Thai Journal of Mathematics
Volume 19 Number 2 (2021)
Pages 457–468
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Ideally Slowly Oscillating Continuity in Abstract Space

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Abstract In this paper, we introduce the notion of ideally slowly oscillating sequences, which is lying between ideal convergent and ideal quasi-Cauchy sequences, and study on ideally slowly oscillating continuous functions in topological vector space valued cone metric space. Also we introduce the notion of strongly continuous on topological vector space valued cone metric space and investigated some new results related to this notion.

MSC: 46A19; 40A05; 26A15; 40G15; 46A50, 54E35

Keywords: ideal convergence; cone metric; continuity; quasi-Cauchy sequence; slowly oscillating sequences

Submission date: 25.03.2016 / Acceptance date: 05.04.2019

1. Introduction

The idea of statistical convergence first appeared, under the name of almost convergence, in the first edition Zygmund [1] of celebrated monograph [2] of Zygmund. Later, this idea was introduced by Fast [3] and Steinhaus [4], Fridy [5] and many authors. Actually, this concept is based on the natural density of subsets of $\mathbb{N}$ of positive integers. A subset E of $\mathbb{N}$ is said to have natural or asymptotic density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \frac{|E(n)|}{n} \text{ exists,}$$

where $E(n) = \{k \leq n : k \in E\}$ and $|E|$ denotes the cardinality of the set E.

A sequence $\mathbf{x} = (x_k)$ is said to be *statistically convergent* to the number L if for each $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

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Published by The Mathematical Association of Thailand.

The notion of the ideal convergence is the dual (equivalent) to the notion of filter convergence introduced by Cartan in 1937 [6]. The notion of the filter convergence is a generalization of the classical notion of convergence of a sequence and it has been an important tool in general topology and functional analysis. Nowadays many authors to use an equivalent dual notion of the ideal convergence. Kostyrko et al. [7] and Nuray and Ruckle [8] independently studied in details about the notion of ideal convergence which is based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$. For more details on ideal convergence we refer to [9–11] and many others.

An ideal I on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under finite unions and subsets of its elements. A filter $\mathcal{F}$ on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under finite intersections and supersets of its elements.

An ideal I is called *non-trivial* ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(I) = \{\mathbb{N} - A : A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$.

Recall that a sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in I$. In this case we write I-$\lim x_n = \ell$. A sequence $\mathbf{x} = (x_n)$ of points in $\mathbb{R}$ is said to be I-quasi-Cauchy if $I - \lim_n (x_{n+1} - x_n) = 0$. We see that I-convergence of a sequence (x_n) implies I-quasi-Cauchyness of (x_n). We note that the definition of a quasi-Cauchy sequence is a special case of an ideal quasi-Cauchy sequences where I is taken as the finite subsets of the set of positive integers. Çakallı and Hazarika [12] introduced the concept of ideal quasi Cauchy sequences and proved some results related to ideal ward continuity and ideal ward compactness.

Throughout this paper we assume I is a non-trivial admissible ideal in $\mathbb{N}$.

A real valued function is continuous on the set of real numbers if and only if it preserves Cauchy sequences.

The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. In [13] Burton and Coleman named these sequences as “quasi-Cauchy” and in [14] Çakallı used the term “ward convergent to 0” sequences.

In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$.

2. Preliminaries and Notations

It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \leq k \leq [\lambda n]} |x_k - x_n| = 0,$$

where $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \leq \frac{m}{n} \to 1$ as $m, n \to \infty$. Using $\varepsilon > 0$ and δ this is equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$

if $n \geq N(\varepsilon)$ and $n \leq m \leq (1+\delta)n$ (see [15]). For more details on slowly oscillating sequences we refer to [16–22].

