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Screen Semi-Slant Lightlike Submanifolds of Indefinite Kaehler Manifolds

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Abstract In this paper, we introduce the notion of screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds giving characterization theorem with some non-trivial examples of such submanifolds. Integrability conditions of distributions D_1, D_2 and $RadTM$ on screen semi-slant lightlike submanifolds of an indefinite Kaehler manifold have been obtained. Further we obtain necessary and sufficient conditions for foliations determined by above distributions to be totally geodesic. We also study mixed geodesic screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds.

MSC: 53C15; 53C40; 53C50

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1. Introduction

The theory of lightlike submanifolds of a semi-Riemannian manifold was introduced by Duggal and Bejancu [1]. Various classes of lightlike submanifolds of indefinite Kaehler manifolds are defined according to the behaviour of distributions on these submanifolds with respect to the action of (1,1) tensor field $\overline{J}$ in Kaehler structure of the ambient manifolds. Such submanifolds have been studied by Duggal and Sahin in [2]. In [3], Sahin studied screen-slant lightlike submanifolds of an indefinite Hermitian manifold. In [4], B.Y. Chen defined slant immersions in complex geometry as a natural generalization of both holomorphic immersions and totally real immersions. The geometry of semi-slant submanifolds of Kaehler manifolds was studied by N. Papaghuic in [5].
The theory of invariant, screen real, screen slant, screen Cauchy-Riemann lightlike submanifolds have been studied in [2]. Thus motivated sufficiently, we introduce the notion of screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds. This new class of lightlike submanifolds of an indefinite Kaehler manifold includes invariant, screen slant, screen real, screen Cauchy-Riemann lightlike submanifolds as its sub-cases. The paper is

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arranged as follows. There are some basic results in Section 2. In Section 3, we study screen semi-slant lightlike submanifolds of an indefinite Kaehler manifold, giving some examples. Section 4 is devoted to the study of foliations determined by distributions on screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds.

2. Preliminaries

A submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a *lightlike submanifold* [1] if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $RadTM$ is of rank r, where $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM, that is

$$TM = RadTM \oplus_{orth} S(TM). \tag{2.1}$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^\perp$. Since for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $[S(TM)]^\perp$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$. Then

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^\perp), \tag{2.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM), \tag{2.3}$$

$$T\overline{M}|_M = S(TM) \oplus_{orth} [RadTM \oplus ltr(TM)] \oplus_{orth} S(TM^\perp). \tag{2.4}$$

Following are four cases of a lightlike submanifold $\big(M, g, S(TM), S(TM^\perp)\big)$:
Case.1 r-lightlike if $r < min\,(m, n)$,
Case.2 co-isotropic if $r = n < m$, $S\left(TM^\perp\right) = \{0\}$,
Case.3 isotropic if $r = m < n$, $S\left(TM\right) = \{0\}$,
Case.4 totally lightlike if $r = m = n$, $S(TM) = S(TM^\perp) = \{0\}$.
The Gauss and Weingarten formulae are given as

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X V = - A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y, A_V X$ belong to $\Gamma(TM)$ and $h(X, Y), \nabla^t_X V$ belong to $\Gamma(tr(TM))$. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (2.5) and (2.6), for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l\left(X, Y\right) + h^s\left(X, Y\right), \tag{2.7}$$

$$\overline{\nabla}_X N = - A_N X + \nabla^l_X N + D^s\left(X, N\right), \tag{2.8}$$

$$\overline{\nabla}_X W = - A_W X + \nabla^s_X W + D^l\left(X, W\right), \tag{2.9}$$

where $h^l(X, Y) = L\left(h(X, Y)\right)$, $h^s(X, Y) = S\left(h(X, Y)\right)$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$. L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ respectively. ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^\perp)$ called the *lightlike connection* and *screen transversal connection* on M respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection $\overline{\nabla}$, we obtain

$$\overline{g}(h^s(X,Y),W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X). \tag{2.11}$$

