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Continuous Spectrum of Robin Nonhomogeneous Elliptic Problems with Variable Exponents

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Abstract By applying two versions of Mountain Pass Theorem and Ekeland's variational principle, we prove three different situations of the existence of solutions for the following Robin problem

$$\begin{aligned} &-\Delta_{p(x)}u = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega,\\ &|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain and $p, q : \overline{\Omega} \to (1, +\infty)$ are continuous functions.

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1. INTRODUCTION

The aim of this article is to analyze the existence of solutions of the nonhomogeneous eigenvalue problem

$$\begin{aligned} &-\Delta_{p(x)}u = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega,\\ &|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, λ is a positive number, p, q are continuous functions on $\overline{\Omega}$ with $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$, $q^- := \inf_{x\in\overline{\Omega}} q(x) > 1$, and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural extension of the classical p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. However, such generalizations are not trivial since the $p(x)$- Laplace operator possesses a more complicated structure than p Laplace operator; for example, it is inhomogeneous.

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The main interest in studying problem (1.1) is given by the presence of the variable exponents $p(.)$ and $q(.)$. Problems involving such kind of growth conditions benefited by a special attention in the last decade since they can model with sufficient accuracy phenomena arising in different branches of science. In this context we remember that the first two models where operators involving variable exponents were considered come from fluid mechanics, more exactly from the study of electrorheological fluids (Acerbi & Mingione [1, 2], Diening [3], Halsey [4], Ruzicka [5, 6], Rajagopal & Ruzicka [7]) and from the study of elastic mechanics (Zhikov [8]). After this pioneering models, many other applications of differential operators with variable exponents have appeared in a large range of fields, such as image restoration (Chen et al. [9]), mathematical biology (Fragnelli [10]), the study of dielectric breakdown, electrical resistivity, and polycrystal plasticity (Bocea & Mihailescu [11], Bocea et al. [12]) or in the study of some models for growth of heterogeneous sandpiles (Bocea et al. [13]).

The Robin boundary conditions appear in the solving Sturm-Liouville problems which are used in many contexts of science and engineering: for example, in electromagnetic problems, in heat transfer problems and for convection-diffusion equations (Fick's law of diffusion). The Robin problem plays a major role in the study of reflected shocks in transonic flow. Important applications of this problem is the capillary problem.

There are many reference papers related to the study of variational problems involving variable exponents, far from being complete, we refer to [14–22]. Recently, Harjulehto et al. gave a survey on the differential equations with non-standard growth conditions and compared different results already obtained on existence and regularity of solutions [23].

In [24], Papageorgiou and Radulescu considered the following problem:

$$\begin{aligned} -div\left(|\nabla u|^{p-2}\nabla u\right) &= \lambda f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $u > 0, \lambda > 0, 1 < p < \infty$. By the variational method and truncation techniques, they proved a bifurcation-type result describing the set of positive solutions as the positive parameter λ varies.

Problem (1.2) may be viewed as a prototype of pattern formation in biology and is related to the steady-state problem for a chemotactic aggregation model introduced by Keller and Segel (1970). Problem (1.2) also plays an important role in the study of activator-inhibitor systems modeling biological pattern formation, as proposed by Gierer and Meihardt (1972).

The $p(x)$-Laplacian problem involving Robin boundary conditions was studied by many authors in recent years, we mention that Deng in [25] considered the following problem:

$$\begin{aligned} -\Delta_{p(x)}u &= \lambda f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Applying the sub-supersolution method and the variational method, under appropriate assumptions on f, the author established the existence of $\lambda_* > 0$ such that the above problem has at least two positive solutions if $\lambda \in (0, \lambda_*)$, has at least one positive solution if $\lambda = \lambda_* < +\infty$ and has no positive solution if $\lambda > \lambda_*$.

Deng et al in [26] investigated problem (1.1) under the particular case when $p(x) \equiv q(x)$, the authors established the existence of infinitely many eigenvalue sequences provided $p(x)$ is non constant and they presented some sufficient conditions for which there is no principal eigenvalue for the problem and the set of all eigenvalues is not closed.