A function defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

Connor and Grosse-Erdman [23] gave sequential definitions of continuity for real functions calling G-continuity instead of A-continuity and their results covers the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, space of all sequences, denoted by c_G, into $\mathbb{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c and st-lim denotes the statistical limit function $st\text{-}\lim \mathbf{x} = st\text{-}\lim_n x_n$ on the linear space $st(\mathbb{R})$. Also I-lim denotes the I-limit function $I\text{-}\lim \mathbf{x} = I\text{-}\lim_n x_n$ on the linear space $I(\mathbb{R})$.

A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$ (for details see [24]).

Now we recall the concept of the cone metric space and topological vector valued cone metric spaces.

The concept of cone metric which has been known since the middle of 20th century (see [25–27]) was introduced by Long-Guang and Xian in [28]. Beg et al., [29] and Azam et al., [30] replaced the set of an ordered Banach space by locally convex Hausdorff topological vector space in the definition of a cone metric and a generalized cone metric space. Let V be a topological vector space with its zero vector $\bar{0}$. A nonempty subset E of V is called a convex cone if $E + E \subseteq E$ and $\lambda E \subseteq E$ for $\lambda > 0$. A convex cone E is said to be pointed if $E \cap (-E) = \{\bar{0}\}$. For a given cone $E \subseteq V$, we can define a partial ordering $\precsim_E$ with respect to E by

$$x \precsim_E y \Leftrightarrow y - x \in E.$$

$x <_E y$ will stand for $x \precsim_E y$ and $x \neq y$, while $x \ll_E y$ stands for $y - x \in E^o$, where E^o denotes the interior of E.

Let X be an nonempty set and $q : X \times X \to E$ be a vector-valued function satisfying the following conditions:

(CM1) $\bar{0} \precsim q(x,y)$ for all $x, y \in X$ and $q(x,y) = \bar{0}$ if and only if $x = y$;
(CM2) $q(x,y) = q(y,x)$ for all $x, y \in X$;
(CM3) $q(x,z) \precsim q(x,y) + q(y,z)$ for all $x, y, z \in X$.

Then the function q is called a topological vector space valued cone metric (TVS-cone metric for short) on X and the pair (X, q) is said to be a topological vector space valued cone metric space (TVS-cone metric space for short).

Throughout the article, we suppose that V is a locally convex Hausdorff topological vector space with its zero vector $\bar{0}$, E is a proper, closed and convex pointed cone in V with $E^o \neq \phi$ and $\precsim_E$ is a partial ordering with respect to E, X a TVS-cone metric space with q, and $\mathbb{N}$ and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of points in X.

3. Ideally Slowly Oscillating Sequences

In this section we introduce the concepts of ideally slowly oscillating continuity and ideally slowly oscillating compactness and establish some interesting results related to these notions.

A sequence $\mathbf{x} = (x_n)$ of points in X is called quasi-Cauchy if for each $c \in E^o$ there exists an $m_0 \in \mathbb{N}$ such that $c - q(x_{n+1}, x_n) \in E^o$ for $n \geq m_0$ (see [31]). It is clear that Cauchy sequences are slowly oscillating not only the real case but also in the TVS-cone metric space setting. It is easy to see that any slowly oscillating sequence of points in X is quasi-Cauchy and therefore Cauchy sequence is quasi-Cauchy. The converses are not always true. There are quasi-Cauchy sequences which are not Cauchy. There are quasi-Cauchy sequences which are not slowly oscillating. Any subsequence of Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences and slowly oscillating sequences as well.

Following Tripathy and Hazarika [11], we say that $\mathbf{x} = (x_k)$ is an ideally Cauchy sequence if for any $\varepsilon > 0$ there exists $m = m(\varepsilon) \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I$. Any sequence is ideally convergent if and only if it is ideally Cauchy.

Now we introduce the notion of ideally-quasi-Cauchy and ideally slowly oscillating sequences in TVS-cone metric space.

Definition 3.1. A sequence $\mathbf{x} = (x_n)$ of points in X is called ideally quasi-Cauchy if for each $c \in E^o$ such that the set $\{n \in \mathbb{N} : c - q(x_{n+1}, x_n) \notin E^o\} \in I$.

