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On Generalized Carleson Operator with Application in Walsh Type Wavelet Packet Expansions

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Abstract In this paper, two new theorems on generalized Carleson operator for a Walsh type wavelet packet system and for periodic Walsh type wavelet packet expansion of a function $f \in L^p\,[0,1)\,,\ 1 < p < \infty$, have been established.

MSC: 40A30; 42C15

Keywords: Walsh function; Walsh-type wavelet packets; periodic Walsh type wavelet packets; logarithmic for single infinite series and for double infinite series; generalized Carleson operator for Walsh type and for periodic Walsh type wavelet packets

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1. Introduction

Wavelet analysis is a cursorily forwarding area of Mathematics and many branches of Science, Engineering and Technology. The wavelet is localized in both frequency and time (space). It gives the new class of orthogonal expansion in Hilbert space with good time-frequency and regularity approximation properties in its area. These expansions play an important role to extract an information from a signal that is not possible by another expansion like Fourier expansion. In transient as well as stationary phenomena, this approach has applications over wavelet and short-time Fourier analysis. The properties of orthonormal bases have been studied in $L^2(\mathbb{R})$ for wavelet expansion. The basic wavelet packet expansions of L^p-functions, $1 < p < \infty$, defined on the real line and the unit interval, have significant importance in wavelet analysis. Walsh system is an example of a system of basic stationary wavelet packets. Billard [1], Chui [2], Daubechies [3], Meyer [4], Morlet [5], Natanson [6], Paley [7], Robert and Wim [8], Sjolin [9], Schipp [10] and Walter [11], have investigated point wise convergent properties of Walsh expansion of $L^p[0,1)$ class functions. At first, Carleson [12] introduced an operator which is presently known as Carleson operator. Nielsen [13–15], Lal and Kumar [16, 17] and Watari [18]

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have already studied important properties of Carleson operator. In this paper, generalized Carleson operators for Walsh type wavelet packet expansion and for periodic Walsh type wavelet packet expansion for any $f \in L^p[0,1)$ are introduced by the help of Logarithmic means $(N, \frac{1}{m+1}, \frac{1}{n+1})$ and two theorems on convergence of Walsh type wavelet packet expansion by this generalized Carleson operator have been established.

2. Definitions and Preliminaries

2.1. Logarithmic Means

The series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots ,$$

is said to be convergent, to the sum s, if the 'partial sum'

$$s_n = a_1 + a_2 + a_3 + \cdots a_n,$$

tends to a finite limit s when $n \to \infty$, and a series which is not convergent is said to be divergent.

If $s_n = a_1 + a_2 + a_3 + \cdots a_n$, and

$$\lim_{n\to\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = s,$$

then we call s the (C,1) sum of $\sum a_n$ and the (C,1) limit of s_n. (Hardy [19], p.7)

The series $1 + 0 - 1 + 1 + 0 - 1 + \cdots$, is not convergent but it is summable (C,1) to the sum $\frac{2}{3}$. (Titchmarsh [20], p.411)

Let $p_n \geq 0, p_0 > 0, P_n = p_0 + p_1 + p_2 + \cdots + p_n, s_n = \sum_{k=0}^{n} s_k$ and define t_m by

$$\begin{aligned} t_m = N_m^{(p)} &= \frac{p_m s_0 + p_{m-1} s_1 + p_{m-2} s_2 + \cdots + p_0 s_m}{p_0 + p_1 + p_2 + \cdots + p_n} \\ &= \frac{1}{P_m} \sum_{k=0}^{m} p_{m-k} s_k. \end{aligned}$$

If $t_m \to s$ when $m \to \infty$, and we write $s_n \to s$, $\sum_{n=0}^{\infty} a_n = s(N, p_n)$ (Hardy [19] p.64), then (N, p_n) is said to be Nörlund means generated by a sequence $\{p_n\}_{n=0}^{\infty}$. If we take $p_n = \frac{1}{n+1}$ in (N, p_n) means then it reduces to $(N, \frac{1}{n+1})$ means which is also known as logarithmic means

$$\text{Now } t_m^{\frac{1}{m+1}} = \frac{1}{\log(m+1)} \sum_{k=0}^{m} \frac{s_k}{m-k+1}.$$

If $t_m^{\frac{1}{m+1}} \to s$ when $m \to \infty$, and we write $s_n \to s$, $\sum_{n=0}^{\infty} a_n = s(N, \frac{1}{n+1})$ means, i.e, Logarithmic Means (Hardy [19] p.59).