Very recently, Ge et al in [27] studied the following problem:

$$\begin{aligned} -\Delta_{p(x)} u &\in \lambda \partial F(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)|u|^{p(x)-2}u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where λ is a positive parameter, $F(x,t)$ is locally Lipschitz function in the t-variable integrand and $\partial F(x,u)$ is the subdifferential with respect to the t-variable in the sense of Clarke. They claimed that problem (1.4) admits at least two nontrivial solutions.

Here, problem (1.1) is stated in the framework of the generalized Sobolev space $X := W^{1,p(x)}(\Omega)$ for which some elementary properties are stated below.

By a weak solution for (1.1) we understand a function $u \in X$ such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv \, d\sigma - \lambda \int_\Omega |u|^{q(x)-2}uv \, d\sigma = 0, \quad \forall v \in X.$$

We point out that in the case when u is nontrivial, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of (1.1) and u is called an associated eigenfunction.

Inspired by the works of Mihăilescu and Rădulescu [28, 29], we study (1.1) in three distinct situations.

This article consists of three sections. Section 2 contains some preliminary properties concerning the generalized Lebesgue-Sobolev spaces and an embedding result. The main results and their proofs are given in Section 3.

2. PRELIMINARIES

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details, see [30, 31]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} p(x) > 1\}.$$

Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)}dx < +\infty\},$$

with the norm

$$|u|_{p(x)} = \inf\{\tau > 0; \int_\Omega |\frac{u}{\tau}|^{p(x)}dx \leq 1\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \inf\{\tau > 0; \int_\Omega (|\frac{\nabla u}{\tau}|^{p(x)} + |\frac{u}{\tau}|^{p(x)})dx \leq 1\},$$
$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

We refer the reader to [20, 30] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1 ([31]). *Both* $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ *and* $(W^{1,p(x)}(\Omega), \|\cdot\|)$ *are separable and uniformly convex Banach spaces.*

Lemma 2.2 ([31]). *Hölder inequality holds, namely*

$$\int_\Omega |uv|dx \leq 2|u|_{p(x)}|v|_{p'(x)} \quad \forall u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Lemma 2.3 ([30]). *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\overline{\Omega})$ *and* $1 \leq q(x) < p^*(x)$ *for* $x \in \overline{\Omega}$, *then there is a compact embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N; \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

Now, we introduce a norm, which will be used later.
Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega} \beta(x) > 0$ and for $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_\beta = \inf\{\tau > 0; \int_\Omega (|\frac{\nabla u}{\tau}|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|\frac{u}{\tau}|^{p(x)})d\sigma \leq 1\}.$$

Then, by Theorem 2.1 in [25], $\|.\|_\beta$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|.\|$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the mapping defined by the following.

Lemma 2.4 ([25]). *Let* $I(u) = \int_\Omega |\nabla u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}d\sigma$ *with* $\beta^- > 0$. *For* $u \in W^{1,p(x)}(\Omega)$ *we have*

- $\|u\|_\beta < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$.
- $\|u\|_\beta \leq 1 \Rightarrow \|u\|_\beta^{p^+} \leq I(u) \leq \|u\|_\beta^{p^-}$.
- $\|u\|_\beta \geq 1 \Rightarrow \|u\|_\beta^{p^-} \leq I(u) \leq \|u\|_\beta^{p^+}$.

The Euler-Lagrange functional associated with (1.1) is defined as $\Phi_\lambda : X \to \mathbb{R}$,

$$\Phi_\lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\, d\sigma - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx.$$

Standard arguments imply that $\Phi_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle \Phi'_\lambda(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\, d\sigma - \lambda \int_\Omega |u|^{q(x)-2}uvdx,$$

for all $u, v \in X$. Thus the weak solutions of (1.1) coincide with the critical points of Φ_λ. If such a weak solution exists and is nontrivial, then the corresponding λ is an eigenvalue of problem (1.1).