In [32] Hazarika introduced the concept of ideally slowly oscillating sequences and ideally slowly oscillating continuity for real number sequences.

Definition 3.2. A sequence $\mathbf{x} = (x_n)$ of points in X is said to be ideally slowly oscillating if for every $c \in E^o$ there exist $\delta > 0$, a positive integer $N = N(c)$ such that

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q(x_k, x_n) \notin E^o\} \in I.$$

It is clear that an ideally convergent sequence is ideally slowly oscillating since an ideally convergent sequence is an ideally Cauchy sequence, but the converse need not to be true as the following example shows.

For $X = \mathbb{R}$, the sequence $(x_n) = \left(\sum_{j=1}^{n} \frac{1}{j}\right)$ is ideally slowly oscillating but not ideally convergent. From definition it is clear that ideally slowly oscillating sequences are not ideally Cauchy in general. Also from definition, it is obvious that a slowly oscillating sequence is ideally slowly oscillating but the converse is not true in general.
Now we introduced the definition of ideally slowly oscillating continuous as follows:

Definition 3.3. A function f defined on a subset K of X is called ideally slowly oscillating continuous if it transforms ideally slowly oscillating sequences to ideally slowly oscillating sequences of points in K, that is, $(f(x_n))$ is ideally slowly oscillating whenever (x_n) is ideally slowly oscillating sequences of points in K.

Proposition 3.4. *The set of the ideally convergent sequences is a proper subset of the set of ideally slowly oscillating sequences.*

Proof. The proof of the result follows from the both the definitions. ∎

Proposition 3.5. *The set of ideally slowly oscillating sequences is a closed subalgebra of* $\ell_\infty(\mathbb{N})$.

Proof. Proof is easy, so omitted. ■

Theorem 3.6. *If f is ideally slowly oscillating continuous on a subset K of X then it is ideally continuous on K.*

Proof. Suppose that f is ideally slowly oscillating continuous on K and let (x_n) be any ideally convergent sequence of points in K with $I - \lim x_n = x_0$. Then the sequence

$$(y_n) = (x_1, x_0, x_2, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$$

is also ideally convergent to x_0 and hence (y_n) is ideally slowly oscillating. Since f is ideally slowly oscillating continuous, the sequence

$$(f(y_n)) = (f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_{n-1}), f(x_0), f(x_n), f(x_0), ...)$$

is also ideally slowly oscillating. Hence $(f(y_n))$ is an ideally quasi-Cauchy sequence. Now it follows that for any $c \in E^o$, there exist a positive integer $N = N(c)$ such that

$$c - q(f(x_n), f(x_0)) \in E^o \text{ for } n \geq N.$$

i.e.

$$\{n \in \mathbb{N} : c - q(f(x_n), f(x_0)) \notin E^o\} \in I.$$

This completes the proof of theorem. ■

In general the converse is not true. For $X = \mathbb{R}$, it follows from the function $f(x) = e^x$ and the sequence $(x_n) = (\ln n)$.

Corollary 3.7. *If f is ideally slowly oscillating continuous, then it is continuous in the ordinary sense.*

Theorem 3.8. *If f and g are ideally slowly oscillating continuous functions on a subset K of X. Then $f + g$ is ideally slowly oscillating continuous in K.*

Proof. Let f and g be ideally slowly oscillating continuous functions on a subset K of X. To prove that $f + g$ is ideally slowly oscillating continuous on K. Let $\mathbf{x} = (x_n)$ is any ideally slowly oscillating sequence in K. Then $(f(x_n))$ and $(g(x_n))$ are ideally slowly oscillating sequences. Since $(f(x_n))$ and $(g(x_n))$ are ideally slowly oscillating sequences, for any $c \in E^o$, there exists a positive integer $n_1 = n_1(c)$ and $\delta > 0$ such that

$$\{n_1(c) \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{2} - q(f(x_k), f(x_n)) \notin E^o\} \in I$$

$$\{n_1(c) \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{2} - q(g(x_k), g(x_n)) \notin E^o\} \in I$$

