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On Semiprime and Quasi-Semiprime Ideals in Ordered $\mathcal{AG}$-Groupoids

Pairote Yiarayong

Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand
e-mail : pairote0027@hotmail.com

Abstract In this paper, we investigate the notion of semiprime ideals in in ordered $\mathcal{AG}$-groupoids as a generalization of prime ideals. The aim of this paper is to investigate the concept of semiprime and quasi-semiprime ideals in ordered $\mathcal{AG}$-groupoids with left identity. Moreover, we investigate relationships between semiprime and quasi-semiprime ideals in ordered $\mathcal{AG}$-groupoids. It is show that an ideal $\prod_{i\in I} P_i$ of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$ is semiprime if and only if $\prod_{i\in I}(a_i(S_i a_i)] \subseteq \prod_{i\in I} P_i$ implies that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$, where $(a_i)_{i\in I} \in \prod_{i\in I} S_i$.

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1. Introduction

In 1972, Kazim and Naseeruddin [1] introduced and studied the notion of an Abel-Grassmann's groupoid. A groupoid $(S, *)$ is called an Abel-Grassmann's groupoid, abbreviated as an $\mathcal{AG}$-groupoid, if its elements satisfy the left invertive law [1], [2] that is: $(a * b) * c = (c * b) * a$, for all $a, b, c \in S$. It is also known [1] that in an AG-groupoid, the medial law holds, that is,

$$(a * b) * (c * d) = (a * c) * (b * d)$$

for all $a, b, c, d \in S$ holds. Several examples and interesting properties of $\mathcal{AG}$-groupoids can be found in [3], [4], [5] and [6]. In 2007, Mushtaq and Khan [7] studied the properties of $\mathcal{AG}$-3-bands. They proved that a subset of an $\mathcal{AG}$-3-band is a left ideal if and only it is right and every ideal of a $\mathcal{AG}$-groupoid S is prime if and only if the set of all ideals of S is totally ordered under inclusion. In 2010, Khan and Ahmad [3] studied the notion of $\mathcal{AG}$-groupoids. They proved that if an $\mathcal{AG}$-groupoid contains a left identity then it is unique. It has been proved also that an $\mathcal{AG}$-groupoid with right identity is a commutative monoid,

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that is, a semigroup with identity element. In 2013, Yiarayong [8] studied the notion of semiprime and quasi-semiprime ideals in $\mathcal{AG}$-groupoids. In 2016, Khan, Yousafzai and Khan [9] characterized $(0,2)$-ideals of an $\mathcal{LA}$-semigroup S and proved that I is a $(0,2)$-ideal of S if and only if I is a left ideal of some left ideal of S. In 2018, Iqbal and Ahmad [10] studied the properties of (left/right) ideals in $\mathcal{CA}$-$\mathcal{AG}$-groupoids.

Let S be a nonempty set, "$\cdot$" a binary operation on S and $\leq$ a relation on S. $(S, \cdot, \leq)$ is called an ordered Abel-Grassmann's groupoid (ordered $\mathcal{AG}$-groupoid) if $(S, \cdot)$ is an $\mathcal{AG}$-groupoid, $(S, \leq)$ is a partially ordered set and for all $a, b, c \in S, a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$. The concept of an ordered $\mathcal{AG}$-groupoid was first given by Shah et. al. in [11, 12] which is infect the generalization of an ordered semigroup. The principal notions of the theory of an ordered $\mathcal{AG}$-groupoid can be found in [13–15]. In 2011, Khan and Faisal [16] introduced and studied some properties of fuzzy ordered $\mathcal{AG}$-groupoids. They proved that the set of all fuzzy two-sided ideals of a left regular ordered $\mathcal{AG}$-groupoid forms a semilattice structure. In 2013, Khan et. al. [17] defined the concept of interval valued fuzzy ordered $\mathcal{LA}$-semigroups and gave characterizations of the intra-regular ordered $\mathcal{LA}$-semigroup in terms of interval valued fuzzy left (right, two-sided) ideals. In 2014, Faisal et. al. [18] studied the properties of fuzzy ordered $\mathcal{AG}$-groupoids and proved that every fuzzy right ideal of an ordered $\mathcal{AG}$-groupoid with left identity becomes a fuzzy left ideal but the converse is not valid. In 2015, Ali, Shi and Khan, [19] introduced and studied the properties of soft intersection left (right, two-sided) ideals, (generalized) bi-ideals, interior ideals and quasi-ideals in ordered $\mathcal{AG}$-groupoids. In 2016, Yousafzai, Yaqoob and Zeb [20] defined the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy (left, right,bi-) ideals of ordered Abel Grassman's groupoids. In 2019, Nasreen [21] introduced and studied the notion of fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$ of an ordered $\mathcal{AG}$-groupoid S.