Next, we write Φ'_λ as

$$\Phi'_\lambda = A - \lambda B,$$

where $A, B : X \to X'$ are defined by

$$\begin{aligned}\langle A(u), v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2}uv\, d\sigma,\\ \langle B(u), v\rangle &= \int_\Omega |u|^{q(x)-2}uv\, dx.\end{aligned}$$

Lemma 2.5 ([27]). *A satisfies condition (S^+), namely, $u_n \rightharpoonup u$, in X and $\limsup\langle A(u_n), u_n - u\rangle \le 0$, imply $u_n \to u$ in X.*

Remark 2.6. Noting that Φ'_λ is still of type (S^+). Hence, any bounded (PS) sequence of Φ_λ in the reflexive Banach space X has a convergent subsequence.

3. MAIN RESULTS AND PROOFS

Theorem 3.1. *Let $p, q \in C_+(\overline{\Omega})$. If*

$$q^+ < p^-, \tag{3.1}$$

then any $\lambda > 0$ is an eigenvalue for problem (1.1). Moreover, for any $\lambda > 0$ there exists a sequence (u_n) of nontrivial weak solutions for problem (1.1) such that $u_n \to 0$ in X.

We want to apply the symmetric mountain pass lemma in [32].

Theorem 3.2. (Symmetric mountain pass lemma) *Let E be an infinite dimensional Banach space and $I \in C^1(E, R)$ satisfy the following two assumptions:*

- *(A1) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS), namely, any sequence u_n in E such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in E as $n \to \infty$ has a convergent subsequence.*
- *(A2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Then, $I(u)$ admits a sequence of critical points u_k such that

$$I(u_k) < 0, u_k \neq 0 \text{ and } \lim_k u_k = 0,$$

where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \ge k$ with $\gamma(A)$ is the genus of A, i.e.,

$$\gamma(K) = \inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k\backslash\{0\} \text{ such that } h \text{ is continuous and odd }\}.$$

We start with two auxiliary results.

Lemma 3.3. *The functional Φ_λ is even, bounded from below and satisfies the (PS) condition; $\Phi_\lambda(0) = 0$.*

Proof. It is clear that Φ_λ is even and $\Phi_\lambda(0)=0$. Since $q^+<p^-$ and X is continuously embedded both in $L^{q^\pm}(\Omega)$, there exist two positive constants $M_1, M_2>0$ such that

$$\int_\Omega |u|^{q^+}dx \le M_1\|u\|^{q^+}, \quad \int_\Omega |u|^{q^-}dx \le M_2\|u\|^{q^-}, \quad \forall u\in X.$$

According to the fact that

$$|u(x)|^{q(x)} \le |u(x)|^{q^+}+|u(x)|^{q^-}, \quad \forall x\in\overline{\Omega}, \tag{3.2}$$

for all $u\in X$, we have

$$\begin{aligned}\Phi_\lambda(u) &\ge \frac{1}{p^+}\Big(\int_\Omega |\nabla u|^{p(x)}dx+\int_{\partial\Omega}\beta(x)|u|^{p(x)}d\sigma\Big)-\frac{\lambda M_1}{q^-}\|u\|^{q^+}-\frac{\lambda M_2}{q^-}\|u\|^{q^-}\\ &\ge \frac{1}{p^+}g(\|u\|_\beta)-\frac{\lambda M_1}{q^-}\|u\|^{q^+}-\frac{\lambda M_2}{q^-}\|u\|^{q^-},\end{aligned}$$

where $g:[0,+\infty[\to\mathbb{R}$ is defined by

$$g(t)=\begin{cases} t^{p^+}, & \text{if } t\le 1,\\ t^{p^-}, & \text{if } t>1.\end{cases} \tag{3.3}$$

As $q^+<p^-$, Φ_λ is bounded from below and coercive. It remains to show that the functional Φ_λ satisfies the (PS) condition to complete the proof. Let $(u_n)\subset X$ be a (PS) sequence of Φ_λ in X, that is,

$$\Phi_\lambda(u_n) \text{ is bounded and } \Phi'_\lambda(u_n)\to 0 \text{ in } X'. \tag{3.4}$$

Then, by the coercivity of Φ_λ, the sequence (u_n) is bounded in X. By the reflexivity of X, for a subsequence still denoted (u_n), we have

$$u_n \rightharpoonup u \quad \text{in } W^{1,p(x)}(\Omega),\ u_n\to u \quad \text{in } L^{p(x)}(\partial\Omega) \text{ and } u_n\to u \quad \text{in } L^{q(x)}(\Omega).$$