Therefore, we have

$$\{n_1(c) \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q((f+g)(x_k), (f+g)(x_n) \notin E^o\}$$

$$\subseteq \{n_1(c) \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{2} - q(f(x_k), f(x_n)) \notin E^o\}$$

$$\cup \{n_1(c) \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{2} - q(g(x_k), g(x_n)) \notin E^o\}$$

Since I is an admissible ideal, so we have

$$\{n_1(c) \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q((f+g)(x_k), (f+g)(x_n) \notin E^o\} \in I.$$

This completes the proof of the theorem. ■

Definition 3.9. [30] Let $q : X \times X \to E$ be a TVS-cone metric, $A \subset X$, and f be a function on A into X. Then f is called uniformly continuous on A if given $c \in E^o$ there is a $\delta_c \in E^o$ such that for all x and y with $q(x,y) \ll \delta_c$ we have $q(f(x), f(y)) \ll c$.

In [33], it was proved that a slowly oscillating continuous function is uniformly continuous on $\mathbb{R}$. We see that is also the case that any ideally slowly oscillating continuous function on a connected subset of X is uniformly continuous.

Theorem 3.10. *If f is a uniformly continuous function defined on a subset K of X, then it is ideally slowly oscillating continuous on K.*

Proof. Let f be uniformly continuous function and $\mathbf{x} = (x_n)$ be any ideally slowly oscillating sequence in K and $c \in E^o$. Since f is uniformly continuous on K, then there exists a $\delta_c \in E^o$ such that for every $x, y \in K$ with $\delta_c - q(x,y) \in E^o$ we have $c - q(f(x), f(y)) \in E^o$. Since (x_n) is ideally slowly oscillating, for the same $\delta_c \in E^o$ there exist an $\delta > 0$ and a positive integer $N = N(\delta_c)$ such that

$$\{N(\delta_c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \delta_c - q(x_k, x_n) \notin E^o\} \in I.$$

Hence we have

$$\{N(\delta_c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q(f(x_k), f(x_n)) \notin E^o\} \in I.$$

It follows that $(f(x_n))$ is ideally slowly oscillating. This completes the proof of theorem. ∎

Definition 3.11. A sequence (x_n) of points in X is called ideally Cesàro slowly oscillating if (t_n) is ideally slowly oscillating, where $t_n = \frac{1}{n}\sum_{k=1}^{n} x_k$, is the Cesàro means of the sequence (x_n). Also a function f defined on a subset K of X is called ideally Cesàro slowly oscillating continuous if it preserves ideally Cesàro slowly oscillating sequences of points in K.

By using the similar argument used in proof of Theorem 3.10, we immediately have the following result.

Theorem 3.12. *If f is a uniformly continuous on a subset K of X and (x_n) is an ideally slowly oscillating sequence in K, then $(f(x_n))$ is ideally Cesàro slowly oscillating.*

Definition 3.13. A sequence of functions (f_n) defined on a subset K of X is said to be uniformly ideally convergent to a function f if for every $c \in E^o$, the set

$$\{x \in K, n \in \mathbb{N} : c - q(f_n(x), f(x)) \notin E^o\} \in I.$$

Note that ordinary uniform convergence implies uniform ideal convergence.

Theorem 3.14. *If (f_n) is a sequence of ideally slowly oscillating continuous functions defined on a subset K of X and (f_n) is uniformly ideally convergent to a function f on K, then f is ideally slowly oscillating continuous on K.*

Proof. Let (x_n) be any ideally slowly oscillating sequence of points in K. By uniform ideal convergence of (f_n), for every $c \in E^o$

$$\{x \in K \text{ and } n \in \mathbb{N} : \frac{c}{3} - q(f_n(x), f(x)) \notin E^o\} \in I.$$

Also since f_{n_1} is ideally slowly oscillating continuous, there exist a positive integer $N = N(c)$ and $\delta > 0$ such that

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f_N(x_k), f_N(x_n)) \notin E^o\} \in I$$