Denote the projection of TM on $S(TM)$ by $\overline{P}$. Then from the decomposition of the tangent bundle of a lightlike submanifold, for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, we have

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi. \tag{2.13}$$

By using above equations, we obtain

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X, \overline{P}Y), \tag{2.14}$$

$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X, \overline{P}Y), \tag{2.15}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{2.16}$$

It is important to note that in general ∇ is not a metric connection. Since $\overline{\nabla}$ is metric connection, by using (2.7), we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{2.17}$$

An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is a 2m-dimensional semi-Riemannian manifold $\overline{M}$ with semi-Riemannian metric $\overline{g}$ of constant index q, $0 < q < 2m$ and a (1, 1) tensor field $\overline{J}$ on $\overline{M}$ such that following conditions are satisfied:

$$\overline{J}^2 X = -X, \quad \forall X \in \Gamma(T\overline{M}), \tag{2.18}$$

$$\overline{g}(\overline{J}X, \overline{J}Y) = \overline{g}(X,Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\overline{M})$.
An indefinite almost Hermitian manifold $(\overline{M}, \overline{g}, \overline{J})$ is called an *indefinite Kaehler manifold* if $\overline{J}$ is parallel with respect to $\overline{\nabla}$, i.e.,

$$(\overline{\nabla}_X \overline{J})Y = 0, \tag{2.20}$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ is Levi-Civita connection with respect to $\overline{g}$.

3. Screen Semi-Slant Lightlike Submanifolds

In this section, we introduce the notion of screen semi-slant lightlike submanifolds of indefinite Kaehler manifolds. At first, we state the following Lemma for later use:

Lemma 3.1. *Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$. Then the screen distribution $S(TM)$ of lightlike submanifold M is Riemannian.*

The proof of above lemma follows as in Lemma 3.1 of [3], so we omit it.

Definition: Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of index $2q$ such that $2q < dim(M)$. Then we say that M is a *screen semi-slant lightlike submanifold* of $\overline{M}$ if the following conditions are satisfied:
(i) $RadTM$ is invariant with respect to $\overline{J}$, i.e. $\overline{J}(RadTM) = RadTM$,

(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2$,
(iii) the distribution D_1 is an invariant distribution, i.e. $\overline{J}D_1 = D_1$,
(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between $\overline{J}X$ and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.
This constant angle θ is called the *slant angle of distribution* D_2. A screen semi-slant lightlike submanifold is said to be *proper* if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$. From the above definition, we have the following decomposition

$$TM = RadTM \oplus_{orth} D_1 \oplus_{orth} D_2. \tag{3.1}$$

In particular, we have
(i) if $D_1 = 0$, then M is a screen slant lightlike submanifold,
(ii) if $D_2 = 0$, then M is an invariant lightlike submanifold,
(iii) if $D_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is a screen real lightlike submanifold,
(iv) if $D_1 \neq 0$ and $\theta = \frac{\pi}{2}$, then M is a screen CR-lightlike submanifold.
Thus the above new class of lightlike submanifolds of an indefinite Kaehler manifold includes invariant, screen slant, screen real, screen Cauchy-Riemann lightlike submanifolds as its sub-cases which have been studied in [2, 3].
Let $(\mathbb{R}^{2m}_{2q}, \overline{g}, \overline{J})$ denote the manifold $\mathbb{R}^{2m}_{2q}$ with its usual Kaehler structure given by

$\overline{g} = \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i)$,

$\overline{J}(\sum_{i=1}^{m}(X_i\partial x_i + Y_i\partial y_i)) = \sum_{i=1}^{m}(Y_i\partial x_i - X_i\partial y_i)$,

where (x^i, y^i) are the cartesian coordinates on $\mathbb{R}^{2m}_{2q}$. Now, we construct some examples of screen semi-slant lightlike submanifolds of an indefinite Kaehler manifold.

