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On Fixed Points in Quasi Partial b-Metric Spaces and an Application to Dynamic Programming

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Abstract In this paper, we prove several fixed point results in quasi partial b-metric spaces. We derive some consequences and corollaries from our obtained results. Some examples are presented making effective the new concepts and results. An application on dynamic programming is also provided.

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1. Introduction

In 1995, Matthews [1] introduced a new class of (generalized) metric spaces called partial metric spaces in order to study the denotational semantics of data of networks which play an important role in constructing models in the theory of computation. Its definition is

Definition 1.1. A partial metric space is a pair (X, p) where X is a non-empty set and $p : X \times X \to [0, \infty)$ is such that

(PM1) $p(x, y) = p(y, x)$ (symmetry);
(PM2) if $p(x, x) = p(x, y) = p(y, y)$, then $x = y$ (equality);
(PM3) $p(x, x) \leq p(x, y)$ (small self-distances);
(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (the triangle inequality),

for all $x, y, z \in X$.

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Fore more details on topological aspects, one can see [2]. Further, Matthews [1] also established the Banach contraction principle in the setting of partial metric spaces. Subsequently, many authors obtained several (coincidence and common) fixed point theorems using the concept of a partial metric. These results have a lot of applications and interesting consequences. For instance, we may refer to [3–13].

On the other hand, the de of a quasi-metric is given as follows:

Definition 1.2. Let X be a non-empty and let $d : X \times X \to [0, \infty)$ be a function which satisfies:
$(d1) d(x, y) = 0$ if and only if $x = y$,
$(d2) d(x, y) \leq d(x, z) + d(z, y)$. Then d is called a *quasi-metric* and the pair (X, d) is called a *quasi-metric space.*

Note that, it misses the symmetry $(d(x, y) \neq d(y, x))$. For some known fixed point results on these spaces, we refer to [14–22].

On the other hand, the concept of a b-metric space was introduced by Czerwik in [23] as a generalization of a metric space where the triangular inequality is replaced by $d(x, y) \leq s[d(x, z) + d(z, y)$ with $s \geq 1$, for all $x, y, z \in X$. Since then several papers deal with fixed point theory for single valued and multivalued operators in b-metric spaces, for instance, see [24–32].

In 2015, Gupta and Gautam [33] combine all above concepts and introduced a generalized metric space called a quasi-partial b-metric space and established some related fixed point results.

The de of a quasi partial b-metric space is given as follows:

Definition 1.3. Let $s \geq 1$. Let X be a non-empty and let $q : X \times X \to [0, \infty)$ be a function which satisfies:
$(q1) q(x, x) = q(y, y) = q(x, y)$ then, $x = y$,
$(q2) q(x, x) \leq q(x, y)$,
$(q3) q(x, x) \leq q(y, x)$,
$(q4) q(x, y) \leq s[q(x, z) + q(z, y)] - q(z, z)$. Then q is called a quasi partial b-metric and the pair (X, q) is called a quasi partial b-metric space.

We have the following simple lemma

Lemma 1.4. *Let (X, q) quasi partial b-metric space. Then,*
(1) *If $q(x, y) = 0$ then, $x = y$,*
(2) *If $x \neq y$ then, $q(x, y) > 0$ and $q(y, x) > 0$.*

Mention that the topological notions of quasi partial b-metric spaces, such as, limit, continuity, completeness and Cauchyness should be reconsidered under the left and right approaches.

If q is a quasi partial b-metric on X, then the function $d_q : X \times X \to [0, \infty)$ given by

$$d_q(x, y) = q(x, y) + q(y, x) - q(x, x) - q(y, y), \tag{1.1}$$

is a b-metric on X [33].

Let $\{x_n\}$ be a sequence in X. It is said to be
(i) *convergent* to a point $x \in X$ if and only if

$$q(x, x) = \lim_{n \longrightarrow +\infty} q(x, x_n) = \lim_{n \longrightarrow +\infty} q(x, x_n).$$

(ii) a *Cauchy sequence* if there exist (and are finite) $\lim_{n,m\longrightarrow+\infty} q(x_n, x_m)$ and $\lim_{n,m\longrightarrow+\infty} q(x_m, x_n)$.