The method $(N, \frac{1}{n+1})$ is more effective than $(C, 1)$ means. If the series $\sum_{n=0}^{\infty} a_n$ is summable to s by $(C, 1)$ method, then it is also summable to s by $(N, \frac{1}{n+1})$ method. it is remarkable to note that the series $\sum n^{-1-ci}$ is not $(C, 1)$ summable but it is $(N, \frac{1}{n+1})$ summable.

For the double infinite series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n},\ S_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}$ Bromwich [21]. Let $q_m \geq 0,\ q_0 > 0,\ Q_m = q_0 + q_1 + q_2 + \cdots + q_m$. Write

$$t_{M,N} = \frac{1}{P_M Q_N}\sum_{m=0}^{M}\sum_{n=0}^{N} p_{M-m} q_{N-n} S_{m,n}$$

If $t_{M,N} \to S$ as $M \to \infty, N \to \infty$, then we write $S_{M,N} \to S$ and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} = S\ (N, p_m, q_n).$$

Taking $p_m = \frac{1}{m+1}, q_n = \frac{1}{n+1}$ in (N, p_m, q_n),

$$t_{M,N}^{(\frac{1}{m+1},\frac{1}{n+1})} = \frac{1}{\log(M+1)\log(N+1)}\sum_{m=1}^{M}\sum_{n=1}^{N} S_{m,n}.$$

If $t_{M,N}^{(\frac{1}{m+1},\frac{1}{n+1})} \to S$ when $M \to \infty, N \to \infty$, we write $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} = S(N, \frac{1}{m+1}, \frac{1}{n+1})$, i.e., double logarithmic means.

For the divergent double infinite series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{(m+n)}, S_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{(i+j)}$ and

$$\begin{aligned}
t_{M,N}^{(\frac{1}{m+1},\frac{1}{n+1})} &= \frac{1}{\log(m+1)\log(n+1)}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{S_{m,n}}{(M-m+1)(N-n+1)} \\
&= \frac{1}{\log(m+1)\log(n+1)}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{1}{4}\frac{(1+(-1)^m)(1+(-1)^n)}{(M-m+1)(N-n+1)} \\
&= \frac{1}{4}\frac{1}{\log(m+1)}\sum_{m=1}^{M}\frac{(1+(-1)^m)}{(M-m+1)}\frac{1}{\log(n+1)}\sum_{n=1}^{N}\frac{(1+(-1)^n)}{(N-n+1)} \\
&= \frac{1}{4}\left\{1 + \frac{\gamma_m}{\log(M+1)} + \frac{(-1)^m}{\log(M+1)(M-m+1)}\right\} \\
&\times \left\{1 + \frac{\gamma_n}{\log(N+1)} + \frac{(-1)^n}{\log(N+1)(N-n+1)}\right\} \\
&\to \frac{1}{4} \quad \text{as} \quad M \to \infty, N \to \infty.
\end{aligned}$$

Hence the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{(m+n)}$ is summable to $\frac{1}{4}$ by $(N, \frac{1}{m+1}, \frac{1}{n+1})$ method.

The method $(N,\frac{1}{m+1},\frac{1}{n+1})$ is more effective than $(C,1,1)$ means. If the series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n}$ is summable to s by $(C,1,1)$ method, then it is also summable to s by $(N,\frac{1}{m+1},\frac{1}{n+1})$ method. The series $\sum\sum m^{-1-ci}n^{-1-dj}$ is not $(C,1,1)$ summable but it is $(N,\frac{1}{m+1},\frac{1}{n+1})$ summable.

2.2. Walsh Functions and Their Properties

The Walsh system $\{W_n\}_{n=0}^{\infty}$ is defined recursively on $[0,1)$ on considering

$$W_0(x)=\begin{cases} 1, & 0\le x<1;\\ 0, & \text{otherwise.}\end{cases}$$

and

$$\begin{aligned} W_{2n}(x) &= W_n(2x)+W_n(2x-1),\\ W_{2n+1}(x) &= W_n(2x)-W_n(2x-1). \end{aligned}$$

Observe that the Walsh system is the family of wavelet packets obtained by considering $\varphi=W_0$,

$$\psi(x)=\begin{cases} 1, & 0\le x<\frac{1}{2};\\ -1, & \frac{1}{2}\le x<1;\\ 0, & \text{otherwise.}\end{cases}$$

and using the Haar filters in the definition of the non-stationary wavelet packets.