Therefore we have

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q(f(x_k), f(x_n)) \notin E^o\}$$
$$\subseteq [\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f(x_k), f_N(x_k)) \notin E^o\}$$
$$\cup \{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f_N(x_k), f_N(x_n)) \notin E^o\}$$
$$\cup \{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f_N(x_n), f(x_n)) \notin E^o\}].$$

Since I is an admissible ideal, it implies that

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q(f(x_k), f(x_n)) \notin E^o\} \in I.$$

It follows that $(f(x_n))$ is an ideally slowly oscillating sequences of points in K which completes the proof of theorem. ■

Corollary 3.15. *If (f_n) is a sequence of ideally slowly oscillating continuous functions defined on a subset K of X and (f_n) is uniformly convergent to a function f on K, then f is slowly oscillating continuous on K.*

Using the same techniques as in the Theorem 3.14, the following result can be obtained easily.

Theorem 3.16. *If (f_n) is a sequence of ideally Cesàro slowly oscillating continuous functions defined on a subset K of X and (f_n) is uniformly ideally convergent to a function f on K, then f is ideally Cesàro slowly oscillating continuous on K.*

Theorem 3.17. *Let X be complete. The set of all ideally slowly oscillating continuous functions defined on a subset K of X is a closed subset of all continuous functions on K, that is $\overline{isoc(K)} = isoc(K)$, where $isoc(K)$ is the set of all ideally slowly oscillating continuous functions defined on K and $\overline{isoc(K)}$ denotes the set of all cluster points of $isoc(K)$.*

Proof. Let f be any element of $\overline{isoc(K)}$. Then there exists a sequence of points in $isoc(K)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is ideally slowly oscillating sequence on K. Now let (x_n) be any ideally slowly oscillating sequence in K. Let $c \in E^o$. Since (f_k) converges to f, there exists a positive integer N such that for all $x \in E$ and for all $n \geq N$,

$$\frac{c}{3} - q(f(x), f_n(x)) \in E^o.$$

By definition of ideal and for all $x \in K$ we have

$$\{n \in \mathbb{N} : \frac{c}{3} - q(f(x), f_n(x)) \notin E^o\} \in I.$$

Also since f_N is ideally slowly oscillating continuous on K, we have

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f_N(x_k), f_N(x_n)) \notin E^o\} \in I.$$

On the other hand we have

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q(f(x_k), f(x_n)) \notin E^o\}$$
$$\subseteq [\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f(x_k), f_N(x_k)) \notin E^o\}$$
$$\cup \{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f_N(x_k), f_N(x_n)) \notin E^o\}$$
$$\cup \{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : \frac{c}{3} - q(f_N(x_n), f(x_n)) \notin E^o\}.]$$

Since I is an admissible ideal, it implies that

$$\{N(c) \leq n \leq k \leq (1+\delta)n \text{ and } k \in \mathbb{N} : c - q(f(x_k), f(x_n)) \notin E^o\} \in I.$$

Thus f is ideally slowly oscillating continuous function on K and this completes the proof of theorem. ∎

Corollary 3.18. *Let X be complete. The set of all ideally slowly oscillating continuous functions defined on a subset K of X is a complete subspace of the space of all continuous functions on K.*

Next we define the ideal version of the concept of slowly oscillating compactness in TVS-cone metric space.

Definition 3.19. A subset K of X is called ideally slowly oscillating compact if any sequence of points in K has an ideally slowly oscillating subsequence.

We see that union of two ideally slowly oscillating compact subsets of $\mathbb{R}$ is ideally slowly oscillating compact. Any subset of ideally slowly oscillating compact set is also ideally slowly oscillating compact and so intersection of any ideally slowly oscillating compact subsets of $\mathbb{R}$ is ideally slowly oscillating compact.