Example 3.2. Let $(\mathbb{R}^{12}_2, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose M is a submanifold of $\mathbb{R}^{12}_2$ given by $x^1 = y^2 = u_1$, $x^2 = -y^1 = u_2$, $x^3 = u_3\cos\beta$, $y^4 = u_3\sin\beta$, $x^4 = u_4\sin\beta$, $y^3 = -u_4\cos\beta$, $x^5 = u_5\sin u_6$, $y^5 = u_5\cos u_6$, $x^6 = \sin u_5$, $y^6 = \cos u_5$.
The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = 2(\partial x_1 + \partial y_2)$, $\quad Z_2 = 2(\partial x_2 - \partial y_1)$,
$Z_3 = 2(\cos\beta\partial x_3 + \sin\beta\partial y_4)$, $Z_4 = 2(\sin\beta\partial x_4 - \cos\beta\partial y_3)$,
$Z_5 = 2(\sin u_6\partial x_5 + \cos u_6\partial y_5 + \cos u_5\partial x_6 - \sin u_5\partial y_6)$,
$Z_6 = 2(u_5\cos u_6\partial x_5 - u_5\sin u_6\partial y_5)$.

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, Z_5, Z_6\}$.
Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \partial y_2$, $\quad N_2 = -\partial x_2 - \partial y_1$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\sin\beta\partial x_3 - \cos\beta\partial y_4)$, $W_2 = 2(\cos\beta\partial x_4 + \sin\beta\partial y_3)$,
$W_3 = 2(\sin u_6\partial x_5 + \cos u_6\partial y_5 - \cos u_5\partial x_6 + \sin u_5\partial y_6)$,
$W_4 = 2(u_5\sin u_5\partial x_6 + u_5\cos u_5\partial y_6)$.

It follows that $\overline{J}Z_1 = Z_2$ and $\overline{J}Z_2 = -Z_1$, which implies that $RadTM$ is invariant, i.e. $\overline{J}RadTM = RadTM$. On the other hand, we can see that $D_1 = span\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = Z_4$ and $\overline{J}Z_4 = -Z_3$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle $\frac{\pi}{4}$. Hence M is a screen semi-slant 2-lightlike submanifold of $\mathbb{R}^{12}_2$.

Example 3.3. Let $(\mathbb{R}_2^{12}, \overline{g}, \overline{J})$ be an indefinite Kaehler manifold, where $\overline{g}$ is of signature $(-,+,+,+,+,+,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6\}$.

Suppose M is a submanifold of $\mathbb{R}_2^{12}$ given by $x^1 = -y^2 = u_1$, $x^2 = y^1 = u_2$, $x^3 = y^4 = u_3$, $x^4 = -y^3 = u_4$, $x^5 = u_5$, $y^5 = u_6$, $x^6 = k\cos u_6$, $y^6 = k\sin u_6$, where k is any constant.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, where

$Z_1 = 2(\partial x_1 - \partial y_2)$, $\quad Z_2 = 2(\partial x_2 + \partial y_1)$,
$Z_3 = 2(\partial x_3 + \partial y_4)$, $\quad Z_4 = 2(\partial x_4 - \partial y_3)$,
$Z_5 = 2(\partial x_5)$, $Z_6 = 2(\partial y_5 - k\sin u_6 \partial x_6 + k\cos u_6 \partial y_6)$.

Hence $RadTM = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, Z_5, Z_6\}$.

Now $ltr(TM)$ is spanned by $N_1 = -\partial x_1 - \partial y_2$, $\quad N_2 = -\partial x_2 + \partial y_1$ and $S(TM^\perp)$ is spanned by

$W_1 = 2(\partial x_3 - \partial y_4)$, $\quad W_2 = 2(\partial x_4 + \partial y_3)$,
$W_3 = 2(k\cos u_6 \partial x_6 + k\sin u_6 \partial y_6)$,
$W_4 = 2(k^2 \partial y_5 + k\sin u_6 \partial x_6 - k\cos u_6 \partial y_6)$.