In this paper, we investigate the notion of semiprime ideals in in ordered $\mathcal{AG}$-groupoids as a generalization of prime ideals. The aim of this paper is to investigate the concept of semiprime and quasi-semiprime ideals in ordered $\mathcal{AG}$-groupoids with left identity. Moreover, we investigate relationships between semiprime and quasi-semiprime ideals in ordered $\mathcal{AG}$-groupoids. It is show that a left ideal $\prod\limits_{i\in I} P_i$ of an ordered $\mathcal{AG}$-groupoid $\prod\limits_{i\in I} S_i$ is quasi-semiprime if and only if $\prod\limits_{i\in I}(a_i(S_i a_i)] \subseteq \prod\limits_{i\in I} P_i$ implies that $(a_i)_{i\in I} \in \prod\limits_{i\in I} P_i$, where $(a_i)_{i\in I} \in \prod\limits_{i\in I} S_i$

2. Preliminaries

In this section we refer to [11, 12] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1 ([11, 12])**.** Let S be a nonempty set, "$\cdot$" a binary operation on S and $\leq$ a relation on S. $(S, \cdot, \leq)$ is called an *ordered AG-groupoid* if $(S, \cdot)$ is a AG-groupoid, $(S, \leq)$ is a partially ordered set and for all $a, b, c \in S, a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$.

Lemma 2.2 ([11])**.** *An ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ is an ordered semigroup if and only if $a(bc) = (cb)a$, for all $a, b, c \in S$.*

Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid. For $\emptyset \neq A \subseteq S$, let

$$(A] = \{x \in S : x \leq a \text{ for some } a \in A\}.$$

The following lemma are similar to the case of ordered $\mathcal{AG}$-groupoids.

Lemma 2.3 ([11])**.** *Let $(S, \cdot, \leq)$ be an ordered AG-groupoid and let A, B be subsets of S. The following statements hold:*

(1) If $A \subseteq B$, then $(A] \subseteq (B]$.
(2) $(A](B] \subseteq (AB]$.
(3) $((A](B]] \subseteq (AB]$.

The following corollary can be easily deduced from the lemma.

Lemma 2.4. *Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid and let A, B be subsets of S. The following statements hold:*

(1) $A \subseteq (A]$.
(2) $((A]] = (A]$.

Proof. Similar to the proof of Lemma 2.3. ∎

A nonempty subset A of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ is called an *$\mathcal{AG}$-subgroupoid* of S if $AA \subseteq A$ (see [11]).

Definition 2.5 ([11])**.** An $\mathcal{AG}$-subgroupoid A of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ is called a *left ideal* of S if $(A] \subseteq A$ and $SA \subseteq A$ and called a *right ideal* of S if $(A] \subseteq A$ and $AS \subseteq A$. An $\mathcal{AG}$-subgroupoid A of S is called an *ideal* of S if A is both left and right ideal of S.

Lemma 2.6 ([11])**.** *Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid with left identity. Then every right ideal of $(S, \cdot, \leq)$ is a left ideal of S.*

Lemma 2.7 ([11])**.** *Let $(S, \cdot, \leq)$ be an ordered $\mathcal{AG}$-groupoid with left identity and $A \subseteq S$. The following statements hold:*

(1) $S(SA) = SA$.
(2) $S(SA] \subseteq (SA]$.
(3) $\langle a \rangle = (Sa]$ for all $a \in S$.