A quasi partial b-metric (X, q) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that

$$q(x,x) = \lim_{n,m\longrightarrow+\infty} q(x_n, x_m) = \lim_{n,m\longrightarrow+\infty} q(x_m, x_n).$$

Lemma 1.5. [33] *Let (X, q) be a quasi partial b-metric space. Then, (X, q) is complete if the b-metric space (X, d_q) is complete.*

The following result is easy to check.

Lemma 1.6. *Let (X, q) be a quasi partial b-metric space and $T : X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then, for all sequence $\{x_n\}$ in X such that $x_n \to u$, we have $Tx_n \to Tu$, that is,*

$$q(Tu, Tu) = \lim_{n\longrightarrow+\infty} q(Tu, Tx_n) = \lim_{n\longrightarrow+\infty} q(Tu, Tx_n).$$

Example 1.7. Let $X = \{0, 1, 2\}$ and $q : X \times X \to \mathbb{R}^+$ defined by

$$q(0,0) = q(1,1) = q(2,2) = 1, \qquad q(2,0) = q(0,2) = 4$$
$$q(1,0) = q(2,1) = 2, \qquad q(1,2) = q(0,1) = 5.$$

Then, (X, q) is a quasi partial $b-$metric space with coefficient $s = 2$. Note that q is not a quasi partial metric since $q(2,0) = 4 > 2 = q(2,1) + q(1,0) - q(1,1)$.

Example 1.8. Let $X = [0, +\infty)$. Define $q : X \times X \to \mathbb{R}^+$ by $q(x, y) = (|x - y| + x)^2$. Then, (X, q) is a quasi partial $b-$metric space with coefficient $s = 2$. Note that q is not a quasi partial metric since $q(3,1) = 25 > 21 = q(3,2) + q(2,1) - q(2,2)$.

Example 1.9. Let $X = [0, +\infty)$. Define $q : X \times X \to \mathbb{R}^+$ by $q(x, y) = (2|x - y| + \max\{x, y\} + x)^2$.

Then, (X, q) is a quasi partial $b-$metric space with coefficient $s = 2$. Note that q is not a quasi partial metric since $q(3,1) = 100 > 84 = q(3,2) + q(2,1) - q(2,2)$.

In 2012, Samet *et al.* [34] introduced the concept of α-admissible maps and suggested a very interesting class of mapping, $\alpha-\psi$ contraction mappings, to investigate the existence and uniqueness of a fixed point.

Definition 1.10. [34] For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that the self-mapping T on X is *α-admissible* if for all $x, y \in X$, we have

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{1.2}$$

Many papers dealing with above notion have been considered to prove some (common) fixed point results, for example see [15, 35–43].

Let Ψ_s be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

i) ψ is nondecreasing;

(ii) $\sum_{n=1}^{+\infty} s^n \psi^n(t) < \infty$ for all $t > 0$.

Remark 1.11. If $\psi \in \Psi_s$, we have $\psi(t) < t$ for all $t > 0$, ψ is continuous at $t = 0$ and $\psi(0) = 0$.

We introduce the following.

Definition 1.12. Let (X, q) be a quasi partial b-metric space and $T : X \to X$ be a given mapping. We say that T is an $\alpha - \psi$ *contractive mapping* if there exist $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_s$ such that, for all $x, y \in X$, with $\alpha(x, y) \geq 1$, we have

$$q(Tx, Ty) \leq \psi(q(x, y)). \tag{1.3}$$

In this paper, we are concerned with a new class of contractions via α-admissible mappings. Our aim is to prove some fixed point theorems involving above contractions in the setting of quasi partial b-metric spaces. Our obtained results are supports by some examples and an application on dynamic programming.

2. Main Results

We state our first fixed point result.

Theorem 2.1. *Let (X, q) be a complete quasi partial b- metric space and $T : X \to X$ be an $\alpha - \psi$ contractive mapping. Suppose that*

(i) T is $\alpha-$admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;
(iii) T is continuous;
(iv) $\alpha(z, z) \geq 1$ for all z verifying $q(z, Tz) \leq sq(Tz, Tz)$ and $q(z, z) = 0$.

