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Remarks on G-Metric Spaces and Related Fixed Point Theorems

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Abstract In this paper we use the notion of (A, B)-weakly increasing mappings in the sense of W. Shatanawi and M. Postolache [W. Shatanawi, M. Postolache, Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces, Fixed Point Theory Appl. 2013 (2013)] to introduce some new common fixed point results in complete G-metric spaces which can not be deduced from fixed point theorems in the setting of standard metric spaces.

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1. Introduction

It is well known that the Banach contraction theorem is the first outstanding result in the field of the fixed point theory that ensure the existence of unique fixed point in complete metric spaces, which is known later as Banach contraction principle. After that, many authors extended the Banach contraction theorem to many directions for example see [1–14].

Recently Mustafa and Sims [15] introduced a new generalization of the standard notion of metric space, named G-metric space. After that many authors proved several fixed point theorems in complete G-metric spaces for example see [16–21].

Cyclic mappings are studied by many authors in the field of fixed point theory for example see [22–38]

Jleli and Samet [39] and Samet *et.al* [40] showed that some of fixed point theorems in G-metric spaces can be deduced from standard metric spaces or quasi metric spaces.

Karapinar and Agarwal [41] proved that the approach of Jleli and Samet [39] and Samet *et.al* [40] cannot be applied if the contraction condition in the statement of the theorem

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can't be reducible to two variables and they introduced and proved diverse results in G-metric spaces.

Saadati et al. [42] initiated the notion of Ω-distance. They employed the notion of Ω-distance to created and proved some fixed point results in G-metric spaces. After that, Some authors obtained many fixed and common fixed point theorems in the setting of G-metric space by using the notion of Ω-distance [43–47]. It is worth mentioning that, the techniques of Jleli and Samet [39] and Samet *et.al* [40] are not working in the notion of Ω-distance.

Kirk et al. [48] gave a new generalization of Banach contraction theorem in very smart way. For this instant, he introduced the notion of cyclic mappings and proved a generalized version of Banach contraction theorem. Recently, many authors studied many fixed and common fixed point theorems for mappings of cyclic form in different matric space, for more details we refer the reader to [22–38]. Recently, Shatanawi and Postolache [8] introduced the notion on (A, B)-weakly increasing mappings and generalized many fixed and common fixed point results of cyclic form.

In this paper, we utilized the notion of (A, B)-weakly increasing mappings in the sense of W. Shatanawi, and M. Postolache to derive fixed point results in the setting of G-metric spaces where the techniques of Samet [39] and Samet *et.al* [40] are not working in our works.

2. Preliminary

In this section we present the important definitions and theorems which is used in sequel.

Definition 2.1 ([15])**.** Let X be a nonempty set, and let $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying:
$(G1)\, G(x, y, z) = 0\, if\, x = y = z$,
$(G2)\, G(x, x, y) > 0\, for\, all\, x, y \in X\, , with\, x \neq y$,
$(G3)\, G(x, y, y) \leq G(x, y, z)\, for\, all\, x, y, z \in X\, , with\, y \neq z$,
$(G4)\, G(x, y, z) = G(p\{x, y, z\})$, where $p\{x, y, z\}$ is the all possible permutation of x,y,z (symmetry in all three variables),
$(G5)\, G(x, y, z) \leq G(x, a, a) + G(a, y, z) \forall\, x, y, z, a \in X (rectangle\, inequality)$. Then the function G is called a *generalized metric space*, or more specifically G-metric on X, and the pair $(X$,G) is called a G-metric space.

Definition 2.2 ([15])**.** Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X, we say that (x_n) is G-convergent to x if
$\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq k$.

Proposition 2.3 ([15])**.** *Let (X, G) be G-metric space then the following are equivalent.*
(1) (x_n) is G-convergent to x.
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$.

Definition 2.4 ([15])**.** Let (X, G) be G-metric space, a sequence $(x_n) \subseteq X$ is said to be G-Cauchy if for every $\epsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$
for all n,m,l $\geq k$.