The Walsh system is closed under point wise multiplication. Define the binary operator $\oplus:\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{N}_0$ by

$$m\oplus n=\sum_{i=0}^{\infty}|m_i-n_i|2^i,$$

where $m=\sum_{i=0}^{\infty}m_i2^i$, $n=\sum_{i=0}^{\infty}n_i2^i$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Then

$$W_m(x)W_n(x)=W_{m\oplus n}(x), \tag{2.1}$$

(Schipp et al. [10]).

We can carry over the operator $\oplus$ to the interval $[0,1]$ by identifying those $x\in[0,1]$ with a unique expansion $x=\sum_{j=0}^{\infty}x_j2^{-j-1}$ (almost all $x\in[0,1]$ has such a unique expansion) by there associated binary sequence $\{x_i\}$. For two such points $x,y\in[0,1]$, define

$$x\oplus y=\sum_{j=0}^{\infty}|x_j-y_j|2^{-j-1}.$$

The operation $\oplus$ is defined for almost all $x,y\in[0,1]$. Using this definition, we have

$$W_n(x\oplus y)=W_n(x)W_n(y) \tag{2.2}$$

for every pair x,y for which $x\oplus y$ is defined (Golubov et al. [22]).

2.3. Walsh Type Wavelet Packets

Let $\{w_n\}_{n\geq 0, k\in\mathbb{Z}}$ be a family of non-stationary wavelet packets constructed by using a family $\left\{h_0^{(p)}(n)\right\}_{p=0}^{\infty}$ of finite filters for which there is a constant $K\in\mathbb{N}$ such that $h_0^{(p)}(n)$ is the Haar filter for every $p\geq K$. If $w_1\in C^1(\mathbb{R})$ and it has compact support then we call $\{w_n\}_{n\geq 0}$ a family of Walsh type wavelet packets.

2.4. Periodic Walsh Type Wavelet Packets

As Meyer [4], an orthonormal basis for $L^2[0,1)$ is obtained periodizing any orthonormal wavelet basis associated with a multiresolution analysis. The periodization works equally well with non-stationary wavelet packets.

Let $\{w_n\}_{n=0}^{\infty}$ be a family of non-stationary basic wavelet packets satisfying $|w_n(x)|\leq C_n(1+|x|)^{-1-\epsilon_n}$ for some $\epsilon_n>0, n\in\mathbb{N}_0$. For $n\in\mathbb{N}_0$ we define the corresponding periodic wavelet packets $\tilde{w_n}$ by

$$\tilde{w}_n(x)=\sum_{k\in\mathbb{Z}} w_n(x-k).$$

It is important to note that the hypothesis about the point-wise decay of the wavelet packets w_n ensures that the periodic wavelet packets are well defined in $L^p[0,1)$ for $1\leq p\leq\infty$. Wickerhauser [13] have proved that,the family $\{\tilde{w_n}\}_{n=0}^{\infty}$ is an orthonormal basis for $L^p[0,1)$.

2.5. Generalized Carleson Operator by $(N,\frac{1}{m+1},\frac{1}{n+1})$ Means

Write

$$\left(S_{N,N}f\right)(x)=\sum_{n=0}^{N}\sum_{k=-N}^{N}\left\langle f,w_n(.-k)\right\rangle w_n(x-k),\, f\in L^p(\mathbb{R}), 1<p<\infty,$$

$$(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}f)(x) = \frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{(S_{m,n}f)(x)}{(N-m+1)(N-n+1)}.$$

The generalized Carleson operator for the Walsh type wavelet packet system, denoted by L_c^*, is defined by

$$\begin{aligned}(L_c^*f)(x) &= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{(S_{m,n}f)(x)}{(N-m+1)(N-n+1)}\right|\\ &= \sup_{N\geq 1}\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}f)(x)\right|\end{aligned} \tag{2.3}$$

Let us define the generalized Carleson operator for periodic Walsh type wavelet packet system $\{\tilde{w}_n\}$. Write

$$(s_Nf)(x)=\sum_{n=0}^{N}\left\langle f,\tilde{w}_n\right\rangle \tilde{w}_n(x), f\in L^p(\mathbb{R}), 1<p<\infty,$$

$$
\begin{aligned}
(t_N f)(x) &= \frac{1}{\log(N+1)} \sum_{n=0}^{N} \frac{(s_n f)(x)}{N-n+1} \\
&= \frac{1}{\log(N+1)} \sum_{n=0}^{N} \sum_{i=0}^{n} \frac{\langle f, \tilde{w}_i \rangle \, \tilde{w}_i(x)}{N-n+1}.
\end{aligned}
$$