It follows that $\overline{J}Z_1 = -Z_2$ and $\overline{J}Z_2 = Z_1$, which implies that $RadTM$ is invariant, i.e. $\overline{J}RadTM = RadTM$. On the other hand, we can see that $D_1 = span\{Z_3, Z_4\}$ such that $\overline{J}Z_3 = Z_4$ and $\overline{J}Z_4 = -Z_3$, which implies that D_1 is invariant with respect to $\overline{J}$ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{1+k^2})$. Hence M is a screen semi-slant 2-lightlike submanifold of $\mathbb{R}_2^{12}$.

Now, for any vector field X tangent to M, we put $\overline{J}X = PX + FX$, where PX and FX are tangential and transversal parts of $\overline{J}X$ respectively. We denote the projections on $RadTM$, D_1 and D_2 in TM by P_1, P_2 and P_3 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2 respectively. Then, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X. \tag{3.2}$$

Now applying $\overline{J}$ to (3.2), we have

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + \overline{J}P_3X, \tag{3.3}$$

which gives

$$\overline{J}X = \overline{J}P_1X + \overline{J}P_2X + fP_3X + FP_3X, \tag{3.4}$$

where fP_3X (resp. FP_3X) denotes the tangential (resp. transversal) component of $\overline{J}P_3X$. Thus we get $\overline{J}P_1X \in \Gamma(RadTM)$, $\overline{J}P_2X \in \Gamma(D_1)$, $fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(S(TM^\perp))$. Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = Q_1W + Q_2W. \tag{3.5}$$

Applying $\overline{J}$ to (3.5), we obtain

$$\overline{J}W = \overline{J}Q_1W + \overline{J}Q_2W, \tag{3.6}$$

which gives

$$\overline{J}W = \overline{J}Q_1W + BQ_2W + CQ_2W, \tag{3.7}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of $\overline{J}Q_2W$. Thus we get $\overline{J}Q_1W \in \Gamma(ltr(TM))$, $BQ_2W \in \Gamma(D_2)$ and $CQ_2W \in \Gamma(S(TM^\perp))$.

Now, by using (2.20), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $RadTM$, D_1, D_2, $ltr(TM)$ and $S(TM^{\perp})$, we obtain

$$\begin{aligned} P_1(\nabla_X \overline{J}P_1Y) + P_1(\nabla_X \overline{J}P_2Y) + P_1(\nabla_X fP_3Y) = P_1(A_{FP_3Y}X) \\ + \overline{J}P_1\nabla_X Y, \end{aligned} \tag{3.8}$$

$$\begin{aligned} P_2(\nabla_X \overline{J}P_1Y) + P_2(\nabla_X \overline{J}P_2Y) + P_2(\nabla_X fP_3Y) = P_2(A_{FP_3Y}X) \\ + \overline{J}P_2\nabla_X Y, \end{aligned} \tag{3.9}$$

$$\begin{aligned} P_3(\nabla_X \overline{J}P_1Y) + P_3(\nabla_X \overline{J}P_2Y) + P_3(\nabla_X fP_3Y) = P_3(A_{FP_3Y}X) \\ + fP_3\nabla_X Y + Bh^s(X,Y), \end{aligned} \tag{3.10}$$

$$h^l(X,\overline{J}P_1Y) + h^l(X,\overline{J}P_2Y) + h^l(X,fP_3Y) = \overline{J}h^l(X,Y) - D^l(X,FP_3Y), \tag{3.11}$$

$$\begin{aligned} h^s(X,\overline{J}P_1Y) + h^s(X,\overline{J}P_2Y) + h^s(X,fP_3Y) =& Ch^s(X,Y) - \nabla^s_X FP_3Y \\ &+ FP_3\nabla_X Y. \end{aligned} \tag{3.12}$$