3. Ideals in Ordered $\mathcal{AG}$-Groupoids

In this section, we concentrate our study on the ideals of an ordered $\mathcal{AG}$-groupoid and investigate their fundamental properties.

First, we consider the cartesian product of ordered $\mathcal{AG}$-groupoids.

Let $\{(S_i, \cdot_i, \leq_i) | i \in I\}$ be a non empty family of ordered $\mathcal{AG}$-groupoids. We consider the cartesian product $\prod_{i \in I} S_i$. Define a mapping

$$\prod_{i\in I} S_i \times \prod_{i\in I} S_i \to \prod_{i\in I} S_i,$$

written as

$$((x_i)_{i\in I}, (y_i)_{i\in I}) \mapsto (x_i)_{i\in I}(y_i)_{i\in I},$$

by

$$(x_i)_{i\in I}(y_i)_{i\in I} = (x_i y_i)_{i\in I}.$$

Then $\prod_{i\in I} S_i$ is an $\mathcal{AG}$-groupoid. Moreover, $\prod_{i\in I} S_i$ is an ordered $\mathcal{AG}$-groupoid with the relation $\leq$ defined by $(x_i)_{i\in I} \leq (y_i)_{i\in I} \Leftrightarrow x_i \leq_i y_i$, for all $i \in I$. We consider the cartesian product of ideals.

Lemma 3.1. *Let $\{(S_i, \cdot_i, \leq_i) | i \in I\}$ be a non empty family of ordered $\mathcal{AG}$-groupoids. For each $i \in I$ if A_i is an ideal of an ordered $\mathcal{AG}$-groupoid S_i, then the set $\prod_{i\in I} A_i$ is an ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.*

Proof. Let A_i be a nonempty subset of an ordered $\mathcal{AG}$-groupoid S_i, for all $i \in I$. It is clear that $\prod_{i\in I} A_i \neq \emptyset$. Since $(x_i)_{i\in I} \in \left(\prod_{i\in I} A_i\right] \Leftrightarrow (x_i)_{i\in I} \leq (a_i)_{i\in I} \Leftrightarrow x_i \leq a_i$, for some $a_i \in A$, we have $\left(\prod_{i\in I} A_i\right] = \prod_{i\in I} (A_i]$. By Lemma 2.4, we get $\left(\prod_{i\in I} A_i\right] = \prod_{i\in I} A_i$. Then

$$\begin{aligned} \left(\prod_{i\in I} A_i\right)\left(\prod_{i\in I} S_i\right) &= \prod_{i\in I} (A_i S_i) \\ &\subseteq \prod_{i\in I} A_i \end{aligned}$$

and

$$\begin{aligned} \left(\prod_{i\in I} S_i\right)\left(\prod_{i\in I} A_i\right) &= \prod_{i\in I} (S_i A_i) \\ &\subseteq \prod_{i\in I} A_i. \end{aligned}$$

Therefore $\prod_{i\in I} A_i$ is an ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$. ∎

The following corollary can be easily deduced from the lemma

Corollary 3.2. *Let $\{(S_i, \cdot_i, \leq_i) | i \in I\}$ be a non empty family of ordered $\mathcal{AG}$-groupoids. The following statements hold:*

(1) For each $i \in I$ if A_i is a left ideal of an ordered $\mathcal{AG}$-groupoid S_i, then the set $\prod_{i\in I} A_i$ is a left ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.

(2) For each $i \in I$ if A_i is a right ideal of an ordered $\mathcal{AG}$-groupoid S_i, then the set $\prod_{i\in I} A_i$ is a right ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.

Proof. Similar to the proof of Lemma 3.1. ∎

The following proposition show that a nonempty subset $\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right]$ of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$ is an ideal of $\prod_{i\in I} S_i$ if $(a_i)_{i\in I} \in \prod_{i\in I} S_i$.