The generalized Carleson Operator for the periodic Walsh type wavelet packet system, denoted by $\mathbb{G}_c^*$, is defined by

$$
\begin{aligned}
(\mathbb{G}_c^* f)(x) &= \sup_{N \geq 1} \left| \frac{1}{\log(N+1)} \sum_{n=0}^{N} \frac{(s_n f)(x)}{N-n+1} \right| \\
&= \sup_{N \geq 1} \left| (t_N^{(N, \frac{1}{n+1})} f)(x) \right|.
\end{aligned} \tag{2.4}
$$

2.6. Strong Type (p, p) Operator

An operator T defined on $L^p(\mathbb{R})$ is of strong type (p, p) if it is sub-linear and there is a constant C such that $\|Tf\|_p \leq C \|f\|_p$ for all $f \in L^p(\mathbb{R})$.

3. Main Results

In this paper, two new theorems have been established in the following forms:

Theorem 3.1. *Let $\{w_n\}$ be a family of Walsh-type wavelet packet system. Then the generalized Carleson operator L_c^* by $(N, \frac{1}{m+1}, \frac{1}{n+1})$ defined by*

$$
\begin{aligned}
(L_c^* f)(x) &= \sup_{N \geq 1} \left| \frac{1}{(\log(N+1))^2} \sum_{m=0}^{N} \sum_{n=0}^{N} \frac{(S_{m,n} f)(x)}{(N-m+1)(N-n+1)} \right| \\
&= \sup_{N \geq 1} \left| (T_{(N,N)}^{(N, \frac{1}{m+1}, \frac{1}{n+1})} f)(x) \right|
\end{aligned}
$$

for any Walsh-type wavelet packet system, is of strong type $(p, p), 1 < p < \infty$.

Theorem 3.2. *Let $\{\tilde{w}_n\}$ be a periodic Walsh-type wavelet packets. Then the generalized Carleson operator $\mathbb{G}_c^*$ by $(N, \frac{1}{m+1}, \frac{1}{n+1})$,defined by*

$$
\begin{aligned}
(\mathbb{G}_c^* f)(x) &= \sup_{N \geq 1} \left| \frac{1}{\log(N+1)} \sum_{n=0}^{N} \frac{(s_n f)(x)}{N-n+1} \right| \\
&= \sup_{N \geq 1} \left| (t_N^{(N, \frac{1}{n+1})} f)(x) \right|.
\end{aligned}
$$

for periodic Walsh type wavelet packet expansion of any $f \in L^p(\mathbb{R}), 1 < p < \infty$, converges a.e..

3.1. Lemmas

For the proof of the theorems following Lemmas are required:

Lemma 3.3 (Zygmund [23], p. 197). *If $v_1, v_2, v_3 \cdots, v_n$ are non-negative and non-increasing,then*

$$|u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n| \leq v_1 \max_k |U_k|,$$

where $U_k = u_1 + u_2 + u_3 + \cdots + u_k$ for $k = 1, 2, 3, \cdots, n$.

Lemma 3.4. *Let $f_1 \in L^2(\mathbb{R})$, and define $\{f_n\}_{n\geq 2}$ recursively by*

$$f_n(x) = \begin{cases} f_m(2x) + f_m(2x-1), & n = 2m; \\ f_m(2x) - f_m(2x-1), & n = 2m+1. \end{cases}$$

Then

$$f_m(x) = \sum_{p=0}^{2^j-1} W_{m-2^j}(p2^{-j}) f_1(2^jx - p),$$

$$\text{for } m, j \in \mathbb{N}, 2^j \leq m < 2^{j+1}.$$

Lemma 3.5. *Let $\{w_n\}_{n\geq 0}$ be a family of Walsh type wavelet packets .If $w_1 \in C^1(\mathbb{R})$ then there exists an isomorphism $\Psi : L^p(\mathbb{R}) \to L^p(\mathbb{R}), 1 < p < \infty$, such that*

$$\Psi w_n(. - k) = W_n(. - k), n \geq 0, k \in \mathbb{Z}.$$

Lemmas 3.4 and 3.5 can be easily proved.

