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Common coincidence points of R-weakly commuting fuzzy maps

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Abstract : The theory of fuzzy sets was introduced by Zadeh [6]. Helprin [2]first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy mappings. Since then, a number of fixed point results for fuzzy mappings have been obtained by several anthers. In this paper, the notion of R-weakly commutativity is for the pair of self mapping has been established. Then this is used as a tool for proving a common coincidence point theorem for fuzzy mappings and self mappings on metric space .These result are in continuation of [4].

Keywords : Fuzzy sets, Fuzzy mappings, R-weakly commutative fuzzy maps, Fixed points.
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1 Introduction

After the introduction of fuzzy sets by Zadeh [6], Heilpern [2] introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy mappings. Since then, a number of fixed point results have be obtained. In this series, recently Rashwan & Ahmed [5] proved a common fixed point theorem for a pair of fuzzy mappings. In this paper, first the coincidence point of a crisp mapping and a fuzzy mapping has been defined. Then R-weakly commutativity is introduced for a pair of crisp mapping & a fuzzy mapping. At last, a common coincidence points theorem has been proved for the combinations of crisp mappings & fuzzy mappings together using the notion of R-weakly commuting mappings.

2 Preliminaries

Here we cite briefly some definition, lemmas and propositions noted in [5]. Let (X, d) be a metric linear space. A fuzzy set in X is a function with domain X and values in [0, 1]. If A if fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of x in A. The α-level set of A denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \forall \alpha \in (0,1] \quad (2.1)$$

$$A_0 = \overline{\{x : A(x) > 0\}}. \quad (2.2)$$

Where $\overline{B}$ denotes the closure of the set B.

Definition 2.1: A fuzzy set A in X is said to be an approximate quantity iff A_α is compact and convex in X for each $\alpha \in [0,1]$ and $sup_{x\in X} A(x) = 1$.

Let $F(X)$ be the collection of all fuzzy sets in X and $W(X)$ be a sub-collection of all approximate quantities.

Definition 2.2: Let $A, B \in W(X), \alpha \in [0,1]$. Then

$$p_\alpha(A,B) = inf_{x\in A_\alpha, y\in B_\alpha} d(x,y)$$

Definition 2.3: Let $A, B \in W(X)$. Then A is said to be more accurate than B (or B includes A), denoted by $A \subseteq B$ iff $A(x) \leq B(x)$ for each $x \in X$.

A fuzzy mapping F is a fuzzy subset on $X \times Y$ with membership function $F(x)(y)$. The function $F(x)(y)$ is the grade of membership of y in Fx.

Lemma 2.1[2]: Let $x \in X$. $A \in W(X)$ and $\{x\}$ be a fuzzy set with membership function equal to characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.

Lemma 2.2[2]: $p_\alpha(x, A) \leq d(x,y) + p_\alpha(y, A)$ for any $x, y \in X$.

Lemma 2.3[2]: If $\{x_0\} \subset A$ then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W(X)$.

Proposition 2.1 [3] : Let (X, d) be a complete metric linear space and $F : X \to W(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.

Remark 2.1: Let $J : X \to X$ and $F : X \to W(X)$ such that $\cup FX_\alpha \subseteq J(X)$ for each $\alpha \in [0,1]$. Suppose $J(X)$ is complete. Then, by an application of Proposition (2.1), it can be easily shown that for any choosen point $x_0 \in X$ there exists a point $x_1 \in X$ such that $\{Jx_1\} \subseteq Fx_0$.

Proposition 2.2 [4] :If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $(a, b) \leq H(A, B)$.

Recently Rashwan and Ahmad [5] introduced the set G of all continuous functions $g : [0,\infty)^5 \to [0,\infty)$ with the following properties :

(i) g is non decreasing in $2^{nd}, 3^{rd}, 4^{th}$ and 5^{th} variables.

(ii) If $u, v \in [0, \infty)$ are such that $u \leq g(v, v, u, u+v, 0)$ or $u \leq g(v, u, v, 0, u+v)$ then $u \leq hv$ where $0 < h < 1$ is a given constant.

(iii) If $u \in [0, \infty)$ is such that $u \leq g(u, 0, 0, u, u)$ then $u = 0$.

Then Rashwan and Ahmad [5] proved the following theorem.