Proposition 3.3. *If $(S_i, \cdot_i, \leq_i)$ is an ordered $\mathcal{AG}$-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, then $\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right]$ is a left ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.*

Proof. Since $(S_i, \cdot_i, \leq_i)$ is an ordered $\mathcal{AG}$-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, we have $\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right] = \left(\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right]\right]$. By Lemma 2.3, we get

$$\begin{aligned}
\left(\prod_{i\in I} S_i\right)\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right] &= \left(\prod_{i\in I} S_i\right]\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right] \\
&\subseteq \left(\prod_{i\in I} S_i\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right)\right] \\
&= \left(\prod_{i\in I} S_i (a_i)_{i\in I} \cup \prod_{i\in I} S_i \prod_{i\in I} S_i a_i\right] \\
&= \left(\prod_{i\in I} S_i a_i \cup \prod_{i\in I} S_i (S_i a_i)\right] \\
&= \left(\prod_{i\in I} S_i a_i \cup \prod_{i\in I} (a_i S_i) S_i\right] \\
&= \left(\prod_{i\in I} S_i a_i \cup \prod_{i\in I} (S_i S_i) a_i\right] \\
&= \left(\prod_{i\in I} S_i a_i \cup \prod_{i\in I} S_i a_i\right] \\
&= \left(\prod_{i\in I} S_i a_i\right] \\
&\subseteq \left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right].
\end{aligned}$$

Therefore $\left(\left(a_i\right)_{i\in I} \cup \prod_{i\in I} S_i a_i\right]$ is a left ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$. ■

The following lemma show that a nonempty subset $\left(\prod_{i\in I} a_i S_i \cup \prod_{i\in I} S_i a_i\right]$ of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$ is an ideal of $\prod_{i\in I} S_i$ if $(a_i)_{i\in I} \in \prod_{i\in I} S_i$.

Lemma 3.4. *If $(S_i, \cdot_i, \leq_i)$ is an ordered $\mathcal{AG}$-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, then $\left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right]$ is an ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.*

Proof. Since $(S_i, \cdot_i, \leq_i)$ is an ordered $\mathcal{AG}$-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, we have

$$\left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right] = \left(\left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right]\right].$$

By Lemma 2.3,

$$\begin{aligned}\left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right] \prod_{i\in I} S_i &\subseteq \left(\left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right)\left(\prod_{i\in I} S_i\right)\right] \\ &= \left(\prod_{i\in I} a_iS_i \prod_{i\in I} S_i \cup \prod_{i\in I} S_ia_i \prod_{i\in I} S_i\right] \\ &= \left(\prod_{i\in I}(a_iS_i)S_i \cup \prod_{i\in I}(S_ia_i)S_i\right] \\ &= \left(\prod_{i\in I}(S_iS_i)a_i \cup \prod_{i\in I} S_i(a_iS_i)\right] \\ &= \left(\prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right].\end{aligned}$$

Therefore $\left(\prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right]$ is a right ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$. By Lemma 2.6, we have $\left(\prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right]$ is an ideal of $\prod_{i\in I} S_i$. ■

Proposition 3.5. *If $(S_i, \cdot_i, \leq_i)$ is an ordered $\mathcal{AG}$-groupoid with left identity and $(a_i)_{i\in I} \in \prod_{i\in I} S_i$, then $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right]$ is an ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.*

Proof. Let $(a_i)_{i\in I} \in \prod_{i\in I} S_i$. By Lemma 2.3,

$$\begin{aligned}\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right] \prod_{i\in I} S_i &\subseteq \left(\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right) \prod_{i\in I} S_i\right] \\ &= \left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I}(S_ia_i)S_i \cup \prod_{i\in I}(a_iS_i)S_i\right]\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_i(a_iS_i) \cup \prod_{i\in I} S_ia_i\right] \\
&= \left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right] \\
&= \left(\prod_{i\in I} a_iS_i \cup \prod_{i\in I} S_ia_i\right] \\
&\subseteq \left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right].
\end{aligned}
$$

Therefore $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right]$ is a right ideal of S. It is clear that

$$\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right] = \left(\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right]\right].$$

By Lemma 2.6, we have $\left((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i\right]$ is an ideal of $\prod_{i\in I} S_i$. ■

4. Semiprime Ideals in Ordered $\mathcal{AG}$-Groupoids

In this section, we concentrate our study on the semiprime and quasi-semiprime ideals in ordered $\mathcal{AG}$-groupoids and investigate their fundamental properties. Moreover, we investigate relationships between semiprime and quasi-semiprime ideals in ordered $\mathcal{AG}$-groupoids.