Lemma 3.6 (Billard [1] and Sjölin [9]). *Let $f \in L^1[0, 1)$ and define*

$$S_n(x, f) = \sum_{k=0}^{n} \int_0^1 f(t)W_k(t)dtW_k(x).$$

Then the Carleson operator G defined by

$$(Gf)(x) = \sup_n |S_n(x, f)| \tag{3.1}$$

is of strong type (p, p) for $1 < p < \infty$.

3.2. Proof of Theorem 3.1

For, $f, g \in L^p(\mathbb{R})$

$$
\begin{aligned}
\left(L_c^*(f+g)\right)(x) &= \sup_{N\geq 1}\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}(f+g))(x)\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{(S_{m,n}(f+g))(x)}{(N-m+1)(N-n+1)}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\sum_{k=-n}^{n}\langle f+g, w_i(.-k)\rangle\, w_i(x-k)}{(N-n+1)(N-n+1)}\right| \\
&\leq \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\sum_{k=-n}^{n}\langle f, w_i(.-k)\rangle\, w_i(x-k)}{(N-n+1)(N-n+1)}\right| \\
&+ \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\sum_{k=-n}^{n}\langle g, w_i(.-k)\rangle\, w_i(x-k)}{(N-n+1)(N-n+1)}\right| \\
&= \sup_{N\geq 1}\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}f(x)\right| + \sup_{N\geq 1}\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}g)(x)\right| \\
&= \left(L_c^* f\right)(x) + \left(L_c^* g\right)(x).
\end{aligned}
$$

Also for $\alpha \in \mathbb{R}$

$$
\begin{aligned}
\left(L_c^*\alpha f\right)(x) &= \sup_{N\geq 1}\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}(\alpha f))(x)\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{(S_{m,n}(\alpha f))(x)}{(N-m+1)(N-n+1)}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\sum_{k=-n}^{n}\langle \alpha f, w_n(.-k)\rangle\, w_n(x-k)}{(N-n+1)(N-n+1)}\right| \\
&= \sup_{N\geq 1}|\alpha|\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})}f)(x)\right| \\
&= |\alpha|\left(L_c^* f\right)(x).
\end{aligned}
$$

Choose $M \in \mathbb{N}$ such that $\operatorname{supp}(w_n) \subset [-M, M]$ for $n \geq 0$. Fix $p \in (1, \infty)$ and take any

$$f(x) = \sum_{n\geq 0, k\in\mathbb{Z}} \langle f, w_n(.-k)\rangle\, w_n(x-k) \in L^p(\mathbb{R}).$$

Define $f_k(x) = \sum_{n\geq 0, k\in\mathbb{Z}} \langle f, w_n(.-k)\rangle\, w_n(x-k), g_k(x) = \sum_{n\geq 0, k\in\mathbb{Z}} \langle f, w_n(.-k)\rangle\, W_n(x-k)$. We have $\|f_k\|_p \approx \|g_k\|_p$, with bounds independent of k (Lemma 3.5). Note that

$$|\{x\in[l,l+1) : |L_c f(x)| > \alpha\}| \leq \frac{C}{\alpha^p}\sum_{k=l-n}^{l+1+n}\int |L_c f_k(x)|^p dx.$$

Using the Marcinkiewicz interpolation theorem, it suffices to prove that

$$\|L_c f_k\|_p \leq C\|f_k\|_p,$$

where C is a constant independent of k. Since

$$\sum_{l\in\mathbb{Z}}\sum_{k=l-n}^{l+1+n}\|f_k\|_p^p \leq 2(n+1)\sum_{k\in\mathbb{Z}}\|f_k\|_p^p \leq 2C(n+1)\sum_{k\in\mathbb{Z}}\|g_k\|_p^p \leq C_1\|f\|_p^p,$$

where C_1 is constant. Without loss of generality, we assume that $k = 0$. Let $K \in \mathbb{N}$ be the scale from which only the Haar filter is used to generate the wavelet packets $(\omega_n)_{n\geq 2^{k+1}}$. Let $N \in \mathbb{N}$ and suppose $2^J \leq N \leq 2^{J+1}$ for some $J > K+1$. Clearly, for each $x \in \mathbb{R}$,

