M(Ax, Qy, (2 - \beta)t) * M(Ax, Sy, t)$,

for all $x, y \in X$, $\beta \in (0, 2)$ with $t > 0$.

If the range one of the subspaces is complete then A, S, P and Q have a unique common fixed point in X.

If we put $A = B = S = T = I$ in Theorem 2.2, we have the following:

Corollary 2.6. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and the condition* (FM-6)*. Let* P *and* Q *be weakly compatible mapping from* X *into itself. If there exists a constant* $k \in (0, 1)$ *such that*

$$\begin{aligned} M(Px, Qy, kt) \quad \geq \quad & M(x, Px, t) * M(y, Qy, t) * M(y, Px, \beta t) \\ & * M(x, Qy, (2-\beta)t) * M(x, y, t), \\ & \qquad \textit{for all } \; x, y \in X, \beta \in (0, 2) \; \textit{and} \; t > 0. \end{aligned}$$

If the range one of the subspaces is complete then P *and* Q *have a unique common fixed point in* X.

If we put $P = Q$, $A = S$ and $B = T = I$ in Theorem 2.2, we have the following:

Corollary 2.7. *Let* $(X, M, *)$ *be a complete fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and the condition* (FM-6)*. Let* P, S *be weakly compatible maps on* X *such that* $P(X) \subset S(X)$ *and satisfy the following condition:*

$$\begin{aligned} M(Px, Py, t) \quad \geq \quad & M(Sx, Px, t) * M(Sy, Py, t) * M(Sy, Px, \beta t) \\ & * M(Sx, Py, (2-\beta)t) * M(Sx, Sy, t), \end{aligned}$$

for all $x, y \in X$, $\beta \in (0, 2)$ *and* $t > 0$*. If the range one of the subspaces is complete then* P *and* S *have a unique common fixed point in* X.

Example 2.8 ([22])**.** *Let* $X = [0, 1]$ *with usual metric* d *and for each* $t \in [0, 1]$ *define*

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad M(x, y, 0) = 0 \textit{ for all } \; x, y \in X.$$

Clearly $(X, M, *)$ is a complete fuzzy metric space where $*$ is defined by $a * b = ab$. Let A, B, S, T, P and Q be defined by $Ax = x$, $Bx = x/2$, $Sx = x/5$, $Tx = x/3$, $Px = x/6$ and $Qx = 0$ for all $x, y \in X$.

Then $P(X) = [0, 1/6] \subset [0, 1/2] = AB(X)$ and $Q(X) = 0 \subset [0, 1/5] = STx$.

If we take $k = 1/2$, $t = 1$ and $\beta = 1$, we see that all conditions of Theorem 2.2 are satisfied.

Moreover, the pair $\{P, AB\}$ and $\{Q, ST\}$ are weakly compatible.

3 Fixed Points for *R*-weakly Commuting Mappings

In this section, we prove some common fixed-point theorem for R-weakly commuting mappings which are not necessarily continuous.

Now, let A, B, S and T be mappings from a fuzzy metric space X into itself satisfying the following conditions:

(3.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(3.2) $M(Ax, By, t) \geq r(M(Sx, Ty, t))$ *where* $r : [0, 1] \to [0, 1]$ *is a continuous function such that* $r(t) > t$ *for each* $0 < t < 1$ *and for all* $x, y,$ *in* X.

Then for an arbitrary point x_0 in X, by (3.1), we choose a point x_1 in X such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

(3.3) $y_{2n} = Tx_{2n+1} = Ax_{2n}, y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ *for* $n = 0, 1, 2, \ldots$.

The sequence $\{x_n\}$ *and* $\{y_n\}$ *in* X *are such that* $x_n \to x, y_n \to y, t > 0$ *implies* $M(x_n, y_n, t) \to M(x, y, t)$.

Lemma 3.1. *Let* A, B, S *and* T *be mappings from a fuzzy metric space* $(X, M, *)$ *into itself satisfying the conditions* (3.1) *and* (3.2)*. Then the sequence* $\{y_n\}$ *defined by* (3.3) *is a Cauchy sequence in* X*.*

Proof. For $t > 0$,

$$\begin{aligned} M(y_{2n}, y_{2n+1}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \geq r(M(Sx_{2n}, Tx_{2n+1}, t)), \\ &= r(M(y_{2n-1}, y_{2n}, t)), \\ &> M(y_{2n-1}, y_{2n}, t) \,. \end{aligned} \tag{3.4}$$

Similarly, we have $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$.