Recall that an ideal P of an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ is called *semiprime* if every ideal A of S such that $AA \subseteq P$ implies that $A \subseteq P$. A left ideal P of S is called quasi-semiprime if every left ideal A of S such that $AA \subseteq P$, we have $A \subseteq P$.

The following proposition show that a left ideal $\prod_{i\in I} P_i$ of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$ is quasi-semiprime if and only if $(a_i)_{i\in I}(a_i)_{i\in I} \in \prod_{i\in I} P_i$ implies that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$, where $(a_i)_{i\in I} \in \prod_{i\in I} S_i$.

Theorem 4.1. *Let P_i be a left ideal of an ordered $\mathcal{AG}^{**}$-groupoid $(S_i, \cdot_i, \leq_i)$. Then the following conditions are equivalent.*

(1) $\prod_{i\in I} P_i$ is a quasi-semiprime ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.

(2) For each $(a_i)_{i\in I} \in \prod_{i\in I} S_i$ if $(a_i)_{i\in I}(a_i)_{i\in I} \in \prod_{i\in I} P_i$, then $(a_i)_{i\in I} \in \prod_{i\in I} P_i$.

Proof. $(1 \Rightarrow 2)$ Assume that $\prod_{i\in I} P_i$ is a quasi-semiprime ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$. Let $(a_i)_{i\in I}$ be any element of $(a_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus we have

$$\begin{aligned}
((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i]((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i] &\subseteq (((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i)((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i)] \\
&= (((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i)(a_i)_{i\in I} \cup \\
&\quad ((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i) \prod_{i\in I} S_i a_i)] \\
&= ((a_i)_{i\in I}(a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i (a_i)_{i\in I} \cup \\
&\quad (a_i)_{i\in I} \prod_{i\in I} S_i a_i \cup \prod_{i\in I} S_i a_i \prod_{i\in I} S_i a_i] \\
&= ((a_i a_i)_{i\in I} \cup \prod_{i\in I} (S_i a_i) a_i \cup \\
&\quad \prod_{i\in I} a_i (S_i a_i) \cup \prod_{i\in I} S_i a_i \prod_{i\in I} S_i a_i].
\end{aligned}$$

Therefore we obtain that $((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i]((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i] \subseteq ((a_i a_i)_{i\in I} \cup \prod_{i\in I} (S_i a_i) a_i \cup \prod_{i\in I} a_i (S_i a_i) \cup \prod_{i\in I} S_i a_i \prod_{i\in I} S_i a_i]$. Since

$$\begin{aligned}
\prod_{i\in I} P_i &= (\prod_{i\in I} P_i] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I} S_i (P_i S_i)] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I} P_i \cup \prod_{i\in I} P_i \cup \prod_{i\in I} S_i ((a_i a_i) S_i)] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I} P_i S_i \cup \prod_{i\in I} S_i P_i \cup \prod_{i\in I} S_i ((a_i a_i) S_i)] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I} (a_i a_i) S_i \cup \prod_{i\in I} S_i (a_i a_i) \cup \prod_{i\in I} S_i ((S_i a_i) a_i)] \\
&= ((a_i a_i)_{i\in I} \cup \prod_{i\in I} (S_i a_i) a_i \cup \prod_{i\in I} a_i (S_i a_i) \cup \prod_{i\in I} S_i a_i \prod_{i\in I} S_i a_i],
\end{aligned}$$

we have $((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i]((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i] \subseteq \prod_{i\in I} P_i$. By hypothesis, $(a_i)_{i\in I} \in ((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i]$ and so that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$.