Therefore

$$<\phi'_\lambda(u_n), u_n-u>\to 0 \quad \text{and} \quad \int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)d\sigma\to 0.$$

Thus

$$<A(u_n),u_n-u>:=\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx+\int_{\partial\Omega}\beta(x)|u_n|^{p(x)-2}u_n(u_n-u)d\sigma\to 0.$$

According to the fact that A satisfies condition (S^+) (see [27]), we have $u_n\to u$ in $W^{1,p(x)}(\Omega)$. The proof is complete. ■

Lemma 3.4. *For each $n\in\mathbb{N}^*$, there exists an $H_n\in\Gamma_n$ such that*

$$\sup_{u\in H_n}\Phi_\lambda(u)<0.$$

Proof. Let $v_1,v_2,\dots,v_n\in C_0^\infty(\mathbb{R}^N)$ such that $\overline{\{x\in\Omega; v_i(x)\ne 0\}}\cap\overline{\{x\in\Omega; v_j(x)\ne 0\}}=\emptyset$ if $i\ne j$ and $\operatorname{meas}(\{x\in\Omega; v_i(x)\ne 0\})>0$ for $i,j\in\{1,2,\dots,n\}$. Take $F_n=\operatorname{span}\{v_1,v_2,\dots,v_n\}$, it is clear that $\dim F_n=n$ and

$$\int_\Omega |v(x)|^{q(x)}dx>0 \quad \text{for all } v\in F_n\setminus\{0\}.$$

Denote $S=\{v\in W^{1,p(x)}(\Omega):\ \|v\|_\beta=1\}$ and $H_n(t)=t(S\cap F_n)$ for $0<t\le 1$. Obviously, $\gamma(H_n(t))=n$, for all $t\in]0,1]$.

Now, we show that, for any $n \in \mathbb{N}^*$, there exists $t_n \in]0,1]$ such that

$$\sup_{u \in H_n(t_n)} \Phi_\lambda(u) < 0.$$

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned}
\sup_{u \in H_n(t)} \Phi_\lambda(u) \leq & \sup_{v \in S \cap F_n} \Phi_\lambda(tv) \\
&= \sup_{v \in S \cap F_n} \Big\{ \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla v(x)|^{p(x)} dx + \int_{\partial\Omega} \frac{\beta(x) t^{p(x)}}{p(x)} |v(x)|^{p(x)} d\sigma \\
&\quad - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)} |v(x)|^{q(x)}\, dx \Big\} \\
&\leq \sup_{v \in S \cap F_n} \Big\{ \frac{t^{p^-}}{p^-} \Big(\int_\Omega |\nabla v(x)|^{p(x)} dx + \int_{\partial\Omega} \beta(x) |v(x)|^{p(x)} d\sigma \Big) \\
&\quad - \frac{\lambda t^{q^+}}{q^+} \int_\Omega |v(x)|^{q(x)}\, dx \Big\} \\
&= \sup_{v \in S \cap F_n} \big\{ t^{p^-} \big(\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t^{p^- - q^+}} \int_\Omega |v(x)|^{q(x)}\, dx \big) \big\}.
\end{aligned}$$

Since $m := \min_{v \in S \cap F_n} \int_\Omega |v(x)|^{q(x)}\, dx > 0$, we may choose $t_n \in]0,1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t_n^{p^- - q^+}} m < 0.$$

This completes the proof. ■

Proof of Theorem 3.1. By Lemmas 3.3, 3.4 and Theorem 3.2, Φ_λ admits a sequence of nontrivial weak solutions $(u_n)_n$ such that for any n, we have

$$u_n \neq 0, \quad \Phi'_\lambda(u_n) = 0, \quad \Phi_\lambda(u_n) \leq 0, \quad \lim_n u_n = 0. \tag{3.5}$$

■

Theorem 3.5. *Let $p, q \in C_+(\overline{\Omega})$. If*

$$q^- < p^- \quad \text{and} \quad q^+ < p^*(x) \quad \text{for all } x \in \overline{\Omega}, \tag{3.6}$$

then there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem (1.1).

For applying Ekeland's variational principle. We start with two auxiliary results.