Then, there exists $u \in X$ such that u is a fixed point of T, that is, $Tu = u$.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. Suppose that $x_{n_0} = x_{n_0+1}$ for some n_0. So the proof is completed since $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$. Consequently, we assume that

$$x_n \neq x_{n+1} \text{ for all } n. \tag{2.1}$$

Since T is $\alpha-$admissible, observe that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By repeating the process above, we derive that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{2.2}$$

From, (1.3) and (2.2), we find that for all $n = 1, 2, \ldots$

$$q(x_n, x_{n+1}) = q(Tx_{n-1}, Tx_n) \leq \psi(q(x_{n-1}, x_n)). \tag{2.3}$$

By induction, we get, for all $n = 0, 1, \ldots$

$$q(x_n, x_{n+1}) =\leq \psi^n(q(x_0, x_1)). \tag{2.4}$$

Since T is $\alpha-$admissible, we also have

$$\alpha(x_1, x_0) = \alpha(Tx_0, Tx_1) \geq 1 \Rightarrow \alpha(Tx_1, Tx_0) = \alpha(x_2, x_1) \geq 1.$$

By repeating the process above, we derive that

$$\alpha(x_{n+1}, x_n) \geq 1, \text{ for all } n = 0, 1, \ldots. \tag{2.5}$$

From, (1.3) and (2.5), we find that for all $n = 0, 1, \ldots$

$$q(x_{n+1}, x_n) \leq \psi^n(q(x_1, x_0)).$$

We will show that $\{x_n\}$ is Cauchy sequence in (X, q).

Using $(q4)$, we have

$$\begin{aligned} q(x_n, x_{n+2}) &\leq s\left[q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})\right] - q(x_{n+1}, x_{n+1}) \\ &\leq s\left[q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})\right] \leq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) \end{aligned}$$

and, similarly,

$$\begin{aligned} q(x_n, x_{n+3}) &\leq s\left[q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+3})\right] - q(x_{n+1}, x_{n+1}) \\ &\leq s\left[q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+3})\right] \\ &\leq sq(x_n, x_{n+1}) + s^2\left[q(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_{n+3})\right] - sq(x_{n+2}, x_{n+2}) \\ &\leq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + s^3 q(x_{n+2}, x_{n+3}). \end{aligned}$$

By induction, we get for all $m > n$

$$q(x_n, x_m) \leq \sum_{k=n}^{m-1} s^{k-n+1} q(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} s^k q(x_k, x_{k+1}) \tag{2.6}$$

$$\leq \sum_{k=n}^{m-1} s^k \psi^k(q(x_0, x_1)) \leq \sum_{k=n}^{\infty} s^k \psi^k(q(x_0, x_1)). \tag{2.7}$$

Using again $(q4)$, we also have

$$\begin{aligned} q(x_{n+2}, x_n) &\leq s\left[q(x_{n+2}, x_{n+1}) + q(x_{n+1}, x_n)\right] - q(x_{n+1}, x_{n+1}) \\ &\leq s^2 q(x_{n+2}, x_{n+1}) + sq(x_{n+1}, x_n) \end{aligned}$$

and, similarly,

$$\begin{aligned} q(x_{n+3}, x_n) &\leq s\left[q(x_{n+3}, x_{n+1}) + q(x_{n+1}, x_n)\right] - q(x_{n+1}, x_{n+1}) \\ &\leq s\left[q(x_{n+3}, x_{n+1}) + q(x_{n+1}, x_n)\right] \\ &\leq s^2\left[q(x_{n+3}, x_{n+2}) + q(x_{n+2}, x_{n+1}\right] + sq(x_{n+1}, x_n) - sq(x_{n+2}, x_{n+2}) \\ &\leq s^3 q(x_{n+3}, x_{n+2}) + s^2 q(x_{n+2}, x_{n+1}) + sq(x_{n+1}, x_n). \end{aligned}$$