Theorem 3.4. *Let M be a $2q$-lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then M is a screen semi-slant lightlike submanifold of $\overline{M}$ if and only if*
(i) $ltr(TM)$ and D_1 are invariant with respect to $\overline{J}$,
(ii) there exists a constant $\lambda \in [0,1)$ such that $P^2X = -\lambda X$.
Moreover, there also exists a constant $\mu \in (0,1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus_{orth} D_2$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then distributions D_1 and $RadTM$ are invariant with respect to $\overline{J}$. Now for any $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(S(TM))$, using (2.19) and (3.4), we obtain $\overline{g}(\overline{J}N, X) = -\overline{g}(N, \overline{J}X) = -\overline{g}(N, \overline{J}P_2X + f\overline{J}P_3X + F\overline{J}P_3X) = 0$. Thus $\overline{J}N$ does not belong to $\Gamma(S(TM))$. For any $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, from (2.19) and (3.7), we have $\overline{g}(\overline{J}N, W) = -\overline{g}(N, \overline{J}W) = -\overline{g}(N, BW + CW) = 0$. Hence, we conclude that $\overline{J}N$ does not belong to $\Gamma(S(TM^{\perp}))$.
Now suppose that $\overline{J}N \in \Gamma(RadTM)$. Then $\overline{J}(\overline{J}N) = \overline{J}^2N = -N \in \Gamma(ltrTM)$, which contradicts that $RadTM$ is invariant. Thus $ltr(TM)$ is invariant with respect to $\overline{J}$. Now for any $X \in \Gamma(D_2)$ we have $|PX| = |\overline{J}X|\cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\overline{J}X|}. \tag{3.13}$$

In view of (3.13), we get $\cos^2\theta = \frac{|PX|^2}{|\overline{J}X|^2} = \frac{g(PX,PX)}{g(\overline{J}X,\overline{J}X)} = \frac{g(X,P^2X)}{g(X,\overline{J}^2X)}$, which gives

$$g(X,P^2X) = \cos^2\theta\, g(X,\overline{J}^2X). \tag{3.14}$$

Since M is a screen semi-slant lightlike submanifold, $\cos^2\theta = \lambda(constant) \in [0,1)$ and therefore from (3.14), we get $g(X,P^2X) = \lambda g(X,\overline{J}^2X) = g(X,\lambda\overline{J}^2X)$, which implies

$$g(X,(P^2 - \lambda\overline{J}^2)X) = 0. \tag{3.15}$$

Since $(P^2 - \lambda\overline{J}^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2 \times D_2}$ is non-degenerate (positive definite), from (3.15), we have $(P^2 - \lambda\overline{J}^2)X = 0$, which implies

$$P^2X = \lambda\overline{J}^2X = -\lambda X. \tag{3.16}$$

For any vector field $X \in \Gamma(D_2)$, we have

$$\overline{J}X = PX + FX, \tag{3.17}$$

where PX and FX are tangential and transversal parts of $\overline{J}X$ respectively.
Now, applying $\overline{J}$ to (3.17) and taking tangential component, we get

$$-X = P^2X + BFX. \tag{3.18}$$

From (3.16) and (3.18), we get

$$BFX = -\mu X, \tag{3.19}$$

where $1 - \lambda = \mu(constant) \in (0, 1]$. This proves (ii).
Conversely suppose that conditions (i) and (ii) are satisfied. We can show that $RadTM$ is invariant in similar way that $ltr(TM)$ is invariant. From (3.18), for any $X \in \Gamma(D_2)$, we get

$$-X = P^2X - \mu X, \tag{3.20}$$

which implies

$$P^2X = -\lambda X, \tag{3.21}$$

where $1 - \mu = \lambda(constant) \in [0, 1)$.
Now $\cos\theta = \frac{g(\overline{J}X,PX)}{|\overline{J}X||PX|} = -\frac{g(X,\overline{J}PX)}{|\overline{J}X||PX|} = -\frac{g(X,P^2X)}{|\overline{J}X||PX|} = -\lambda\frac{g(X,\overline{J}^2X)}{|\overline{J}X||PX|} = \lambda\frac{g(\overline{J}X,\overline{J}X)}{|\overline{J}X||PX|}$.
From above equation, we get

$$\cos\theta = \lambda\frac{|\overline{J}X|}{|PX|}. \tag{3.22}$$

Therefore (3.13) and (3.22) give $\cos^2\theta = \lambda(constant)$.
Hence M is a screen semi-slant lightlike submanifold. ∎