Lemma 3.6. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, a > 0$ such that $\Phi_\lambda(u) \geq a > 0$ for any $u \in X$ with $\|u\|_\beta = \rho$.*

Proof. Since $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, it follows that X is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant C_1 such that

$$|u|_{L^{q(x)}(\Omega)} \leq C_1 \|u\|_\beta, \quad \text{for all } u \in X. \tag{3.7}$$

Fix $\rho \in]0,1[$ such that $\rho < \frac{1}{C_1}$. Then relation (3.7) implies $|u|_{L^{q(x)}(\Omega)} < 1$, for all $u \in X$ with $\|u\|_\beta = \rho$. Thus,

$$\int_\Omega |u|^{q(x)} dx \leq |u|^{q^-}_{L^{q(x)}(\Omega)}, \quad \text{for all } u \in X \text{ with } \|u\|_\beta = \rho. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\int_\Omega |u|^{q(x)}dx \leq C_1^{q^-}\|u\|_\beta^{q^-}, \quad \text{for all } u \in X \text{ with } \|u\|_\beta = \rho. \tag{3.9}$$

Hence, from (3.9) we deduce that for any $u \in X$ with $\|u\|_\beta = \rho$, we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{p^+}\Big(\int_\Omega |\nabla u|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)}d\sigma\Big) - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx \\ &\geq \frac{1}{p^+}\|u\|_\beta^{p^+} - \frac{\lambda}{q^-}C_1^{q^-}\|u\|_\beta^{q^-} \\ &= \frac{1}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}C_1^{q^-}\rho^{q^-} \\ &= \rho^{q^-}(\frac{1}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-}C_1^{q^-}).\end{aligned}$$

Putting

$$\lambda^* = \frac{\rho^{p^+-q^-}}{2p^+}\frac{q^-}{c_1^{q^-}}, \tag{3.10}$$

for any $u \in X$ with $\|u\|_\beta = \rho$, there exists $a = \rho^{p^+}/(2p^+)$ such that

$$\Phi_\lambda(u) \geq a > 0.$$

This completes the proof. ■

Lemma 3.7. *There exists $\xi \in X$ such that $\xi \geq 0$, $\xi \neq 0$ and $\Phi_\lambda(t\xi) < 0$, for $t > 0$ small enough.*

Proof. Since $q^- < p^-$, there exists $\varepsilon_0 > 0$ such that

$$q^- + \varepsilon_0 < p^-.$$

Since $q \in C(\overline{\Omega})$, there exists an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| < \varepsilon_0, \quad \text{for all } \ x \in \Omega_0.$$

Thus, we deduce

$$q(x) \leq q^- + \varepsilon_0 < p^-, \quad \text{for all } x \in \Omega_0. \tag{3.11}$$

Take $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\overline{\Omega_0} \subset \operatorname{supp}\xi$, $\xi(x) = 1$ for $x \in \overline{\Omega_0}$ and $0 \leq \xi \leq 1$ in Ω. Without loss of generality, we may assume $\|\xi\|_\beta = 1$, that is

$$\int_\Omega |\nabla\xi|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|\xi|^{p(x)}d\sigma = 1. \tag{3.12}$$

By using (3.11), (3.12) and the fact

$$\int_{\Omega_0} |\xi|^{q(x)}dx = \text{meas}(\Omega_0),$$

for all $t \in]0,1[$, we obtain

$$\begin{aligned}
\Phi_\lambda(t\xi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\xi|^{p(x)}dx + \int_{\partial\Omega} \frac{t^{p(x)}\beta(x)}{p(x)}|\xi|^{p(x)}\,d\sigma - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\xi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-}\Big(\int_\Omega |\nabla\xi|^{p(x)}dx + \int_{\partial\Omega}\beta(x)|\xi|^{p(x)}\,d\sigma\Big) - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\xi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|\xi|^{q(x)}dx \\
&\le \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^-+\varepsilon_0}}{q^+}\,\mathrm{meas}(\Omega_0).
\end{aligned}$$

