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Hermite-Hadamard Type Inequalities for g-GA-Convex Dominated Functions

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Abstract In this paper, the author introduces the concept of g-GA-convex dominated function and gives a version of Hermite-Hadamard-type inequalities for g-GA-convex dominated functions. Applications are also given involving a functional and some common means.

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1. Introduction

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{1.1}$$

holds for all $x,y \in I$ and $t \in [0,1]$.

It is general knowledge that if $f : I \to \mathbb{R}$ is a convex function and $a,b \in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

This inequality is well known in the literature as Hermite-Hadamard's inequality for convex functions. For various some results in recent years based on integral inequalities, you may see the papers [1–3].

The following defnitions and results are well known in the literature.

Definition 1.1 ([4, 5]). A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq tf(x)+(1-t)f(y)$$

for all $x,y \in I$ and $t \in [0,1]$.

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The following proposition is obvious from definition.

Proposition 1.2. *The function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is GA-convex if and only if $f \circ \exp$ is convex on the interval $\ln I = \{\ln x : x \in I\}$.*

As a result of Proposition 1.2, we can give a analogous of Hermite-Hadamard's inequality for GA-convex functions as follows:

If $f : I \subseteq (0, \infty) \to \mathbb{R}$ is a GA-convex function and $a, b \in I$ with $a < b$, then

$$f\left(\sqrt{ab}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x)}{x}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

For GA-convex (concave) function, R.A. Satnoianu [6] obtained the following result.

Lemma 1.3. *If $f : I \subseteq (0, \infty) \to (0, \infty)$ is a twice differentiable function, then f is a GA- convex (concave) function in I if and only if $xf'(x) + x^2 f''(x) \geq (\leq) 0$ for all $x \in I$.*

Definition 1.4 ([1])**.** Let $g : I \to \mathbb{R}$ be a given convex function. The function $f : I \to \mathbb{R}$ is called g-convex dominated on I if

$$\begin{aligned} & |tf(x) + (1-t)f(y) - f(tx + (1-t)y)| \\ \leq\ & tg(x) + (1-t)g(y) - g(tx + (1-t)y) \end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For a mapping $f : [a, b] \to \mathbb{R}$ with $f \in L[a, b]$, we can define the mapping $H_f : [0, 1] \to \mathbb{R}$

$$H_f(t) := \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\left(\frac{a+b}{2}\right)\right)dx.$$

The following theorem contains some results of this type for convex-dominated functions:

Theorem 1.5 ([7])**.** *Let $g : [a, b] \to \mathbb{R}$ be a convex function and $f : [a, b] \to \mathbb{R}$ a g-convex dominated on $[a, b]$. Then*

(i) H_f is H_g -convex dominated on $[0, 1]$.

(ii) One has the inequalities

$$0 \leq |H_f(t_2) - H_f(t_1)| \leq H_g(t_2) - H_g(t_1)$$

for all $0 \leq t_1 \leq t_2 \leq 1$.

(iii) One has the inequalities

$$0 \leq \left|f\left(\frac{a+b}{2}\right) - H_f(t)\right| \leq H_g(t) - g\left(\frac{a+b}{2}\right)$$

and

$$0 \leq \left|\frac{1}{b-a}\int_a^b f(x)\,dx - H_f(t)\right| \leq \frac{1}{b-a}\int_a^b g(x)\,dx - H_g(t)$$

for all $t \in [0, 1]$.

2. g-GA-Convex Dominated Functions

Definition 2.1. Let $g : I \subseteq (0, \infty) \to \mathbb{R}$ be a given GA-convex function. The function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is called g-GA-convex dominated on I if

$$\begin{aligned} & \left| tf(x) + (1-t)f(y) - f\left(x^t y^{1-t}\right) \right| \\ \leq \quad & tg(x) + (1-t)g(y) - g\left(x^t y^{1-t}\right) \end{aligned} \tag{2.1}$$

for all $x, y \in I$ and $t \in [0, 1]$.

The class of g-GA-convex dominated functions on an interval I is manifestly nonempty. If g is GA-convex function on $I \subseteq (0, \infty)$ and $f : I \to \mathbb{R}$ is defined by $f(x) := x$, then f and g are both g-GA-convex dominated on I. Indeed there are GA-concave functions which are g-GA-convex dominated (for example $-g$).

The following proposition is obvious from definition.

Proposition 2.2. *Let $g : I \subseteq (0, \infty) \to \mathbb{R}$ be a given GA-convex function. The function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is g-GA-convex dominated on I if and only if $f \circ \exp$ is $g \circ \exp$-convex dominated on the interval $\ln I = \{\ln x : x \in I\}$.*

The next simple characterization of g-GA-convex dominated functions holds.

Lemma 2.3. *Let $g : I \subseteq (0, \infty) \to \mathbb{R}$ be a GA-convex function and $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a real function. Then the following statements are equivalent:*

(1) f is g-GA-convex dominated on I.
(2) The mappings $g - f$ and $g + f$ are GA-convex on I.
(3) There exist two GA-convex dominated φ, ψ defined on I such that

$$f = \frac{1}{2}(\varphi - \psi) \text{ and } g = \frac{1}{2}(\varphi + \psi).$$

Proof. $1 \Longleftrightarrow 2$: The condition (2.1) is equivalent to

$$\begin{aligned} & g\left(x^t y^{1-t}\right) - tg(x) - (1-t)g(y) \\ \leq \quad & tf(x) + (1-t)f(y) - f\left(x^t y^{1-t}\right) \\ \leq \quad & tg(x) + (1-t)g(y) - g\left(x^t y^{1-t}\right) \end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$. The two inequalities may be rearranged as

$$(g + f)\left(x^t y^{1-t}\right) \leq t(g + f)(x) + (1-t)(g + f)(y)$$

and

$$(g - f)\left(x^t y^{1-t}\right) \leq t(g - f)(x) + (1-t)(g - f)(y)$$

which are equivalent to the GA-convexity of $g + f$ and $g - f$, respectively.