Corollary 3.5. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ with slant angle θ, then for any $X, Y \in \Gamma(D_2)$, we have*
(i) $g(PX, PY) = \cos^2\theta\, g(X, Y)$,
(ii) $g(FX, FY) = \sin^2\theta\, g(X, Y)$.

The proof of above Corollary follows by using similar steps as in proof of Corollary 3.2 of [3].

Theorem 3.6. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ is integrable if and only if*
(i) $h^s(Y, \overline{J}P_1X) = h^s(X, \overline{J}P_1Y)$ and $P_2(\nabla_X\overline{J}P_1Y) = P_2(\nabla_Y\overline{J}P_1X)$,
(ii) $P_3(\nabla_X\overline{J}P_1Y) = P_3(\nabla_Y\overline{J}P_1X)$,
for all $X, Y \in \Gamma(RadTM)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(RadTM)$. From (3.12), we have $h^s(X, \overline{J}P_1Y) = Ch^s(X, Y) + FP_3\nabla_X Y$, which gives $h^s(X, \overline{J}P_1Y) - h^s(Y, \overline{J}P_1X) = FP_3[X, Y]$. In view of (3.9), we obtain $P_2(\nabla_X \overline{J}P_1Y) = \overline{J}P_2\nabla_X Y$, which implies $P_2(\nabla_X \overline{J}P_1Y) - P_2(\nabla_Y \overline{J}P_1X) = \overline{J}P_2[X, Y]$. Also from (3.10), we get $P_3(\nabla_X \overline{J}P_1Y) = fP_3\nabla_X Y + Bh^s(X, Y)$, which gives $P_3(\nabla_X \overline{J}P_1Y) - P_3(\nabla_Y \overline{J}P_1X) = fP_3[X, Y]$. ■

Theorem 3.7. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 is integrable if and only if*
(i) $h^s(Y, \overline{J}P_2X) = h^s(X, \overline{J}P_2Y)$, $P_1(\nabla_X \overline{J}P_2Y) = P_1(\nabla_Y \overline{J}P_2X)$,
(ii) $P_3(\nabla_X \overline{J}P_2Y) = P_3(\nabla_Y \overline{J}P_2X)$,
for all $X, Y \in \Gamma(D_1)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_1)$. From (3.12), we have $h^s(X, \overline{J}P_2Y) = Ch^s(X, Y) + FP_3\nabla_X Y$, which gives $h^s(X, \overline{J}P_2Y) - h^s(Y, \overline{J}P_2X) = FP_3[X, Y]$. In view of (3.8), we have $P_1(\nabla_X \overline{J}P_2Y) = \overline{J}P_1\nabla_X Y$, which gives $P_1(\nabla_X \overline{J}P_2Y) - P_1(\nabla_Y \overline{J}P_2X) = \overline{J}P_1[X, Y]$. In view of (3.10), we get $P_3(\nabla_X \overline{J}P_2Y) = fP_3\nabla_X Y + Bh^s(X, Y)$, which implies $P_3(\nabla_X \overline{J}P_2Y) - P_3(\nabla_Y \overline{J}P_2X) = fP_3[X, Y]$. ■