Then, for any $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$, with $0 < \delta < \min\{1, \lambda p^-\,\mathrm{meas}(\Omega_0)/q^+\}$, we conclude that

$$\Phi_\lambda(t\xi) < 0.$$

The proof is complete. ■

Proof of Theorem 3.5. By Lemma 3.6, we have

$$\inf_{\partial B_\rho(0)} \Phi_\lambda > 0, \tag{3.13}$$

where $\partial B_\rho(0) = \{u \in X;\ \|u\|_\beta = \rho\}$.
On the other hand, from Lemma 3.7, there exists $\xi \in X$ such that $\Phi_\lambda(t\xi) < 0$ for $t > 0$ small enough. Using (3.9), it follows that

$$\Phi_\lambda(u) \ge \frac{1}{p^+}\|u\|_\beta^{p^+} - \frac{\lambda}{q^-}C_1^{q^-}\|u\|_\beta^{q^-} \quad \text{for } u \in B_\rho(0).$$

Thus,

$$-\infty < \underline{c}_\lambda := \inf_{\overline{B_\rho(o)}} \Phi_\lambda < 0,$$

Let

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda - \inf_{\overline{B_\rho(0)}} \Phi_\lambda.$$

Then, by applying Ekeland's variational principle to the functional

$$\Phi_\lambda : \overline{B_\rho(0)} \to \mathbb{R},$$

there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned}
&\Phi_\lambda(u_\varepsilon) \le \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon, \\
&\Phi_\lambda(u_\varepsilon) < \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_\beta \ \text{ for } u \ne u_\varepsilon.
\end{aligned}$$

Since $\Phi_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda$, we deduce $u_\varepsilon \in B_\rho(0)$.
Now, define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_\beta.$$

It is clear that u_ε is an minimum of I_λ. Therefore, for $t > 0$ and $v \in B_1(0)$, we have

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \ge 0$$

for $t > 0$ small enough and $v \in B_1(0)$; that is,

$$\frac{\Phi_\lambda(u_\varepsilon + tv) - \Phi_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\|_\beta \geq 0$$

for t positive and small enough, and $v \in B_1(0)$. As $t \to 0$, we obtain

$$\langle \Phi'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\|_\beta \geq 0 \quad \text{for all } v \in B_1(0).$$

Hence, $\|\Phi'_\lambda(u_\varepsilon)\|_{X'} \leq \varepsilon$. We deduce that there exists a sequence $(u_n)_n \subset B_\rho(0)$ such that

$$\Phi_\lambda(u_n) \to \underline{c}_\lambda \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0. \tag{3.14}$$

It is clear that (u_n) is bounded in X. By a standard arguments and the fact A is type of (S^+), for a subsequence we obtain $u_n \to u$ in X as $n \to +\infty$. Thus, by (3.14) we have

$$\Phi_\lambda(u) = \underline{c}_\lambda < 0 \quad \text{and} \quad \Phi'_\lambda(u) = 0 \quad \text{as } n \to \infty. \tag{3.15}$$

The proof is complete. ■

Theorem 3.8. *Let $p, q \in C_+(\overline{\Omega})$. If*

$$p^+ < q^- \leq q^+ < p^*(x) \quad \textit{for all } x \in \overline{\Omega}, \tag{3.16}$$

then for any $\lambda > 0$, problem (1.1) possesses a nontrivial weak solution.

We want to construct a mountain geometry, and first need two lemmas.

Lemma 3.9. *There exist $\eta, b > 0$ such that $\Phi_\lambda(u) \geq b$, for $u \in X$ with $\|u\| = \eta$.*

Proof. Since $q^+ < p^*(x)$, in view the Theorem 3.2, there exist $M_1, M_2 > 0$ such that

$$|u|_{L^{q^+}(\Omega)} \leq M_1\|u\| \quad \text{and} \quad |u|_{L^{q^-}(\Omega)} \leq M_2\|u\|.$$