Proposition 2.5 ([15]). *In a G-metric space, the following are equivalent.*
(1) The sequence (x_n) is G-Cauchy.
(2)For every $\epsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_m) < \epsilon$ for all n,m,l $\geq k$.

Definition 2.6 ([17]). A G-metric space (X, G) is said to be G-complete or complete G-metric space if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 2.7 ([17]). Let (X, G) and (X', G') be two G-metric spaces and let $f : X \to X'$ be a function, then f is said to be G-continuous at a point a $\in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that x,y $\in X$; $G(a, x, y) < \delta$ implies $G'(fa, fx, fy) < \epsilon$. A function f is G-continuous on X if and only if it is G-continuous at every point a $\in X$.

Proposition 2.8 ([17]). *Let (X, G) and (X', G') be two G-metric spaces and let $f : X \to X'$ be a function, then f is said to be G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G'-convergent to f(x).*

Proposition 2.9 ([15]). *Let (X, G) be G-metric space. Then the function G is jointly continuous in all three of it's variables.*

3. Main Result

We start with the following definitions:

Definition 3.1 ([8]). Let $(X, \preceq)$ be a partially ordered set and A, B be two nonempty subsets of X with $X = A \cup B$. Let $f, g : X \to X$ be two mappings. Then the pair (f, g) is said to be (A, B)-weakly increasing if $fx \preceq gfx$ for all $x \in A$ and $gx \preceq fgx$ for all $x \in B$

Definition 3.2 ([49]). The function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied
(1) ϕ is continuous and nondecreasing.
(2) $\phi(t) = 0$ if and only if $t = 0$.

For more acquaintance on theorems related to altering distance functions, see [8, 50–54].

Theorem 3.3. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X,G) is complete G-metric space. Let A,B be two nonempty closed subsets of X with $X = A \cup B$. Let $f, g : A \cup B \to A \cup B$ such that the pair (f, g) is (A, B)-weakly increasing with $f(A) \subseteq B$, $g(B) \subseteq A$. Let ϕ, ψ be an altering distance functions. Also suppose that*

$$\begin{aligned} \phi G(fx, gfx, gy) \quad &\leq \phi \max\{G(x, y, y), G(fx, fx, x), G(gy, gy, y)\} \\ &-\psi \max\{G(x, y, y), G(fx, fx, x), G(gy, gy, y)\} \\ &\textit{for all comparative } x, y \in X \textit{ with } x \in A, y \in B \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \phi G(gx, fgx, fy) \quad &\leq \phi \max\{G(x, y, y), G(gx, gx, x), G(fy, fy, y)\} \\ &-\psi \max\{G(x, y, y), G(gx, gx, x), G(fy, fy, y)\} \\ &\textit{for all comparative } x, y \in X \textit{ with } x \in B, y \in A. \end{aligned} \tag{3.2}$$

If f or g is continuous then f and g have a common fixed point in $A \cap B$

Proof. Let $x_0 \in A$. Since $f(A) \subseteq B$, then $fx_0 = x_1 \in B$. Also, since $g(B) \subseteq A$, then $gx_1 = x_2 \in A$. Continuing this process we obtain a sequence (x_n) in X such that $fx_{2n} = x_{2n+1}$, $x_{2n} \in A$ and $gx_{2n+1} = x_{2n+2}$, $x_{2n+1} \in B \;\; n \in \mathbb{N}$. Since (f, g) is (A, B) weakly increasing, then $x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 \preceq fgx_1 = fx_2 = x_3 \cdots$. Therefore $x_n \preceq x_{n+1} \forall n \in \mathbb{N}$. If $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}$, then x_{2n+1} is a fixed point for $g \, in \, A \cap B$. To prove that x_{2n+1} is a fixed point for f it is equivalent to prove that $x_{2n+1} = x_{2n+2} = x_{2n+3}$.
Since $x_{2n+1} \preceq x_{2n+2}$, then by (3.2) we get

$$\begin{aligned} \phi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) &= \phi G(gx_{2n+1}, fgx_{2n+1}, fx_{2n+2}) \\ &\leq \phi \max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+3}, x_{2n+3}, x_{2n+2})\} \\ &-\psi \max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+3}, x_{2n+3}, x_{2n+2})\}. \end{aligned}$$