$$\begin{aligned}
(L_c^* f)(x) &= \sup_{N\geq 1}\left|(T_{(N,N)}^{(N,\frac{1}{m+1},\frac{1}{n+1})} f)(x)\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{m=0}^{N}\sum_{n=0}^{N}\frac{(S_{m,n}f)(x)}{(N-m+1)(N-n+1)}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{n=0}^{N}\sum_{n=0}^{N}\frac{(S_{n,n}f)(x)}{(N-n+1)(N-n+1)}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))^2}\sum_{n=0}^{N}\frac{1}{N-n+1}\sum_{n=0}^{N}\frac{(S_{n,n}f)(x)}{(N-n+1)}\right| \\
&\leq \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))}\sum_{n=0}^{N}\frac{(S_{n,n}f)(x)}{(N-n+1)}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\sum_{k=-n}^{n}\langle f, w_i(.-k)\rangle\, w_i(x-k)}{(N-n+1)}\right|. \\
&= \sup_{N\geq 1}\left|\frac{1}{(\log(N+1))}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\langle f, w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right| \quad (k=0) \\
&\leq \sup_{1\leq N<2^{K+1}}\left|\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\langle f, w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right|
\end{aligned}$$

$$+\sup_{J>K+1}\left|\frac{1}{(\log(N+1))}\sum_{n=2^{K+1}}^{2^J-1}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle w_i(x)}{(N-n+1)}\right|$$

$$+\sup_{J>K+1}\sup_{2^J\leq N<2^{J+1}}\left|\frac{1}{(\log(N+1))}\sum_{n=2^J}^{N}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle w_i(x)}{(N-n+1)}\right|$$

$$= J_1+J_2+J_3. \tag{3.2}$$

Using Lemma 3.3 we have

$$\begin{aligned}
J_1 &= \sup_{1\leq N<2^{K+1}}\left|\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle w_i(x)}{(N-n+1)}\right| \\
&\leq \sup_{1\leq N<2^{K+1}}\max\sum_{0=n\leq 2^{K+1}-1}|\langle f,w_n(.)\rangle|\,|w_n(x)| \\
&\leq \sum_{n=0}^{2^{K+1}-1}|\langle f,w_n(.)\rangle|\,|w_n(x)| \\
&\leq |\langle f,w_n(.)\rangle|\,\|w_n(x)\|_\infty\,\chi_{[-N,N]}(x) \\
&\leq \|f_0\|_p\sum_{n=0}^{2^{K+1}-1}\|w_n\|_q\,\|w_n(x)\|_\infty\,\chi_{[-N,N]}(x).
\end{aligned} \tag{3.3}$$

Next,

$$\begin{aligned}
J_2 &= \sup_{J>K+1}\left|\frac{1}{\log(N+1)}\sum_{n=2^{K+1}}^{2^J-1}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle w_i(x)}{(N-n+1)}\right| \\
&\leq \sup_{J>K+1}\max\left|\sum_{2^{K+1}=n\leq 2^J-1}\langle f,w_n(.)\rangle w_n(x)\right|.
\end{aligned}$$

Also,

$$\begin{aligned}
\left\|\sup_{J>K+1}\frac{1}{\log(N+1)}\sum_{n=2^{K+1}}^{2^J-1}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle w_i(x)}{(N-n+1)}\right\|_p &\leq \sup_{J>K+1}\max\sum_{2^{K+1}=n\leq 2^j-1}|\langle f,w_n(.)\rangle|\,\|w_n(x)\|_p \\
&\leq C\sum_{n=0}^{\infty}|\langle f,w_n(.)\rangle|\,\|w_n(x)\|_p \\
&= C\,\|f_0\|_p\,.
\end{aligned} \tag{3.4}$$

Consider J_3,

$$J_3 = \sup_{J>K+1} \sup_{2^J \le N < 2^{J+1}} \left| \frac{1}{\log(N+1)} \sum_{n=2^J}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle w_i(x)}{(N-n+1)} \right|$$

$$\le \sum_{j=0}^{2^{K-1}} \left(\sup_{2^J+j2^{J-K} \le N < 2^J+(j+1)2^{J-K}} \left| \frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle w_i(x)}{(N-n+1)} \right| \right)$$

so it suffices to prove that

$$\left\| \sup_{J>K+1,} \sup_{2^J+j2^{J-K} \le N < 2^J+(j+1)2^{J-K}} \frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \left| \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle w_i(x)}{(N-n+1)} \right| \right\|_p \le C \|f_0\|_p$$

for $j = 0, 1, 2, \cdots, 2^{K-1}$. Fix $J > K+1, 0 \le j \le 2^{2^k-1}$ and $2^J + j2^{JK} \le N < 2^J + (j+1)2^{J-K}$. Write,

$$J_3' = \frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle w_i(x)}{(N-n+1)}.$$