$2 \Longleftrightarrow 3$: Let we define the mappings f, g as $f = \frac{1}{2}(\varphi - \psi)$ and $g = \frac{1}{2}(\varphi + \psi)$. Then if we sum and subtract f and g, respectively, we have $g + f = \varphi$ and $g - f = \psi$. By the condition 2 in Lemma 2.3, the mappings $g + f$ and $g - f$ are GA-convex on I, so, φ, ψ are GA-convex on I, also. ∎

Proposition 2.4. *Suppose f'', g'' exist and satisfy*

$$\left| xf'(x) + x^2 f''(x) \right| \leq xg'(x) + x^2 g''(x)$$

on an interval I. Then f is g–GA-convex dominated on I.

Proof. By the given condition

$$xg'(x)+x^2g''(x),\ x(g-f)'(x)+x^2(g-f)''(x),\ x(g+f)'(x)+x^2(g+f)''(x)$$

are all nonnegative on I, so g, $g-f$, $g+f$ are all GA-convex on I by Lemma 1.3, whence the stated result follows by Lemma 2.3. ∎

Theorem 2.5. *Let $g : I \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function and the function $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a g-GA-convex dominated on I. Then for all $a,b \in I$ with $a<b$,*

$$\left| f(\sqrt{xy}) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \right| \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{g(x)}{x}dx - g(\sqrt{xy})$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \right| \leq \frac{g(a)+g(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{g(x)}{x}dx.$$

Proof. Since f is a g-GA-convex dominated on I, we have by Lemma 2.3 that $g+f$ and $g-f$ are GA-convex on I, and so by the Hermite-Hadamard inequality (1.2)

$$(f+g)(\sqrt{xy}) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{(f+g)(x)}{x}dx \leq \frac{(f+g)(a)+(f+g)(b)}{2}$$

and

$$(g-f)(\sqrt{xy}) \leq \frac{1}{\ln b - \ln a}\int_a^b \frac{(g-f)(x)}{x}dx \leq \frac{(g-f)(a)+(g-f)(b)}{2},$$

which are equivalent to desired inequalities. ∎

3. Means

For a pair x,y of positive numbers, we define the the arithmetic, geometric and logarithmic means by

$$A(x,y) := \frac{x+y}{2},\ G(x,y) := \sqrt{xy} \text{ and } L(x,y) := \frac{y-x}{\ln y - \ln x}$$

respectively. We now give several corollaries that provide examples of GA-convex dominated functions.

Corollary 3.1. *Suppose $[a,b] \subseteq (0,\infty)$ and $p \in \mathbb{R}\backslash\{0\}$. Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that $\left|xf'(x)+x^2f''(x)\right| \leq Mx^p$ $(M>0)$ for $x \in [a,b]$. Then*

$$\left| f(\sqrt{xy}) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \right| \leq \frac{M}{p^2}\left[L(a^p,b^p) - G^p(a,b)\right]$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx \right| \leq \frac{M}{p^2}\left[A(a^p,b^p) - L(a^p,b^p)\right].$$

Proof. Define the mapping $f : [a, b] \to \mathbb{R}$ by

$$g(x) = \frac{Mx^p}{p^2}.$$

Then

$$\left|xf'(x) + x^2 f''(x)\right| \le Mx^p$$

on $[a, b]$. The stated results follow from Proposition 2.4 and Theorem 2.5. ■

In particular we derive the following in the case $p = 0$.

4. Functionals

If $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ with $f \in L[a, b]$, we define the induced mapping $I_f : [0, 1] \to \mathbb{R}$ by

$$I_f(t) := \frac{1}{\ln b - \ln a} \int_a^b \frac{f\left(x^t(\sqrt{ab})^{1-t}\right)}{x} dx.$$

The following theorem holds.

Theorem 4.1. *Let $g : [a, b] \to \mathbb{R}$ be a GA-convex function and $f : [a, b] \to \mathbb{R}$ a g-GA-convex dominated on $[a, b]$. Then*

(i) I_f is I_g -convex dominated on $[0, 1]$.

(ii) One has the inequalities

$$0 \le |I_f(t_2) - I_f(t_1)| \le I_g(t_2) - I_g(t_1)$$

for all $0 \le t_1 \le t_2 \le 1$.

(iii) One has the inequalities

$$0 \le \left|f\left(\sqrt{ab}\right) - I_f(t)\right| \le I_g(t) - g\left(\sqrt{ab}\right)$$

and

$$0 \le \left|\frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}dx - I_f(t)\right| \le \frac{1}{\ln b - \ln a}\int_a^b \frac{g(x)}{x}dx - I_g(t)$$

for all $t \in [0, 1]$.

Proof. Since g is GA-convex on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a g-GA-convex dominated on $[a, b]$, by Proposition 2.2 $f \circ \exp$ is $g \circ \exp$-convex dominated on the on the interval $\ln I$. Then the function $H_f : [0, 1] \to \mathbb{R}$

$$\begin{aligned} H_{f\circ\exp}(t) \quad : \quad &= \frac{1}{\ln b - \ln a}\int_a^b (f \circ \exp)\left(t \ln x + (1 - t)\left(\frac{\ln a + \ln b}{2}\right)\right) d \ln x \\ &= I_f(t) \end{aligned}$$

holds Theorem 1.5. Thus desired results are obtained from Theorem 1.5. We shall omit the details. ■

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