Since $x_{2n+1} = x_{2n+2}$, then

$$G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = G(x_{2n+2}, x_{2n+2}, x_{2n+1}) = 0,$$

and so

$$\phi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) \leq \phi G(x_{2n+3}, x_{2n+3}, x_{2n+2}) - \psi G(x_{2n+3}, x_{2n+3}, x_{2n+2}).$$

This implies that

$$\psi G(x_{2n+3}, x_{2n+3}, x_{2n+2}) = 0$$

and hence $G(x_{2n+3}, x_{2n+3}, x_{2n+2}) = 0$. Thus, $x_{2n+3} = x_{2n+2}$, and so $x_{2n+3} = x_{2n+2} = x_{2n+1}$. Therefore x_{2n+1} is also a fixed point for f. Hence x_{2n+1} is a common fixed point for f and $g \, \text{in} \, A \cap B$.
Now assume that $x_{n+1} \neq x_n$, $\forall n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Since $x_{2n+1} \preceq x_{2n+2}$, then by (3.2) we have

$$\begin{aligned} \phi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) &= \phi G(gx_{2n+1}, fgx_{2n+1}, fx_{2n+2}) \\ &\leq \phi \max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &\; G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+3}, x_{2n+3}, x_{2n+2})\} \\ &-\psi \max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &\; G(x_{2n+2}, x_{2n+2}, x_{2n+1}), G(x_{2n+3}, x_{2n+3}, x_{2n+2})\} \\ &= \phi \max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &\; G(x_{2n+3}, x_{2n+3}, x_{2n+2})\} \\ &-\psi \max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), \\ &\; G(x_{2n+3}, x_{2n+3}, x_{2n+2})\}. \end{aligned}$$

If $\max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), G(x_{2n+3}, x_{2n+3}, x_{2n+2})\} = G(x_{2n+3}, x_{2n+3}, x_{2n+2})$, then

$$\phi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) \leq \phi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) - \psi G(x_{2n+2}, x_{2n+3}, x_{2n+3}).$$

Therefore $\psi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) = 0$, and so $G(x_{2n+2}, x_{2n+3}, x_{2n+3}) = 0$.
Hence $x_{2n+3} = x_{2n+2}$ a contradiction. So,
$\max\{G(x_{2n+1}, x_{2n+2}, x_{2n+2}), G(x_{2n+3}, x_{2n+3}, x_{2n+2})\} = G(x_{2n+1}, x_{2n+2}, x_{2n+2})$.
Hence

$$\phi G(x_{2n+2}, x_{2n+3}, x_{2n+3}) \leq \phi G(x_{2n+1}, x_{2n+2}, x_{2n+2}) - \psi G(x_{2n+1}, x_{2n+2}, x_{2n+2}). \tag{3.3}$$

$$G(x_{2n+2}, x_{2n+3}, x_{2n+3}) \leq G(x_{2n+1}, x_{2n+2}, x_{2n+2}). \tag{3.4}$$

Again, since $x_{2n} \preceq x_{2n+1}$, then by (3.1) we have

$$\begin{aligned} \phi G(x_{2n+1}, x_{2n+2}, x_{2n+2}) &= \phi G(fx_{2n}, gfx_{2n}, gx_{2n+1}) \\ &\leq \phi \max\{G(x_{2n}, x_{2n+1}, x_{2n+1}), \\ &\quad G(x_{2n+1}, x_{2n+1}, x_{2n}), G(x_{2n+2}, x_{2n+2}, x_{2n+1})\} \\ &\quad -\psi \max\{G(x_{2n}, x_{2n+1}, x_{2n+1}), \\ &\quad G(x_{2n+1}, x_{2n+1}, x_{2n}), G(x_{2n+2}, x_{2n+2}, x_{2n+1})\} \\ &= \phi \max\{G(x_{2n}, x_{2n+1}, x_{2n+1}), \\ &\qquad G(x_{2n+2}, x_{2n+2}, x_{2n+1})\} \\ &\quad -\psi \max\{G(x_{2n}, x_{2n+1}, x_{2n+1}), \\ &\qquad G(x_{2n+2}, x_{2n+2}, x_{2n+1})\}. \end{aligned}$$