Using Lemma 3.4, we have

$$\left|j_3'\right| = \left| \sum_{s=0}^{2^{J-K}-1} \frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle}{(N-n+1)} W_{n-2^J-j2^{J-K}}(s2^{-(J-K)}) w_{2^K+j}(2^{J-K}x - s) \right|.$$

Define

$$F_N(t) = \frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle}{(N-n+1)} W_{n-2^J-j2^{J-K}}(t),$$

and

$$F(t) = \sup_{N<2^J+(j+1)2^{J-K}} |F_N(t)|.$$

We have

$$\begin{aligned} \left|j_3'\right| &= \frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle w_i(x)}{(N-n+1)} \\ &\le \max \left| \sum_{2^J+j2^{J-K}=n\le N} \langle f, w_n(.)\rangle w_n(x) \right| \\ &\le \sum_{s=0}^{2^{J-K}-1} F(s2^{-(J-K)}) \left| w_{2^K+j}((2^{J-K})x - s) \right|, \end{aligned}$$

and using the compact support of the wavelet packets,

$$\begin{aligned} \left|j_3^{'}\right| &= \left|\frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right| \\ &\leq \max\left|\sum_{2^J+j2^{J-K}=n\leq N} \langle f, w_n(.)\rangle\, w_n(x)\right| \\ &\leq \|w_{2^K}+j\|_\infty \sum_{l=-M}^{M+1} F((\lfloor 2^{J-K}x\rfloor + l)2^{-(J-K)}). \end{aligned}$$

Note that F is constant on dyadic intervals of type $[l2^{-(J-K)}, (l+1)2^{-(J-K)})$, and taking

$$\Delta_l = \left(\left(\left\lfloor 2^{J-K}x + l\right\rfloor\right) 2^{-(J-K)}, \left(\left\lfloor 2^{J-K}x + l + 1\right\rfloor\right) 2^{-(J-K)}\right),$$

we have

$$\begin{aligned} \left|j_3^{'}\right| &= \left|\frac{1}{\log(N+1)} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\sum_{i=0}^{n} \langle f, w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right| \\ &\leq \max\left|\sum_{2^J+j2^{J-K}=n\leq N} \langle f, w_n(.)\rangle\, w_n(x)\right| \\ &\leq \|w_{2^K}+j\|_\infty \sum_{l=-M}^{M+1} |\Delta_l|^{-1} \int_{\Delta_l} F(t)dt. \end{aligned}$$

We need an estimate of F that does not depend on J. Note that for k,
$0 \leq k < 2^{J-K}$,

$$W_{2^J+j2^{J-K}}(t)W_k(t) = W_{2^J+j2^{J-K}+k}(t),$$

since the binary expansions of $2^J + j2^{J-K}$ and of k have no $1^{'s}$ in common. Hence,

$$\begin{aligned} |F_N(t)| &= \left|W_{2^J+j2^{J-K}}(t)F_N(t)\right| \\ &= \left|\frac{1}{(\log(N+1))^2} \sum_{n=2^J+j2^{J-K}}^{N} \frac{\langle f, w_n(.)\rangle}{(N-n+1)^2} W_n(t)\right| \\ &\leq \max\left|\sum_{2^J+j2^{J-K}=n<N} \langle f, w_n(.)\rangle\, W_n(t)\right|. \end{aligned}$$

Using Lemma 3.6 we have $F(t) \leq 2(Gg_0)(t)$. Thus,

$$\left|\frac{1}{\log(N+1)}\sum_{n=2^J+j2^{J-K}}^{N}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right| \leq \max\left|\sum_{2^J+j2^{J-K}=n<N}\langle f,w_n(.)\rangle\, w_n(x)\right|$$
$$\leq 2\left\|w_{2^K+j}\right\|_\infty \sum_{l=-M}^{M+1}|\Delta_l|^{-1}\int_{\Delta_l}(Gg_0)(t)dt.$$