$(2 \Rightarrow 1)$ It is obvious. ■

Theorem 4.2. *Let P_i be a left ideal of an ordered $\mathcal{AG}$-groupoid $(S_i, \cdot_i, \leq_i)$ with left identity. Then the following conditions are equivalent.*

(1) $\prod_{i\in I} P_i$ is a quasi-semiprime ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.

(2) For each $(a_i)_{i\in I} \in \prod_{i\in I} S_i$ *if* $\prod_{i\in I}(a_i(S_ia_i)] \subseteq \prod_{i\in I} P_i$*, then* $(a_i)_{i\in I} \in \prod_{i\in I} P_i$.

Proof. $(1 \Rightarrow 2)$ Assume that $\prod_{i\in I} P_i$ is a quasi-semiprime ideal of $\prod_{i\in I} S_i$. Then we have,

$$\begin{aligned}
(a_i)_{i\in I}(a_i)_{i\in I} \in \left(\prod_{i\in I} S_ia_i\right)\left(\prod_{i\in I} S_ia_i\right) &= \prod_{i\in I}(S_ia_i)(S_ia_i)\\
&= \prod_{i\in I}((S_ia_i)a_i)S_i\\
&= \prod_{i\in I} S_i(a_i(S_ia_i))\\
&\subseteq \prod_{i\in I} S_i(a_i(S_ia_i)]\\
&\subseteq \prod_{i\in I} S_i \prod_{i\in I}(a_i(S_ia_i)]\\
&\subseteq \prod_{i\in I} S_i \prod_{i\in I} P_i\\
&\subseteq \prod_{i\in I} P_i.
\end{aligned}$$

Therefore we obtain that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$.

$(2 \Rightarrow 1)$ It is obvious. ■

The following proposition show that an ideal $\prod_{i\in I} P_i$ of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$ is semiprime if and only if $(a_i)_{i\in I}(a_i)_{i\in I} \in \prod_{i\in I} P_i$ implies that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$, where $(a_i)_{i\in I} \in \prod_{i\in I} S_i$.

Theorem 4.3. *Let* P_i *be an ideal of an ordered* $\mathcal{AG}^{**}$*-groupoid* $(S_i, \cdot_i, \leq_i)$*. Then the following conditions are equivalent.*

(1) $\prod_{i\in I} P_i$ *is a semiprime ideal of an ordered* $\mathcal{AG}$*-groupoid* $\prod_{i\in I} S_i$.

(2) For each $(a_i)_{i\in I} \in \prod_{i\in I} S_i$ *if* $(a_i)_{i\in I}(a_i)_{i\in I} \in \prod_{i\in I} P_i$*, then* $(a_i)_{i\in I} \in \prod_{i\in I} P_i$.

Proof. $(1 \Rightarrow 2)$ Assume that $\prod_{i\in I} P_i$ is a semiprime ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$. Let $(a_i)_{i\in I}$ be any element of $(a_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus we have

$$\begin{aligned}
&((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i]((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i]\\
\subseteq\ &(((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i)((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i)]
\end{aligned}$$

$$
\begin{aligned}
&= (((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i)(a_i)_{i\in I} \cup \\
&\quad ((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i)\prod_{i\in I} S_ia_i \cup \\
&\quad ((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i)\prod_{i\in I} a_iS_i] \\
&= ((a_i)_{i\in I}(a_i)_{i\in I} \cup (\prod_{i\in I} S_ia_i)(a_i)_{i\in I} \cup (\prod_{i\in I} a_iS_i)(a_i)_{i\in I} \cup \\
&\quad (a_i)_{i\in I}(\prod_{i\in I} S_ia_i) \cup (\prod_{i\in I} S_ia_i)(\prod_{i\in I} S_ia_i) \cup (\prod_{i\in I} a_iS_i)(\prod_{i\in I} S_ia_i) \cup \\
&\quad (a_i)_{i\in I}(\prod_{i\in I} a_iS_i) \cup (\prod_{i\in I} S_ia_i)(\prod_{i\in I} a_iS_i) \cup (\prod_{i\in I} a_iS_i)(\prod_{i\in I} a_iS_i)] \\
&\subseteq (\prod_{i\in I} P_i \cup \prod_{i\in I}(S_ia_i)a_i \cup \prod_{i\in I}(a_iS_i)a_i \cup \\
&\quad \prod_{i\in I} a_i(S_ia_i) \cup \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(a_iS_i)(S_ia_i) \cup \\
&\quad \prod_{i\in I} a_i(a_iS_i) \cup \prod_{i\in I}(S_ia_i)(a_iS_i) \cup \prod_{i\in I}(a_iS_i)(a_iS_i)]
\end{aligned}
$$