Theorem 3.8. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 is integrable if and only if*
(i) $P_1(\nabla_X fP_3Y - \nabla_Y fP_3X) = P_1(A_{FP_3Y}X - A_{FP_3X}Y)$,
(ii) $P_2(\nabla_X fP_3Y - \nabla_Y fP_3X) = P_2(A_{FP_3Y}X - A_{FP_3X}Y)$,
for all $X, Y \in \Gamma(D_2)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Let $X, Y \in \Gamma(D_2)$. In view of (3.8), we get $P_1(\nabla_X fP_3Y) = P_1(A_{FP_3Y}X) + \overline{J}P_1\nabla_X Y$, which gives $P_1(\nabla_X fP_3Y) - P_1(\nabla_Y fP_3X) - P_1(A_{FP_3Y}X) + P_1(A_{FP_3X}Y) = \overline{J}P_1[X, Y]$. From (3.9), we obtain $P_2(\nabla_X fP_3Y) = P_2(A_{FP_3Y}X) + \overline{J}P_2\nabla_X Y$, which gives $P_2(\nabla_X fP_3Y) - P_2(\nabla_Y fP_3X) - P_2(A_{FP_3Y}X) + P_2(A_{FP_3X}Y) = \overline{J}P_2[X, Y]$. ■

Theorem 3.9. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ is a metric connection if and only if*
(i) $Bh^s(X, Y) = 0$,
(ii) A^*_Y vanishes on $\Gamma(TM)$,
for all $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ on M is a metric connection if and only if $RadTM$ is parallel distribution with respect to ∇ ([4]). From (2.7), (2.13) and (2.20), for any $X \in \Gamma(TM)$ and $Y \in \Gamma(RadTM)$, we have $\overline{\nabla}_X \overline{J}Y = \overline{J}\nabla^{*t}_X Y - \overline{J}A^*_Y X + \overline{J}h^l(X, Y) + \overline{J}h^s(X, Y)$. On comparing tangential components of both sides of above equation, we obtain $\nabla_X \overline{J}Y = \overline{J}\nabla^{*t}_X Y - \overline{J}A^*_Y X + Bh^s(X, Y)$. ■

4. Foliations Determined by Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold to be totally geodesic.

Definition: A screen semi-slant lightlike submanifold M of an indefinite Kaehler manifold $\overline{M}$ is said to be a *mixed geodesic* if its second fundamental form h satisfies $h(X,Y)=0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is mixed geodesic screen semi-slant lightlike submanifold if $h^l(X,Y)=0$ and $h^s(X,Y)=0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

Theorem 4.1. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $RadTM$ defines a totally geodesic foliation if and only if* $\overline{g}(h^l(X,PZ),\overline{J}Y) = -\overline{g}(D^l(X,FZ),\overline{J}Y)$*, for all $X,Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$.*

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. It is easy to see that $RadTM$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(RadTM)$, for all $X,Y \in \Gamma(RadTM)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.20), for any $X,Y \in \Gamma(RadTM)$ and $Z \in \Gamma(S(TM))$, we get $\overline{g}(\nabla_X Y,Z) = -\overline{g}(\overline{\nabla}_X PZ + \overline{\nabla}_X FZ, \overline{J}Y)$, which implies $\overline{g}(\nabla_X Y,Z) = -\overline{g}(h^l(X,PZ)+D^l(X,FZ),\overline{J}Y)$. ∎

Theorem 4.2. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_1 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(A_{FZ}X,\overline{J}Y) = \overline{g}(\nabla_X fZ,\overline{J}Y)$,

(ii) $A_{\overline{J}N}X$ has no component in D_1,

for all $X,Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. To prove the distribution D_1 defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_1)$, for all $X,Y \in \Gamma(D_1)$. Since $\overline{\nabla}$ is metric connection, From (2.7), (2.19) and (2.20), for any $X,Y \in \Gamma(D_1)$, and $Z \in \Gamma(D_2)$, we obtain $\overline{g}(\nabla_X Y,Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z,\overline{J}Y)$, which gives $\overline{g}(\nabla_X Y,Z) = \overline{g}(A_{FZ}X - \nabla_X fZ,\overline{J}Y)$. In view of (2.7), (2.19) and (2.20), for any $X,Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$, we get $\overline{g}(\nabla_X Y,N) = -\overline{g}(\overline{J}Y,\overline{\nabla}_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y,N) = \overline{g}(\overline{J}Y, A_{\overline{J}N}X)$. ∎