Theorem 2.1 : Let X be a complete metric linear space and let F_1 and F_2 be fuzzy mappings from X into $W(X)$. If there is a $g \in G$ such that for all $x, y \in X$

$$D(F_1x, F_2y) \leq g(d(x,y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_2(y)), p(y, F_1(x)))$$

then there exists $z \in X$ such that $\{z\} \subseteq F_1(z)$ and $\{z\} \subseteq F_2(z)$.

In this paper, first we extend the concept of R-weakly commuting mappings to the setting of single valued (crisp) mapping and fuzzy mapping and then give examples of fuzzy mappings and crisp mappings, which are R-weakly commuting and not. Then we study the structure of common coincidence point theorem for two pairs of mappings, generalizing Theorem 2.1.

3 MAIN RESULT

First of all, we introduce the following definitions and examples :

Definition 3.1 : Let $I : X \to X$ be a self mapping and $F : X \to W(X)$ a fuzzy mapping. Then a point $u \in X$ is said to be coincidence point of I and F. If $Iu \subseteq Fu$ that is, $Iu \in Fu_1$.

Definition 3.2 : The mappings $I : X \to X$ and $F : X \to W(X)$ are said to be R-weakly commuting if for all x in X, $IFx_\alpha \in CPX$ and there exists a positive number R such that

$$H(IFx_\alpha, FIx_\alpha) \leq Rd(Ix, Fx_\alpha), \forall[0,1]. \tag{3.1}$$

Example 3.1 : Let (X, d) be a metric space where $X = [0,1]$ and d denote usual metric. Define mapping $I : X \to X$ such that $Ix = \frac{x}{2}$ for all $x \in X$ and $F : X \to W(X)$ a fuzzy mapping such that for all $x \in [0,1]$, Fx is a fuzzy set on X given by, for all $x, y \in [0,1]$,

$$F(x)(y) = \begin{cases} 0 & \text{if } 0 \leq y < \frac{x+1}{2} \\ \frac{y-x}{1-x} & \text{if } \frac{x+1}{2} \leq y \leq 1 \end{cases}$$

When $0 \leq \alpha < \frac{1}{2}$ then $\{Fx\}_\alpha = [\frac{1}{2}(1+x), 1], \{Ix\} = \{\frac{x}{2}\}$,

$$\{FIx\}_\alpha = \{f(\frac{x}{2})\}_\alpha = [\frac{1}{2}(1 + x/2), 1]$$

$$I\{Fx\}_\alpha = [\frac{1}{4}(1+x), \frac{1}{2}]$$

and so, $H(I\{Fx\}_\alpha, \{FIx\}_\alpha) = \text{Max}\{\mid \frac{1}{2}(1+\frac{x}{2}) - \frac{1}{4}(1+x) \mid, \mid 1-\frac{1}{2} \mid\} = \frac{1}{2}$ and $d(Ix, \{Fx\}) = \frac{1}{2}$.

When $\frac{1}{2} \leq \alpha \leq 1$ then $\{Fx\}_\alpha = [x+\alpha(1-x), 1], I\{Fx\}_\alpha = [\frac{x}{2} + \frac{\alpha}{2}(1-x), \frac{1}{2}]$ and $\{FIx\}_\alpha = [\frac{x}{2} + \alpha(1-\frac{x}{2}), 1]$.

Now we have

$$\begin{aligned} H(I\{Fx\}_\alpha, \{FIx\} - \alpha) &= \text{Max}\{\mid (\frac{x}{2} + \alpha(1-\frac{x}{2})) - (\frac{x}{2} + \frac{\alpha}{2}(1-x)) \mid, \frac{1}{2}\} \\ &= \text{Max}\{\frac{\alpha}{2}, \frac{1}{2}\} = \frac{1}{2} \qquad (3.2) \end{aligned}$$

and

$$d(Ix, \{Fx\}_\alpha) = \frac{x}{2} + \alpha(1+x) \geq \frac{x}{2} + \frac{1}{2}(1-x) = \frac{1}{2}\{\frac{1}{2} \leq \alpha \leq 1\}$$

Hence for $R = 1$, we have $H(I\{Fx\}_\alpha, \{FIx\}_\alpha) \leq Rd(Ix, \{Fx\}_\alpha)$ and so $\{F, I\}$ are R-weakly commuting.