Therefore we obtain that $((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i]((a_i)_{i\in I} \cup \prod_{i\in I} S_ia_i \cup \prod_{i\in I} a_iS_i] \subseteq (\prod_{i\in I} P_i \cup \prod_{i\in I}(S_ia_i)a_i \cup \prod_{i\in I}(a_iS_i)a_i \cup \prod_{i\in I} a_i(S_ia_i) \cup \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(a_iS_i)(S_ia_i) \cup \prod_{i\in I} a_i(a_iS_i) \cup \prod_{i\in I}(S_ia_i)(a_iS_i) \cup \prod_{i\in I}(a_iS_i)(a_iS_i)]$. Since

$$
\begin{aligned}
\prod_{i\in I} P_i &= (\prod_{i\in I} P_i] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I}(P_iS_i)S_i \cup \prod_{i\in I} S_i(S_iP_i) \cup \prod_{i\in I} S_i(S_iP_i) \cup \prod_{i\in I} P_iS_i] \\
&= (\prod_{i\in I} P_i \cup \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(a_iS_i)(S_ia_i) \cup \\
&\quad \prod_{i\in I}(S_ia_i)(a_iS_i) \cup \prod_{i\in I}(a_iS_i)(a_iS_i)] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(a_iS_i)(S_ia_i) \cup \\
&\quad \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(a_iS_i)(S_ia_i) \cup \\
&\quad \prod_{i\in I}(S_ia_i)(a_iS_i) \cup \prod_{i\in I}(S_ia_i)(a_iS_i) \cup \prod_{i\in I}(a_iS_i)(a_iS_i)] \\
&\supseteq (\prod_{i\in I} P_i \cup \prod_{i\in I}(S_ia_i)a_i \cup \prod_{i\in I}(a_iS_i)a_i \cup \\
&\quad \prod_{i\in I} a_i(S_ia_i) \cup \prod_{i\in I}(S_ia_i)(S_ia_i) \cup \prod_{i\in I}(a_iS_i)(S_ia_i) \cup \\
&\quad \prod_{i\in I} a_i(a_iS_i) \cup \prod_{i\in I}(S_ia_i)(a_iS_i) \cup \prod_{i\in I}(a_iS_i)(a_iS_i)],
\end{aligned}
$$

we have $((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i \cup \prod_{i\in I} a_i S_i]((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i \cup \prod_{i\in I} a_i S_i] \subseteq \prod_{i\in I} P_i$. By hypothesis, $((a_i)_{i\in I} \cup \prod_{i\in I} S_i a_i \cup \prod_{i\in I} a_i S_i] \subseteq \prod_{i\in I} P_i$ and so that $(a_i)_{i\in I} \in \prod_{i\in I} P_i$.

$(2 \Rightarrow 1)$ It is obvious. ■

Recall that an ordered $\mathcal{AG}$-groupoid in which $(aa)a = a(aa) = a$ holds for all a is called an *ordered $\mathcal{AG}$-3-band.*

Theorem 4.4. *Let P_i be a left ideal of an ordered $\mathcal{AG}$-3-band $(S_i, \cdot_i, \leq_i)$ with left identity. Then the following conditions are equivalent.*

(1) $\prod_{i\in I} P_i$ is a semiprime ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.
(2) $\prod_{i\in I} P_i$ is a quasi-semiprime ideal of an ordered $\mathcal{AG}$-groupoid $\prod_{i\in I} S_i$.

Proof. This follows from Theorem 4.1 and Theorem 4.3. ■

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