By induction, we get for all $m > n$

$$q(x_m, x_n) \leq \sum_{k=n}^{m-1} s^{k-n+1} q(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} s^k q(x_{k+1}, x_k) \tag{2.8}$$

$$\leq \sum_{k=n}^{\infty} s^k \psi^k(q(x_1, x_0)). \tag{2.9}$$

Since $\sum_{n=0}^{\infty} s^n \psi^n(q(x_0, x_1)) < \infty$ and $\sum_{n=0}^{\infty} s^n \psi^n(q(x_1, x_0)) < \infty$, then from (2.6) and (2.8), we obtain

$$\lim_{n,m\to\infty} q(x_n, x_m) = 0. \tag{2.10}$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X,q). Since (X,q) is complete, so by (2.10), there exists $u \in X$ such that

$$\lim_{n\to\infty} q(x_n,u) = \lim_{n\to\infty} q(u,x_n) = q(u,u) = \lim_{n,m\to\infty} q(x_n,x_m) = 0.$$

We will show that u is a fixed point of T.

Since T is continuous, we obtain that $\lim_{n\to\infty} q(Tx_n,Tu) = q(Tu,Tu)$.

Using $(q4)$, we have for all $n \in \mathbb{N}$,

$$q(u,Tu) \le sq(u,x_{n+1}) + sq(x_{n+1},Tu) - q(x_{n+1},x_{n+1}) \le sq(u,x_{n+1}) + sq(Tx_n,Tu)$$

Then, letting $n \to \infty$, we have

$$q(u,Tu) \le sq(Tu,Tu) \tag{2.11}$$

Having $q(u,u)=0$ and (2.11), so by by (iv), we obtain $\alpha(u,u) \ge 1$. Using now (1.3) and the fact that $\psi \in \Psi_s$, we have

$$0 \le q(Tu,Tu) \le \psi(q(u,u)) = \psi(0) = 0$$

Then, $q(Tu,Tu)=0$. So, by (2.11), we get $q(u,Tu)=0$. Hence, by Lemma 1.5, we have $Tu=u$ and hence, u is a fixed point of T. ■

In the following, we state some consequences and corollaries of our obtained result.

Corollary 2.2. *Let (X,q) be a complete quasi partial b-metric space and $T : X \to X$ be a given mapping. Suppose there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi_s$ such that*

$$\alpha(x,y)q(Tx,Ty) \le \psi(q(x,y)) \tag{2.12}$$

for all $x,y \in X$. Suppose also that

(i) T is $\alpha-$admissible;
(ii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(Tx_0,x_0) \ge 1$;
(iii) T is continuous;
(iv) $\alpha(z,z) \ge 1$ for all z verifying $q(z,Tz) \le sq(Tz,Tz)$ and $q(z,z)=0$.

Then, there exists $u \in X$ such that u is a fixed point of T, that is, $Tu=u$.

Proof. Let $x,y \in X$ such that $\alpha(x,y) \ge 1$. Then, if (2.12) holds, we have

$$q(Tx,Ty) \le \alpha(x,y)q(Tx,Ty) \le \psi(q(x,y)).$$

Then, the proof is concluded by Theorem 2.1. ■

Corollary 2.3. *Let (X,q) be a complete quasi partial b-metric space and $T : X \to X$ be a given continuous mapping. Suppose there exists a function $\psi \in \Psi_s$ such that*

$$q(Tx,Ty) \le \psi(q(x,y)) \tag{2.13}$$

for all $x,y \in X$. Then, there exists $u \in X$ such that u is a fixed point of T, that is, $Tu=u$.

Proof. It suffices to take $\alpha(x,y)=1$ in Corollary 2.2. ■

Corollary 2.4. *Let (X, q) be a complete quasi partial b-metric space and $T : X \to X$ be a given continuous mapping. Suppose there exists a constant $0 < k < \frac{1}{s}$ such that*

$$q(Tx, Ty) \leq kq(x, y) \tag{2.14}$$

for all $x, y \in X$. Then, there exists $u \in X$ such that u is a fixed point of T, that is, $Tu = u$.