Theorem 3.20. *An ideally slowly oscillating continuous image of an ideally slowly oscillating compact subset of X is ideally slowly oscillating compact.*

Proof. Let f be an ideally slowly oscillating continuous function on X and K be an ideally slowly oscillating compact subset of X. Let $\mathbf{y} = (y_n)$ be a sequence of points in $f(K)$. Then we can write $y_n = f(x_n)$ where x_n is sequence of points in K for each $n \in \mathbb{N}$. Since K is ideally slowly oscillating compact, there is an ideally slowly oscillating subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n). Then, ideally slowly oscillating continuity of f implies that $f(z_k)$ is an ideally slowly oscillating subsequence of $f(x_n)$. Hence $f(K)$ is ideally slowly oscillating compact. ∎

Corollary 3.21. *An ideally slowly oscillating continuous image of any compact subset of X is ideally slowly oscillating compact.*

Proof. The proof follows for the preceding theorem. ∎

We say that a subset K of X is called Cauchy compact if any sequence of points of K has a Cauchy subsequence. We see that any Cauchy compact subset of X is also ideally slowly oscillating compact and ideally slowly oscillating continuous image of any Cauchy compact subset of X is Cauchy compact.

Corollary 3.22. *For any regular subsequential method G, if K is G-sequentially compact subset of X, then it is ideally slowly oscillating compact.*

Theorem 3.23. *Let E be an ideally slowly oscillating compact subset of X and let $f : K \to X$ be an ideally slowly oscillating continuous function. Then f is uniformly continuous on K.*

Proof. Suppose that f is not uniformly continuous on K so there exists an $c_o \in E^o$ and sequences (x_n) and (y_n) of points in K such that $\frac{c}{n} - q(x_n, y_n) \in E^o$ for all $c \in E^o$, but

$$c_o - q(f(x_n), f(y_n)) \notin E^o \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Since K is ideally slowly oscillating compact, there is an ideally slowly oscillating subsequence (x_{n_k}) of (x_n).

It is clear from the result

$$q(y_{n_k}, x_{n_k}) + q(x_{n_k}, x_{n_m}) + q(x_{n_m}, y_{n_m}) - q(y_{n_k}, y_{n_m}) \in E^o$$

that the corresponding subsequence (y_{n_k}) of (y_n) is also ideally slowly oscillating. Then from the result (3.1) we observe that the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ are not ideally slowly oscillating.

This contradiction completes the proof of the theorem. ∎

Corollary 3.24. *Let K be an ideally slowly oscillating compact subset of X and let $f : K \to X$ be an ideally slowly oscillating continuous function. Then f is uniformly continuous if and only if it is ideally slowly oscillating continuous.*

Corollary 3.25. *A real valued function defined on a bounded subset of $\mathbb{R}$ is uniformly continuous if and only if it is ideally slowly oscillating continuous.*

Proof. The proof of the result follows from the fact that totally boundedness coincides with ideally slowly oscillating compactness and boundedness coincides with totally boundedness in $\mathbb{R}$. ∎

4. Ideally Sequentially Continuous

An element x_0 in X is called an ideal limit point of a subset K of X if there is an K-valued sequence of points with ideal limit x_0. It follows that the set of all ideal limit points of K is equal to the set of all limit points of K in the ordinary sense. An element x_0 in X is called an ideal accumulation point of a subset K if it is an ideal limit point of the set $K - \{x_0\}$. The set of all ideal accumulation points of K is equal to the set of all accumulation points of K in the ordinary sense.

A function f on X is said to have an ideally sequential limit at a point x_0 of X if the image sequence $(f(x_n))$ is ideally convergent to x_0 for any ideally convergent sequence $\mathbf{x} = (x_n)$ with ideal limit x_0 and a function f is to be ideally sequentially continuous at a point x_0 of X if the sequence $(f(x_n))$ is ideally convergent to $f(x_0)$ for any ideally convergent sequence $\mathbf{x} = (x_n)$ with ideal limit x_0 (for details see [12]). Then f is ideally sequentially continuous on X if f is ideally sequentially continuous at every point in X.