If $\max\{G(x_{2n}, x_{2n+1}, x_{2n+1}), G(x_{2n+2}, x_{2n+2}, x_{2n+1})\} = G(x_{2n+2}, x_{2n+2}, x_{2n+1})$, then
$\phi G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq \phi G(x_{2n+1}, x_{2n+2}, x_{2n+2}) - \psi G(x_{2n+1}, x_{2n+2}, x_{2n+2})$.
Therefore $\psi G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = 0$ and so $G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = 0$.
Hence $x_{2n+2} = x_{2n+1}$ a contradiction.
Thus

$$\phi G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq \phi G(x_{2n}, x_{2n+1}, x_{2n+1}) - \psi G(x_{2n}, x_{2n+1}, x_{2n+1}). \tag{3.5}$$

$$G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \leq G(x_{2n}, x_{2n+1}, x_{2n+1}). \tag{3.6}$$

From (3.4) and (3.6) we conclude that $\forall n \in \mathbb{N}$

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1}). \tag{3.7}$$

which means that $(G(x_n, x_{n+1}, x_{n+1}) : n \in \mathbb{N})$ is a nonnegative decreasing sequence. Therefore $\exists r \geq 0$ such that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r$. From (3.3) and (3.5) we conclude that $\forall n \in \mathbb{N}$

$$\phi G(x_{n+1}, x_{n+2}, x_{n+2}) \leq \phi G(x_n, x_{n+1}, x_{n+1}) - \psi G(x_n, x_{n+1}, x_{n+1}). \tag{3.8}$$

By taking the limit as $n \to \infty$ in (3.8) and using the fact that $\phi\, and\, \psi$ are continuous we get
$\phi r \leq \phi r - \psi r$. Therefore $\psi r = 0$ and so $r = 0$.
Hence

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.9}$$

Also, it can be proved that

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{3.10}$$

To show that (x_n) is G-Cauchy sequence it is sufficient to show that (x_{2n}) is G-Cauchy sequence.
Suppose to the contrary that (x_{2n}) is not G-Cauchy sequence. Then $\exists \epsilon > 0$ and two subsequences (x_{2n_k}),(x_{2m_k}) of (x_{2n}) such that m_k is chosen as the smallest index for which

$$G(x_{2n_k}, x_{2m_k}, x_{2m_k}) \geq \epsilon \qquad m_k > n_k, \tag{3.11}$$

this means that

$$G(x_{2n_k}, x_{2m_k-2}, x_{2m_k-2}) < \epsilon. \tag{3.12}$$

From (3.11) and (3.12), we get

$$\begin{aligned}\epsilon \leq G(x_{2n_k}, x_{2m_k}, x_{2m_k}) &\leq G(x_{2n_k}, x_{2m_k-2}, x_{2m_k-2}) + G(x_{2m_k-2}, x_{2m_k}, x_{2m_k})\\ &\leq G(x_{2n_k}, x_{2m_k-2}, x_{2m_k-2}) + G(x_{2m_k-2}, x_{2m_k-1}, x_{2m_k-1})\\ &\quad +G(x_{2m_k-1}, x_{2m_k}, x_{2m_k})\\ &< \epsilon + G(x_{2m_k-2}, x_{2m_k-1}, x_{2m_k-1}) + G(x_{2m_k-1}, x_{2m_k}, x_{2m_k}).\end{aligned}$$

On letting $k \to \infty$ and using (3.9) and (3.10), we get

$$\lim_{k\to\infty} G(x_{2n_k}, x_{2m_k}, x_{2m_k}) = \epsilon. \tag{3.13}$$

Also, by (G5) we have

$$\begin{aligned}G(x_{2n_k}, x_{2n_k}, x_{2m_k}) - G(x_{2n_k}, x_{2n_k}, x_{2n_k-1}) &\leq G(x_{2n_k-1}, x_{2n_k-1}, x_{2m_k})\\ &\leq G(x_{2n_k}, x_{2n_k-1}, x_{2n_k})\\ &\quad +G(x_{2n_k}, x_{2m_k}, x_{2m_k})\end{aligned}$$