Proof. It suffices to take $\psi(t) = kt$ with $k \in (0, \frac{1}{s})$ in Corollary 2.3. ∎

We replace the continuity of T and the condition (iv) given in Theorem 2.1 by the following hypothesis.

(H) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ and $\alpha(x, x_{n(k)}) \geq 1$ for all $k \in \mathbb{N}$.

We also need to consider for $T : X \to X$, the mapping $f_T : X \to [0, \infty)$ defined by

$$f_T(x) = \frac{1}{2}(q(x, Tx) + q(Tx, x)).$$

Now, we state the following result.

Theorem 2.5. *Let (X, q) be a complete quasi partial b- metric space and $T : X \to X$ be an $\alpha - \psi$ contractive mapping. Suppose that*

(i) T is $\alpha-$admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;
(iii) (H) is verified or f_T is lower semi-continuous.

Then, there exists $u \in X$ such that u is a fixed point of T, that is, $Tu = u$.

Proof. Proceeding as in the proof of Theorem 2.1, we construct a Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to u$ as $n \to \infty$. We show that $Tu = u$.
Suppose that (H) is verified (that is, $\alpha(x_{n(k)}, u) \geq 1$), then, by (1.3) and using $(q4)$, we have for all $k \in \mathbb{N}$,

$$\begin{aligned} 0 \leq q(u, Tu) &\leq sq(u, x_{n(k)+1}) + sq(x_{n(k)+1}, Tu) - q(x_{n(k)+1}, x_{n(k)+1}) \\ &\leq sq(u, x_{n(k)+1}) + sq(Tx_{n(k)}, Tu) \\ &\leq sq(u, x_{n(k)+1}) + s\psi(q(x_{n(k)}, u)). \end{aligned}$$

We know that

$$\lim_{k\to\infty} q(u, x_{n(k)+1}) = \lim_{k\to\infty} q(x_{n(k)}, u) = q(u, u) = 0.$$

Moreover, since $\psi \in \Psi_s$, so we get

$$\lim_{n\to\infty} \psi(q(x_{n(k)}, u)) = \psi(0) = 0.$$

Thus, $q(u, Tu) = 0$. So, by Lemma 1.5, we have $Tu = u$.

Now, we pass to the case where f_T is lower semi-continuous. We have

$$f_T(u) \leq \lim_{n\to\infty} \frac{1}{2}(q(x_n, Tx_n) + q(Tx_n, x_n)) = \frac{1}{2} \lim_{n\to\infty} (q(x_n, x_{n+1}) + q(x_{n+1}, x_n)) = 0.$$

Thus, $q(u, Tu) = 0$, and so $Tu = u$. ∎

Analogously, we can derive the following results.

Corollary 2.6. *Let (X, q) be a complete quasi partial b-metric space and $T : X \to X$ be a given mapping. Suppose there exist two functions $\alpha : X \times X \to [1, \infty)$ and $\psi \in \Psi_s$ such that*

$$\alpha(x, y)q(Tx, Ty) \leq \psi(q(x, y)) \tag{2.15}$$

for all $x, y \in X$. Suppose also that

(i) *T is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;*
(iii) *(H) is verified or f_T is lower semi-continuous.*

Then, there exists $u \in X$ such that u is a fixed point of T, that is, $Tu = u$.

To prove uniqueness of the fixed point given in Theorem 2.1 (resp. Theorem 2.5), we need to take the following additional condition:

(U): For all $x, y \in Fix(T)$, we have $\alpha(x, y) \geq 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 2.7. *Adding condition (U) to the hypotheses of Theorem 2.1 (resp. Theorem 2.5), we obtain that u is the unique fixed point of T.*

Proof. We argue by contradiction, that is, there exist $u, v \in X$ such that $u = Tu$ and $v = Tv$ with $u \neq v$. By assumption (U), we have $\alpha(u, v) \geq 1$, so by (1.3)

$$0 < q(u, v) = q(Tu, Tv) \leq \psi(q(u, v)) < q(u, v)$$

which is a contradiction. Thus, $u = v$, so the uniqueness of the fixed point of T. ■

Corollary 2.8. *Let (X, q) be a complete quasi partial b-metric space and $T : X \to X$ be a given mapping. Suppose there exists a function $\psi \in \Psi_s$ such that*

$$q(Tx, Ty) \leq \psi(q(x, y)) \tag{2.16}$$

for all $x, y \in X$. Then there exists $u \in X$ such that u is the unique fixed point of T, that is, $Tu = u$.