Since $r(t) > t$ for $0 < t < 1$. Therefore, for every $n \in N$, $\{M(y_n, y_{n+1}, t), n \geq 0\}$ is a increasing sequence of positive real numbers in $[0, 1]$ and therefore tends to limit $l \geq 1$. We claim that $l = 1$. For if $l < 1$, on letting $n \to \infty$ in (3.4), we have $l \geq r(l) > l$, a contradiction. Hence $l = 1$.

Now for any positive integer p,

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, t/p) * \ldots * M(y_{n+p}, y_{n+p}, t/p) \\ &\geq M(y_n, y_{n+1}, t/p) * \ldots * M(y_n, y_{n+1}, t/p), \end{aligned}$$

for any positive integer n.

Since $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$, for $t > 0$, it follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * \ldots * 1 \geq 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. □

Theorem 3.2. *Let A, B, S and T be mappings from a fuzzy metric space $(X, M, *)$ into itself satisfying the conditions (3.1) and (3.2). Suppose that*

(3.5) *One of A, B, S and T is continuous,*

(3.6) *pairs (A, S) and (B, T) are R-weakly commuting on X.*

Then sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x, y_n \to y$, $t > 0$ implies $M(x_n, y_n, t) \to M(x, y, t)$.

Proof. By Lemma 3.1, $\{y_n\}$ is a Cauchy sequence and by the completeness of $X, \{y_n\}$ converges to some point $z \in X$. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n+2}\}$, $\{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ of y_n also converge to z.

Assume that S is continuous. Since A and S are R-weakly commuting it follows that

(3.7) $M(ASx_n, SAx_n, t) \geq M(Ax_n, Sx_n, t/R)$

On letting $n \to \infty$ in (3.7) we get $ASx_n \to Sz$.

By (3.2), we obtain

(3.8) $M(ASx_{2n}, Bx_{2n+1}, t) \geq r(M(SSx_{2n}, Tx_{2n+1}, t))$

Proceeding $\lim_{n\to\infty}$, from (3.8) we have

$$M(Sz, z, t) \geq r(M(Sz, z, t)) > M(Sz, z, t), \text{ a contradiction.}$$

Therefore, $Sz = z$. By (3.2), we also obtain

(3.9) $M(Az, Bx_{2n+1}, t) \geq r(M(Sz, Tx_{2n+1}, t))$

Taking $\lim_{n\to\infty}$ in (3.9), we have

$$M(Az, z, t) \geq r(M(Sz, z, t)) = r(M(z, z, t)) = r(1) = 1$$

since $r(t)$ 1 for $t = 1$, which implies $Az = z$.

Since $A(X) \subset T(X)$. For u in X there exists a point z in X such that $Az = Tu$. Hence $z = Az = Tu$.

$M(z, Bu, t) = M(Az, Bu, t) \geq r(M(Sz, Tu, t))$, implies $Bu = z$. Since B and T are R-weakly commuting on X and $Tu = Bu = z$.

¿From (3.2), we have

$M(BTu, TBu, t) \geq M(Bu, Tu, t/R)$, implies $Tz = TBu = BTu = Bz$.

Moreover, by (3.2), we obtain

$$M(z, Tz, t) = M(Az, Bz, t) \geq r(M(Sz, Tz, t)) = r(M(z, Tz, t)).$$

Hence $z = Tz$. Therefore, z is common fixed point of A, B, S and T.

Similarly, we can also complete the proof by assuming any one of the mappings A, B, and T is continuous.

Now to prove the uniqueness, let if possible $z' \neq z$ be another common fixed point of A, B, S and T. Then there exists $t > 0$ such that

$$M(z, z', t) < 1$$

and

$$\begin{aligned} M(z,z',t) &= M(Az,Bz',t) \geq r(M(Sz,Tz',t)) \\ &= r(M(z,z',t)) \\ &> M(z,z',t) \end{aligned}$$

Since $r(t) > t$ for $0 < t < 1$, which is a contradiction. Therefore, $z = z'$ i.e. z is a unique common fixed point of A, B, S and T. □

Example 3.3. *Let $X = [2, 20]$ with the metric d defined by $d(x, y) = |x - y|$. For each $t \in [0, \infty)$, define*

$$M(x,y,t) = \begin{cases} 0, & \text{if } t = 0 \\ \dfrac{t}{(t + |x-y|)}, & \text{if } t > 0, \ x,y,z \in X. \end{cases}$$

Clearly $(x, M, *)$ is a fuzzy metric on X, where $*$ is defined by $a * b = ab$. Also $(x, M, *)$ is a complete fuzzy metric space [1994].