Theorem 4.3. *Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{g}(\nabla_X fY,\overline{J}Z) = \overline{g}(A_{FY}X,\overline{J}Z)$,

(ii) $\overline{g}(fY,A_{\overline{J}N}X) = \overline{g}(FY,D^s(X,\overline{J}N))$,

for all $X,Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X,Y \in \Gamma(D_2)$. From (2.7), (2.19) and (2.20) for any $X,Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y,Z) = \overline{g}(\overline{\nabla}_X \overline{J}Y,\overline{J}Z)$, which implies $\overline{g}(\nabla_X Y,Z) = \overline{g}(\nabla_X fY - A_{FY}X,\overline{J}Z)$. Since $\overline{\nabla}$ is metric connection, From (2.7), (2.19) and (2.20), for any $X,Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we get $\overline{g}(\nabla_X Y,N) = -\overline{g}(\overline{J}Y,\overline{\nabla}_X \overline{J}N)$, which gives $\overline{g}(\nabla_X Y,N) = \overline{g}(fY,A_{\overline{J}N}X) - \overline{g}(FY,D^s(X,\overline{J}N))$. ∎

Theorem 4.4. *Let M be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then D_2 defines a totally geodesic foliation if and only if*
(i) $\nabla_X \overline{J}Z$ has no component in D_2,
(ii) $\overline{g}(fY, A_{\overline{J}N}X) = \overline{g}(FY, D^s(X, \overline{J}N))$,
for all $X, Y \in \Gamma(D_2)$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. The distribution D_2 defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(D_2)$, for all $X, Y \in \Gamma(D_2)$. Since $\overline{\nabla}$ is metric connection, From (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\overline{\nabla}_X \overline{J}Z, \overline{J}Y)$, which gives $\overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X \overline{J}Z, fY) - \overline{g}(h^s(X, \overline{J}Z), FY)$. In view of (2.7), (2.19) and (2.20), for any $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$, we get $\overline{g}(\nabla_X Y, N) = -\overline{g}(\overline{J}Y, \overline{\nabla}_X \overline{J}N)$, which implies $\overline{g}(\nabla_X Y, N) = \overline{g}(fY, A_{\overline{J}N}X) - \overline{g}(FY, D^s(X, \overline{J}N))$. ■

Theorem 4.5. *Let M be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then the induced connection ∇ on $S(TM)$ is a metric connection if and only if*
(i) $A^*_\xi X$ has no component in D_1,
(ii) $\overline{g}(fW, A^*_{\overline{J}\xi}Z) = \overline{g}(FW, h^s(Z, \overline{J}\xi))$,
for all $X \in \Gamma(D_1)$, $Z, W \in \Gamma(D_2)$, and $\xi \in \Gamma(RadTM)$.

Proof. Let M be a mixed geodesic screen semi-slant lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$. Then $h^l(X, Z) = 0$, for all $X \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$. In view of (2.14), for any $X, Y \in \Gamma(D_1)$ and $\xi \in \Gamma(RadTM)$, we have $\overline{g}(h^l(X, Y), \xi) = g(Y, A^*_\xi X)$. Since $\overline{\nabla}$ is metric connection, using (2.7), (2.19) and (2.20), for any $Z, W \in \Gamma(D_2)$ and $\xi \in \Gamma(RadTM)$, we obtain $\overline{g}(h^l(Z, W), \xi) = -\overline{g}(fW, \nabla_Z \overline{J}\xi) - \overline{g}(FW, h^s(Z, \overline{J}\xi))$, which implies $\overline{g}(h^l(Z, W), \xi) = \overline{g}(fW, A^*_{\overline{J}\xi}Z) - \overline{g}(FW, h^s(Z, \overline{J}\xi))$. ■

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