Proof. It suffices to take $\alpha(x, y) = 1$ in Corollary 2.6. The uniqueness of u follows from Theorem 2.7. ■

Corollary 2.9. *Let (X, q) be a complete quasi partial b-metric space and $T : X \to X$ be a given mapping. Suppose there exists a constant $0 < k < \frac{1}{s}$ such that*

$$q(Tx, Ty) \leq kq(x, y) \tag{2.17}$$

for all $x, y \in X$. Then, there exists a unique fixed point of T.

Proof. It suffices to take $\psi(t) = kt$ in Corollary 2.8. ■

Now, we give the following example to support our result.

Example 2.10. We go back to example 1.8 where $X = [0, +\infty)$. Define $q : X \times X \to \mathbb{R}^+$ by $q(x, y) = (|x - y| + x)^2$. Mention that (X, q) is a complete quasi partial b-metric space . Define the map $T : X \to X$ by $T(x) = \frac{1}{2}x$ and

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{1}{3}t$. It's clear that $\psi \in \Psi_s$. Note that T is $\alpha-$admissible. In fact, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So $x, y \in [0, 1]$. Owing the definition of T, we have $Tx, Ty \in [0, 1]$ and hence, $\alpha(Tx, Ty) \geq 1$. Then, T is $\alpha-$admissible. Now, we show that (H) is verified. Let $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\{x_n\} \subset [0, 1]$. If $x_n \to u$ as $n \to \infty$, we have $q(u, x_n) = (|x_n - u| + u)^2 \to q(u, u) = u^2$ as $n \to \infty$. Then, $|x_n - u| \to 0$ as $n \to \infty$. Hence, $u \in [0, 1]$ and hence, $\alpha(x_n, u) = \alpha(u, x_n) = 1$.
Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. In fact, for $x_0 = \frac{1}{2}$, we have

$$\alpha(\frac{1}{2}, T\frac{1}{2}) = \alpha(\frac{1}{2}, \frac{1}{4}) = 1, \quad \alpha(T\frac{1}{2}, \frac{1}{2}) = \alpha(\frac{1}{4}, \frac{1}{2}) = 1.$$

Now, we show that T is $\alpha - \psi-$contractive mapping. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$. So, $x, y \in [0, 1]$. We have

$$q(Tx, Ty){=}(|Tx - Ty| + Tx)^2 = \frac{1}{4}(|x - y| + x)^2 \leq \frac{1}{3}(|x - y| + x)^2{=}\frac{1}{3}q(x, y) = \psi(q(x, y))$$

Then, all the required hypothesis of Theorem 2.5 are satisfied. Here , $u = 0$ is the unique fixed point of T.

Remark 2.11. Note that Theorem 2.1 is true for the Example 2.10, since T is continuous in (X, q) and condition (iv) of this theorem is verified.

3. Application

In this section, we present an application on dynamic programming. In particular, we assume that U and V are Banach space and $W \subset U$ is a state space and $D \subset V$ is a decision space. it is well known that the dynamic programming provides useful tools for mathematical optimization and computer programming as well. In particular, we are interested in resolving the following functional equation

$$r(x) = \sup_{y \in D}\{f(x, y) + G(x, y, r(\tau(x, y)))\}, \quad x \in W, \tag{3.1}$$

where $\tau : W \times D \to W, f : W \times D \to \mathbb{R}$ and $G : W \times D \times \mathbb{R} \to \mathbb{R}$. Here, we study the existence and uniqueness of the bounded solution of the functional equation (3.1).
Let $B(W)$ denote the set of all bounded real valued functions on W and, for an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |h(x)|$. Clearly, $(B(W), \|.\|)$ is a Banach space. Let $B(W)$ be endowed with the quasi partial $b-$metric q (with $s = 2$) defined by

$$q(h, k) = [\sup_{x \in W} |h(x) - k(x)| + \sup_{x \in W} |h(x)|]^2 = (\|h - k\| + \|h\|)^2.$$