Define

$$Ax = x \text{ for all } x \in X, Sx = x \text{ if } x = 2, Sx = 6 \text{ if } x > 2,$$

$$Bx = x \text{ if } x = 2 \text{ or } x > 5 \text{ and } Bx = 6 \text{ if } 2 < x \leq 5,$$

$$Tx = 12 \text{ if } 2 < x \leq 5 \text{ and } Tx = x - 3 \text{ if } x > 5.$$

It is evident that $A(X) \subset T(X)$ and $B(X) \subset S(X)$ and A is continuous.

Define $r = [0, 1] \to [0, 1]$ by $r(t) = \sqrt{t}$ for $0 < t < 1$ and $r(t) = 1$ for $t = 1$.

Then $r(t) > t$ for $0 < t < 1$. Also $M(Ax, By, t) \geq r(M(Sx, ty, t))$ for all $x, y,$ in X. Pairs (A, S) and (B, T) are R-weakly commuting on X.

Thus all the conditions of theorem (3.1) are satisfied and 2 is common fixed point of A, B, S and T.

4 Fixed Points for *R*-weakly Commuting Type (Ag) Mappings

As an application of the notion of R-weak commutativity of type (Ag), we prove a common fixed point theorem for the class of nonexpansive or lipschitz type mapping pairs even without assuming continuity of mappings involved or completeness of the space.

Theorem 4.1. *Let f and g be noncompatible self mappings of a fuzzy metric space $(X, M,^{*})$ satifying the following:*

(4.1) $\overline{f(X)} \subset g(X)$, *where* $\overline{f(X)}$ *denotes the closure of* $f(X)$

(4.2) $M(fx, fy, kt) \geq M(gx, gy, t), k \neq 0$ *and*

(4.3) $M(x, fx, t) > M(x, gx, t)$ *whenever* $x \neq fx$. *If f and g are pointwise R-weakly commuting of type(Ag) then f and g have a unique common fixed point and the fixed point is a point of discontinuity.*

Proof. Non compatibility of f and g implies that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X but $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonunit or non existent. Since $p \in \overline{f(X)}$ and $\overline{f(X)} \subset g(X)$, there exists u in X such that $p = gu$. From (4.2), we have

$$M(fu, fx_n, kt) \geq M(gu, gx_n, t).$$

Proceeding limit as $n \to \infty$, we have

$$\lim_{n\to\infty} fx_n = fu, \text{ i.e., } fu = gu.$$

Since f and g are R-weakly commuting of type(Ag), we have

$$M(ffu, gfu, t/R) = M(fu, gu, t) = 1, \text{ i.e., } ffu = gfu$$

If $fu \neq ffu$, using(4.3), we get

$$M(fu, ffu, t) > M(fu, gfu, t) = M(fu, ffu, t),$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Uniqueness follows easily.

We now show that f and g are discontinuous at the common fixed-point $p = fu = gu$. If possible, suppose f is continuous. Then considering the sequence $\{x_n\}$ of (4.1), we get

$$\lim_{n\to\infty} ffx_n = fp = p \text{ and } \lim_{n\to\infty} fgx_n = fp.$$

R-weak commutativity of type(Ag) implies that

$$M(ffx_n, gfx_n, t/R) \geq M(fx_n, gx_n, t).$$

Letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} gfx_n = fp = p.$$

Further, this in turn, implies

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = M(fp, gp, t) = 1.$$

This contradicts the fact that $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonunit or nonexistent for the sequence $\{x_n\}$ of (4.1). Hence f is discontinuous at the fixed point.

Therefore, g is also discontinuous at the fixed point, by virtue of (4.2), continuity of g implies continuity of f.

This establishes our theorem. $\square$

We now give an example to illustrate the above theorem.

Example 4.2. *Let* $X = [2, 20]$ *and* d *be the usual metric on* X.