On letting $k \to \infty$ and using (3.9), (3.10) and (3.13) we get

$$\lim_{k\to\infty} G(x_{2n_k-1}, x_{2n_k-1}, x_{2m_k}) = \epsilon. \tag{3.14}$$

Again, by (G5) we have

$$\begin{aligned}G(x_{2n_k}, x_{2n_k+1}, x_{2m_k+1}) &\leq G(x_{2n_k}, x_{2n_k}, x_{2n_k+1}) + G(x_{2n_k}, x_{2n_k}, x_{2m_k})\\ &\quad +G(x_{2m_k}, x_{2m_k}, x_{2m_k+1}).\end{aligned}$$

On letting $k \to \infty$ and using (3.10) and (3.13) we get

$$\lim_{k\to\infty} G(x_{2n_k}, x_{2n_k+1}, x_{2m_k+1}) \leq \epsilon.$$

Also,

$$\begin{aligned}G(x_{2n_k}, x_{2n_k}, x_{2m_k}) &\leq G(x_{2n_k}, x_{2n_k+1}, x_{2n_k+1}) + G(x_{2n_k+1}, x_{2n_k}, x_{2m_k+1})\\ &\quad +G(x_{2m_k+1}, x_{2m_k+1}, x_{2m_k}).\end{aligned}$$

On letting $k \to \infty$ and using (3.9) and (3.13) we get

$$\epsilon \leq \lim_{k\to\infty} G(x_{2n_k}, x_{2n_k+1}, x_{2m_k+1}).$$

Thus

$$\lim_{k\to\infty} G(x_{2n_k}, x_{2n_k+1}, x_{2m_k+1}) = \epsilon. \tag{3.15}$$

Now, since $x_{2n_k-1} \preceq x_{2m_k}$, then by using (3.2) we get

$$\begin{aligned}\phi G(x_{2n_k}, x_{2n_k+1}, x_{2m_k+1}) &= \phi G(gx_{2n_k-1}, fgx_{2n_k-1}, fx_{2m_k})\\ &\leq \phi \max\{G(x_{2n_k-1}, x_{2m_k}, x_{2m_k}),\\ &\qquad G(x_{2n_k}, x_{2n_k}, x_{2n_k-1}),\\ &\qquad G(x_{2m_k+1}, x_{2m_k+1}, x_{2m_k})\}\\ &\quad -\psi \max\{G(x_{2n_k-1}, x_{2m_k}, x_{2m_k}),\\ &\qquad G(x_{2n_k}, x_{2n_k}, x_{2n_k-1}),\\ &\qquad G(x_{2m_k+1}, x_{2m_k+1}, x_{2m_k})\}.\end{aligned}$$

Taking the limit as $k \to \infty$ and using the fact that ϕ and ψ are continuous and using (3.9), (3.14) and (3.15), we get

$$\phi\epsilon \leq \phi\epsilon - \psi\epsilon.$$

Therefore $\psi\epsilon = 0$ and so $\epsilon = 0$ a contradiction. Hence (x_{2n}) is G-Cauchy sequence, and so (x_n) is G-Cauchy sequence. Since (X, G) is a complete G-metric space, then there exists $u \in X$ such that (x_n) is G-converges to u. Therefore the subsequences $(gx_{2n+1}), f(x_{2n}), (x_{2n+1})$, and (x_{2n}) are G-converge to u. Since $(x_{2n}) \subseteq A$ and A closed then $u \in A$, and since $(x_{2n+1}) \subseteq B$ and B closed, then $u \in B$. First suppose (without lose of generality) that f is continuous. Then
$\lim_{n\to\infty} fx_{2n} = fu$ and $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} x_{2n+1} = u$, by uniqueness of the limit we have $fu = u$. Since $u \in A \cap B$ and $u \preceq u$ then by (3.1) we have

$$\begin{aligned}\phi G(u, gu, gu) = \phi G(fu, gfu, gu) &\leq \phi \max\{G(u, u, u), G(fu, fu, u),\\ &\qquad G(gu, gu, u)\}\\ &\quad -\psi \max\{G(u, u, u), G(fu, fu, u),\\ &\qquad G(gu, gu, u)\}\\ &\leq \phi G(gu, gu, u) - \psi G(gu, gu, u).\end{aligned}$$