We also define $T : B(W) \to B(W)$ by

$$T(h)(x) = \sup_{y \in D}\{f(x,y) + G(x,y,h(\tau(x,y)))\}, \tag{3.2}$$

for all $h \in B(W)$ and $x \in W$. Obviously, if the functions f and G are bounded, then T is well-defined. We shall prove the following theorem.

Theorem 3.1. *Assume that there exists* $0 \leq r < \frac{1}{\sqrt{2}}$ *such that for every* $(x,y) \in W \times D$

$$|G(x,y,h(\tau(x,y))) - G(x,y,k(\tau(x,y)))| \leq r \sup_{x \in W} |h(x) - k(x)| \tag{3.3}$$

$$f(x,y) + G(x,y,h(\tau(x,y))) \leq r \sup_{x \in W} |h(x)| \tag{3.4}$$

where the functions $G : W \times D \times \mathbb{R} \to \mathbb{R}$ *and* $f : W \times D \to \mathbb{R}$ *are bounded. Then, the functional equation* (3.1) *has a unique bounded solution.*

Proof. Let $\lambda > 0$ be an arbitrary positive real number, $x \in W$, $h \in B(W)$ and $k \in B(W)$. Then, by (3.2), there exist $y_1, y_2 \in D$ such that

$$T(h)(x) < f(x,y_1) + G(x,y_1,h(\tau(x,y_1))) + \lambda, \tag{3.5}$$

$$T(k)(x) < f(x,y_2) + G(x,y_2,k(\tau(x,y_2))) + \lambda, \tag{3.6}$$

and

$$T(k)(x) \geq f(x,y_1) + G(x,y_1,k(\tau(x,y_1))), \tag{3.7}$$

$$T(h)(x) \geq f(x,y_2) + G(x,y_2,h(\tau(x,y_2))). \tag{3.8}$$

Then, by (3.5) and (3.7)

$$\begin{aligned} T(h)(x) - T(k)(x) &\leq G(x,y_1,h(\tau(x,y_1))) - G(x,y_1,k(\tau(x,y_1))) + \lambda \\ &\leq |G(x,y_1,h(\tau(x,y_1))) - G(x,y_1,k(\tau(x,y_1)))| + \lambda. \end{aligned}$$

Also, by (3.6) and (3.8)

$$\begin{aligned} T(k)(x) - T(h)(x) &\leq G(x,y_2,h(\tau(x,y_2))) - G(x,y_2,h(\tau(x,y_2))) + \lambda \\ &\leq |G(x,y_2,h(\tau(x,y_2))) - G(x,y_2,h(\tau(x,y_2)))| + \lambda. \end{aligned}$$

Thus, by (3.3), we have

$$|T(h)(x) - T(k)(x)| \leq r \sup_{x \in W} |h(x) - k(x)| + \lambda,$$

and with (3.4) we have

$$T(h)(x) \leq r \sup_{x \in W} |h(x)|. \tag{3.9}$$

Therefore, for $h, k \in B(W)$,

$$\begin{aligned} q(T(h), T(k)) &= [\sup_{x \in W} |T(h)(x) - T(k)(x)| + \sup_{x \in W} |T(h)(x)|]^2 \\ &\leq [r \sup_{x \in W} |h(x) - k(x)| + \lambda + r \sup_{x \in W} |h(x)|]^2. \end{aligned}$$

Since $\lambda > 0$ be an arbitrary positive real number, so

$$q(T(h), T(k)) \leq [r \sup_{x \in W} |h(x) - k(x)| + r \sup_{x \in W} |h(x)|]^2 = r^2 q(h,k).$$

Therefore, all conditions of Corollary 2.9 are verified with $s = 2$ and hence the operator T has a unique fixed point. Then, the functional equation (3.1) has a unique bounded solution. ■

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