Define $M(x, y, t) = \{t/t + d(x, y)\}$ *for* $t > 0$. *Clearly* $(X, M, ^*)$ *is a fuzzy metric space. Also Define*

$f, g : X \to X$ *by*

$$fx = 2 \text{ if } x = 2 \text{ or } x > 5, fx = 6 \text{ if } 2 < x \leq 5,$$

$$g2 = 2, gx = 9 \text{ if } 2 < x \leq 5, gx = (x + 1)/3 \text{ if } x > 5.$$

Then f *and* g *satisfy all the condition of the theorem and have a unique common fixed point at* $x = 2$.

Moreover, $fX = \{2\} \cup \{6\}$, $gX = [2, 7] \cup \{9\}$ *and* $fX \subset gX$. *It can also be verified that* f *and* g *are point wise R-weakly commuting type (Ag).*

To see that f *and* g *are noncompatible maps, consider the sequence* $\{x_n = 5 + 1/n; n \geq 1\}$ *in* X. *Then* $\lim_{n\to\infty} fx_n = 2$, $\lim_{n\to\infty} gx_n = 2$, $\lim_{n\to\infty} fgx_n = 6$ *and* $\lim_{n\to\infty} gfx_n = 2$. *Hence* f *and* g *are noncompatible maps. Clearly* f *and* g *satisfy the Lipschitz type condition*

$M(fx, fy, kt) = M(gx, gy, t)$ *with* $k = 2$, *together with the condition,* $M(x, fx, t) > M(x, gx, t)$.

Moreover, both f *and* g *are discontinuous at the common fixed point* $x = 2$.

Theorem 4.3. *Let f and g be noncompatible selfmappings of a fuzzy metric space $(X, M,^*)$ satifying the following:*

(4.4) $fX \subset gX$, *where $\overline{fX}$ denotes the closure of the range of the mapping f*

(4.5) $M(fx, fy, t) \geq M(gx, gy, t)$ *and*

(4.6) $M(x, fx, t) > M(x, gx, t)$ *whenever $x \neq fx$. If f and g are point wise R-weakly commuting type(Ag) then f and g have a common fixed point and the fixed point is a point of discontinuity.*

The above theorem can be proved on similar lines by setting $k = 1$ in Theorem 4.1.

Theorem 4.4. *Let f and g be noncompatible self mappings of a fuzzy metric space $(X, M,^*)$ such that $\overline{f(X)} \subset g(X)$, where $\overline{f(X)}$ denotes the closure of $f(X)$ and satisfying the following:*

(4.7) $M(fx, fy, t) > M(gx, gy, t), fx \neq fy.$

If f and g be R-weakly commuting of type (Ag)*, then f and g have a unique common fixed point and the fixed point is a point of discontinuity.*

Proof. Noncompatibility of f and g implies that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some p in X but $\lim_{n\to\infty} M(fgxn, gfxn, t) \neq 1$ or nonexistent. Since $p \in \overline{f(X)}$ and $\overline{f(X)} \subset g(X)$, there exists u in X such that $p = gu$.

By (4.7), $M(fu, fx_n, t) = M(gu, gx_n, t)$.

Proceeding limit as $n \to \infty$, we have

$$\lim_{n\to\infty} fx_n = fu, \text{ i.e, } fu = gu.$$

Since f and g are R-weakly commuting of type (Ag), we have

$$M(ffu, gfu, t/R) = M(fu, gu, t) = 1, \text{ i.e } ffu = gfu.$$

If $fu \neq ffu$, using (4.7), we have

$$M(fu, ffu, t/R) > M(gu, gfu, t) = M(fu, ffu, t),$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness follows easily from (4.7).

Now we show that f and g are discontinuous at the common fixed point $p = fu = gu$. If possible, suppose f is continuous. Let $\{x_n\}$ be a sequence of $f(X) \subset g(X)$ we have $\lim_{n\to\infty} ffx_n = fp = p$ and $\lim_{n\to\infty} fgx_n = fp$.

By R-weak commutativity of type (Ag), we have

$$M(ffxn, gfx_n, t/R) = M(fxn, gx_n, t),$$

Proceeding limit as $n \to \infty$, we have

$$\lim_{n\to\infty} gfx_n = ft = t.$$

Further, $M(fgx_n, gfx_n, t) = 1$, this contradicts the fact that

$\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or non existent for the sequence

$\{x_n\}$ of $\overline{f(X)} \subset g(X)$. Hence f is discontinuous at the fixed point.

Therefore by virtue of (4.7), g is also discontinuous at the fixed point, since continuity of g implies the continuity of f. □

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