Therefore $\psi G(gu, gu, u) = 0$ and so $G(gu, gu, u) = 0$. Hence $gu = u$. Hence u is a common fixed point for $f \, and \, g$ in $A \cap B$. ∎

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X, G) is complete G-metric space. Let A,B be two nonempty closed subsets of X with $X = A \cup B$. Let $f : A \cup B \to A \cup B$ be a continuous function, such that $fx \preceq f^2x \, \forall x \in X$, with $f(A) \subseteq B$, $f(B) \subseteq A$. Let ϕ, ψ be an altering distance functions, and suppose that*

$$\begin{aligned}\phi G(fx, f^2x, fy) &\leq \phi \max\{G(x, y, y), G(fx, fx, x), G(fy, fy, y)\}\\ &\quad -\psi \max\{G(x, y, y), G(fx, fx, x), G(fy, fy, y)\}\end{aligned} \tag{3.16}$$

for all comparative $x, y \in X$ with $x \in A, y \in B$ or $x \in B, y \in A$. Then f has a fixed point in $A \cap B$.

Proof. It follows from Theorem 3.3 by taking $g = f$. ∎

Theorem 3.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X, G) is complete G-metric space. Let A,B be two nonempty closed subsets of X with $X = A \cup B$. Let $f, g : A \cup B \to A \cup B$ such that the pair (f, g) is (A, B)-weakly increasing with $f(A) \subseteq B$ and $g(B) \subseteq A$. Suppose that $\exists\, r \in [0, 1)$ such that*

$$G(fx, gfx, gy) \leq r \, \max\{G(x, y, y), G(fx, fx, x), G(gy, gy, y)\} \tag{3.17}$$

for all comparative $x, y \in X$ with $x \in A, y \in B$,and

$$G(gx, fgx, fy) \leq r \, \max\{G(x, y, y), G(gx, gx, x), G(fy, fy, y)\} \tag{3.18}$$

for all comparative $x, y \in X$ with $x \in B, y \in A$
If f or g is continuous then f and g have a common fixed point in $A \cap B$.

Proof. Define $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\phi(t) = t$ and $\psi(t) = (1 - r)t$. Then the proof follows from Theorem 3.3. ∎

Corollary 3.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a G-metric on X such that (X,G) is complete G-metric space. Let A,B be two nonempty closed subsets of X with $X = A \cup B$. Let $f : A \cup B \to A \cup B$ be a continuous function such that $fx \preceq f^2x \, \forall x \in X$ with $f(A) \subseteq B$ and $f(B) \subseteq A$. Suppose that $\exists\ r \in [0, 1)$ such that*

$$G(fx, f^2x, fy) \le r\, max\{G(x, y, y), G(fx, fx, x), G(fy, fy, y)\} \tag{3.19}$$

for all comparative $x, y \in X$ with $x \in A,\ y \in B$ or $x \in B,\ y \in A$.

Then f has a fixed point in $A \cap B$.

Proof. The proof follows from Theorem (3.5) by taking $g = f$. ∎

Conclusion

Samet et al [39, 40] in their outstanding paper proved that some fixed point theorems in G-metric spaces in sense of Mustafa and Sims [15] can be reduce to some known fixed point theorems in standard metric space. While, Karapinar and Agarwal [41] in their nice paper introduced some theorems where the method of Samet et al [39, 40] can not be used in their theorems. In this paper, we showed that the study of fixed point theorems in G-metric spaces in the sense of Sims and Mustafa [15] is significant and real generalization of fixed point theorems in standard fixed point theorems. Our result supported the conclusion of Karapinar and Agarwal in their paper [41] about the fixed point theorems in G-metric spaces.

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