Thus, from (3.2) we obtain

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+}\Big(\int_\Omega |\nabla u(x)|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u(x)|^{p(x)}d\sigma\Big) \\
&\quad - \frac{\lambda}{q^-}\Big[(M_1\|u\|)^{q^+} + (M_2\|u\|)^{q^-}\Big] \\
&\geq \frac{1}{p^+}g(\|u\|_\beta) - \frac{\lambda M_1^{q^+}}{q^-}\|u\|^{q^+} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^-} \\
&\geq \frac{C_1}{p^+}g(\|u\|) - \frac{\lambda M_1^{q^+}}{q^-}\|u\|^{q^+} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^-} \\
&= \begin{cases} \big(\frac{C_1}{p^+} - \frac{\lambda M_1^{q^+}}{q^-}\|u\|^{q^+-p^+} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^--p^+}\big)\|u\|^{p^+} & \text{if } \|u\| \leq 1, \\ \big(\frac{C_1}{p^+} - \frac{\lambda M_1^{q^+}}{q^-}\|u\|^{q^+-p^-} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^--p^-}\big)\|u\|^{p^-} & \text{if } \|u\| > 1, \end{cases}
\end{aligned}$$

where C_1 is a positive constant. Since $p^+ < q^- \leq q^+$, the functional $h : [0,1] \to \mathbb{R}$ defined by

$$h(s) = \frac{C_1}{p^+} - \frac{\lambda M_1^{q^+}}{q^-}s^{q^+-p^+} - \frac{\lambda M_2^{q^-}}{q^-}s^{q^--p^+}$$

is positive on neighborhood of the origin. So, the result of Lemma 3.9 follows. ■

Lemma 3.10. *There exists $e \in X$ with $\|e\| \geq \eta$ such that $\Phi_\lambda(e) < 0$, where η is given in Lemma 3.9.*

Proof. Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$. For $t > 1$, we have

$$\Phi_\lambda(t\varphi) \leq \frac{t^{p^+}}{p^-}\Big(\int_\Omega |\nabla\varphi(x)|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|\varphi(x)|^{p(x)}d\sigma\Big) - \frac{\lambda t^{q^-}}{q^+}\int_\Omega |\varphi(x)|^{q(x)}dx.$$

Then, since $p^+ < q^-$, we deduce that

$$\lim_{t\to\infty} \Phi_\lambda(t\varphi) = -\infty.$$

Therefore, for $t > 1$ large enough, there is $e = t\varphi$ such that $\|e\| \geq \eta$ and $\Phi_\lambda(e) < 0$. This completes the proof. ■

Lemma 3.11. *Let $p, q \in C_+(\overline{\Omega})$. Assume that $p^+ < q^-$. Then the functional Φ_λ satisfies the condition (PS).*

Proof. Let $(u_n) \subset X$ be a sequence such that $M := \sup_n \Phi_\lambda(u_n) < \infty$ and $\Phi'_\lambda(u_n) \to 0$ in X'. By contradiction suppose that

$$\|u_n\|_\beta \to +\infty \text{ as } n \to \infty \quad \text{and} \quad \|u_n\|_\beta > 1 \quad \text{for any } n.$$

Thus,

$$\begin{aligned}
M + 1 + \|u_n\|_\beta &\geq \Phi_\lambda(u_n) - \frac{1}{q^-}\langle \Phi'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}d\sigma \\
&\quad - \frac{1}{q^-}\Big(\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}d\sigma\Big) \\
&\quad + \lambda\int_\Omega (\frac{1}{q^-} - \frac{1}{q(x)})|u_n|^{q(x)}\,dx \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{q^-}\Big)\Big(\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \beta(x)|u_n|^{p(x)}d\sigma\Big) \\
&\geq (\frac{1}{p^+} - \frac{1}{q^-})\|u_n\|_\beta^{p^-}.
\end{aligned}$$

Since $p^+ < q^-$, this contradicts the fact that $p^- > 1$. So, the sequence (u_n) is bounded in X and similar arguments as those used in the proof of Lemma 3.4 completes the proof. ■

Proof of Theorem 3.8. From Lemmas 3.9 and 3.10, we deduce

$$\max(\Phi_\lambda(0), \Phi_\lambda(e)) = \Phi_\lambda(0) < \inf_{\|u\|=\eta} \Phi_\lambda(u) =: \alpha.$$

By Lemma 3.11 and the mountain pass theorem, we deduce the existence of critical points u of Φ_λ associated of the critical value given by

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \geq \alpha, \tag{3.17}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. This completes the proof. ■

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