Lemma 4.1. *A function f on X has an ideally sequential limit at a point x_0 of X if and only if it has an ideal limit at a point x_0 of X in ordinary sense.*

Proof. The proof follows from the fact that any ideally convergent sequence has a convergent subsequence (also see [12]). ∎

Theorem 4.2. *A function f on X is ideally sequentially continuous on X if and only if it is continuous in ordinary sense.*

Proof. The proof follows from the fact that any ideally convergent sequence has a convergent subsequence and from the above lemma. ■

Theorem 4.3. *If a function is slowly oscillating continuous on a subset K of X, then it is ideally sequentially continuous on K.*

Proof. Let f be any slowly oscillating continuous on K. By [31, Theorem 3.1], we have f is continuous on K. Also from Theorem 4.2, we see that f is ideally sequentially continuous on K. This completes the proof. ■

Theorem 4.4. *If a function is δ-ward continuous on a subset K of X, then it is ideally sequentially continuous on K.*

Proof. Let f be any δ-ward continuous function on K. It follows from [34, Corollary 2] that f is continuous. By Theorem 4.2 we obtain that f is ideally sequentially continuous on K. This completes the proof of the theorem. ■

We introduced the definition of strongly continuity on topological vector space valued cone metric space as follows:

Definition 4.5. Let (X, q) and (Y, p) be two TVS-cone metric spaces. A function $f : X \to Y$ is called strongly continuous at a point $x_0 \in X$ for each $c \in E^o$ there exists $d \in E^o$ such that for each $x \in X$

$$c - p(f(x), f(x_0)) \gtrsim d - q(x, x_0) \in E^o.$$

Then f is strongly continuous on X if it is strongly continuous at every point in X.

Theorem 4.6. *Let (X, q) and (Y, p) be two TVS-cone metric spaces and $f : X \to Y$ be a map. If f is strongly continuous, then it is sequentially continuous.*

Proof. We assume that f is strongly continuous at a point $x_0 \in X$, then for each $c \in E^o$ there exists $d \in E^o$ such that for each $x \in X$

$$c - p(f(x), f(x_0)) \gtrsim d - q(x, x_0) \in E^o. \tag{4.1}$$

Let (x_n) be a sequence of points in X such that $x_n \to x_0$. Then for each $d \in E^o$ we have

$$d - q(x_n, x_0) \in E^o \text{ for all } n \in \mathbb{N} \tag{4.2}$$

From relation (4.1) we have

$$c - p(f(x_n), f(x_0)) \gtrsim d - q(x_n, x_0) \in E^o. \tag{4.3}$$

From relations (4.2) and (4.3), we have

$$c - p(f(x_n), f(x_0)) \in E^o \text{ for all } n \in \mathbb{N}.$$

This proves that f is sequentially continuous. ■

Theorem 4.7. *Let (X, q) and (Y, p) be two TVS-cone metric spaces and $f : X \to Y$ be a map. If f is strongly continuous, then it is ideally sequentially continuous.*

Proof. Let f be any strongly continuous function on X. We know that a function is continuous if and only if sequentially continuous. By Theorem 4.2, we see that f is ideally sequentially continuous. This completes the proof. ∎

Corollary 4.8. *Let (X, q) and (Y, p) be two TVS-cone metric spaces and $f : X \to Y$ be a map. If f is strongly continuous, then it is ideally continuous.*

5. Conclusions

In this paper, the concept of ideally slowly oscillating continuity and ideally slowly oscillating compactness in TVS-cone metric spaces are introduced and investigated. In this investigation we have obtained theorems related to ideally slowly oscillating continuity , ideally slowly oscillating compactness, sequential continuity, uniform continuity and uniformly ideally continuity. Finally, we note that the results of this paper can be obtained by defining the ideas of ideally quasi-slowly oscillating and ideally Δ-quasi-slowly oscillating sequences. It seems that an investigation of the present work taking "nets" instead of "sequences" could be done using the properties of "nets" instead of using the properties of "sequences" in different abstract spaces.

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