Let Δ_l^* be the smallest dyadic interval containing Δ_l and x, and note that $|\Delta_l^*| \leq (M+1)\,|\Delta_l|$ since $x \in (\Delta_0 - D)$.
We have

$$\left|\frac{1}{\log(N+1)}\sum_{n=2^J+j2^{J-K}}^{N}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right| \leq \max\left|\sum_{2^J+j2^{J-K}=n<N}\langle f,w_n(.)\rangle\, w_n(x)\right|$$
$$\leq 2\left\|w_{2^K+j}\right\|_\infty \sum_{l=-M}^{M+1}|\Delta_l|^{-1}\int_{\Delta_l^*}(Gg_0)(t)dt$$
$$\leq 4\left\|w_{2^K+j}\right\|_\infty (M+1)^2(M^*(Gg_0)(x), \quad (3.5)$$

where M^* is the maximal operator of Hardy and Little wood. The right hand side of (3.5) neither depend on N nor J so we may conclude that

$$J_3 \leq \sup_{J>K+1}\sum_{j=0}^{2^{K-1}}\left(\sup_{2^J+j2^{J-K}\leq N<2^J+(j+1)2^{J-K}}\left|\frac{1}{\log(N+1)}\sum_{n=2^J+j2^{J-K}}^{N}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right|\right)$$
$$\leq \sum_{j=0}^{2^{K-1}}\left(\sup_{2^J+j2^{J-K}\leq N<2^J+(j+1)2^{J-K}}\left(\max\left|\sum_{2^J+j2^{J-K}=n<N}\langle f,w_n(.)\rangle\, w_n(x)\right|\right)\right)$$
$$\leq 4\left\|w_{2^K+j}\right\|_\infty (M+1)^2(M^*(Gg_0))(x), \qquad a.e., \quad (3.6)$$

Using Sjolin [9], M^* and G both are bounded and of strong type (p,p). Now,

$$\left\|\sup_{J>K+1} J_3\right\|_p \leq \left\|\sup_{J>K+1}\sum_{j=0}^{2^{K-1}}\left(\sup_{t\leq N<2^J+(j+1)2^{J-K}}\left|\frac{1}{\log(N+1)}\sum_{n=t}^{N}\frac{\sum_{i=0}^{n}\langle f,w_i(.)\rangle\, w_i(x)}{(N-n+1)}\right|\right)\right\|_p$$
$$\leq C\left\|g_0\right\|_p$$
$$\leq C_1\left\|f_0\right\|_p,$$

where $j = 0,1,2,\cdots,2^K-1$ and $t = 2^J + j2^{J-K}$. Hence the operator L_C^* is of strong type (p,p) $1 < p < \infty$. Thus the Theorem 3.1 is completely established.

3.3. Proof of Theorem 3.2

Let $f \in L^p[0,1)$, and choose $M \in \mathbb{N}$ such that $supp(w_n) \subset [-M, M]$ for $n \geq 0$. Then

$$\begin{aligned}
(\mathbb{G}_c^* f)(x) &= \sup_{N\geq 1}\left|(t_N^{(N,\frac{1}{n+1})} f)(x)\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{(s_n f)(x)}{N-n+1}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{\sum_{i=0}^{n}\langle f,\tilde{\omega}_i\rangle\,\tilde{\omega}_i(x)}{N-n+1}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{\left\langle f,\sum_{k_1=-M}^{M+1} w_n(.-k_1)\right\rangle \sum_{k_2=-M}^{M+1} w_n(x-k_2)}{N-n+1}\right| \\
&= \sup_{N\geq 1}\left|\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{\sum_{k_1=-M}^{M+1}\langle f, w_n(.-k_1)\rangle \sum_{k_2=-M}^{M+1} w_n(x-k_2)}{N-n+1}\right| \\
&= \sup_{N\geq 1}\left|\sum_{k_1=-M}^{M+1}\sum_{k_2=-M}^{M+1}\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{\langle f, w_n(.-k_1)\rangle\, w_n(x-k_2)}{N-n+1}\right| \\
&= \sup_{N\geq 1}\left|\sum_{k_1=-M}^{M+1}\sum_{k_2=-M}^{M+1}\frac{1}{\log(N+1)}\sum_{n=0}^{N}\frac{1}{N-n+1}\int_0^1 f(y)\overline{w_n(y-k_1)}dy\, w_n(x-k_2)\right|.
\end{aligned}$$

Following the proof of Theorem 3.1, it can be proved that the generalized Carleson operator $\mathbb{G}_c^*$ for the periodic Walsh type wavelet packet expansions converges a.e.. Thus the Theorem 3.2 is completely established.

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