Example 3.2: Let $I : X \to X$ be such that $Ix = \frac{x}{2}$ and $F : X \to W(X)$ be defined as

$$F(x)(y) = \begin{cases} 0 & \text{if } 0 \leq y < x \\ \frac{y-x}{1-x} & \text{if } x \leq y \leq 1. \end{cases}$$

Now, we have for all $0 \leq \alpha \leq 1$,

$$\{Fx\}_\alpha = [x+\alpha(1-x), 1]$$

$$I\{Fx\}_\alpha = [\frac{x}{2} + \frac{\alpha}{2}(1-x), \frac{1}{2}]$$

and

$$\{FIx\}_\alpha = [\frac{x}{2} + \alpha(1-\frac{x}{2}), 1].$$

then similarly as in Example 3.1, we get $H(I\{Fx\}_\alpha, \{FIx\} - \alpha) = \frac{1}{2}$.

But $d(ix, \{Fx\} - \alpha) = \frac{x}{2} + \alpha(1-x)$ which can be made as small as possible by taking α and x very small. Thus no $R > 0$ can serve the purpose. Hence F and I are not R-weakly commuting.

Now, we prove our main theorem as follows

Theorem 3.1: Let I, J be mapping of a metric space X into itself and let $F_1, F_2 : X \to W(X)$ be fuzzy mappings such that

(iv) $(a) \cup F_1 X_\alpha \subset J(X)$
$(b) \cup F_2 X_\alpha \subset I(X)$ for each $\alpha \in [0,1]$,

(v) there is a $g \in G$ such that for all $x, y \in X$
$D(F_1x, F_2y) \leq g(d(Ix, Jy), p(Ix, F_1x), p(Jy, F_2y), p(Ix, F_2y), p(Jy, F_1x))$.

(vi) the pairs $\{F_1, I\}$ and $\{F_2, J\}$ are R - weakly commuting.

Suppose that one of $I(X)$ or $J(X)$ is complete, then there exists $z \in X$ such that $Iz \subseteq F_1z$ and $Jz \subseteq F_2z$.

Proof: Let $x_0 \in X$ and suppose that $J(X)$ is complete. Taking $y_0 = Ix_0$. Then by Remark (2.1) and (iv)(a) there exists point $x_1, y_1 \in X$ such that $\{y_1\} = \{Jx_1\} \subseteq F_1x_0$. For this point y_1, by proposition (2.1), there exists a point $y_2 \in F_2x_{11}$. But, by (iv), (b) there exists $x_2 \in X$ such that $\{y_2\} = \{Ix_2\} \subseteq F_2x_1$. Now by proposition (2.2) and condition (v), we obtain

$$d(y_1, y_2) \leq D_1(F_1x_0, F_2x_1) \leq D(F_[1x_0, F_2x_1)$$

$$\leq g(d(Ix_0, Jx_1), p(Ix_0, F_1x_0), p(Jx_1, F_2x_1), p(Ix0, F_2x_1), p(Jx_1, F_1x_0))$$

$$\leq g(d(y_0, y_1), d(y_0, y_1), d(y_1, y_2), d(y_0, y_1) + d(y_1, y_2), 0) \quad (3.3)$$

which, by (ii) gives $d(y_[1, y_2) \leq hd(y_0, y_1)$. Since $F_2x_{11}, F_1x_{21} \in CP(X)$ and $y_2 = Ix_2 \in \{F_2x_1\}_1$ therefore, by proposition (2.2), there exists $y_3 \in \{F_1x_2\}_1 \subseteq J(X)$ and hence there exists $x_3 \in X$ such that $\{y_3\} = \{Jx_3\} \subseteq \{F_1x_2\}_1$. Again $d(y_2, y_3) \leq hd(y_1, y_2)$. Thus, by repeating application of Proposition (2.2) and (iv) (a) - (b), we construct a sequence $\{yk\}$ in X such that, for each $k = 0, 1, 2, ...$
$\{y_{2k+1}\} = \{Jx_{2k+1} \subseteq F_1(x_{2k})$ and $\{y_{2k+2}\} = Ix_{2k+2}F_2(x_{2k+1})$ and $d(y_k, y_{k+1}) \leq hd(y_{k-1}, y_k)$. Then, as in proof of Theorem 3.1 in [5], the sequence $\{yk\}$, and hence any subsequence thereof, is Cauchy. Since $J(X)$ is complete. Then $Jx_{2k+1} \to z = Jv$ for some $v \in X$. Then

$$d(Ix_{2k}, Jv) \leq d(Ix_{2k}, Jx_{2k+1}) + d(jx_{2k+1, Jv) \to 0 as k \to \infty}.$$

Hence $Ix_{2k} \to Jv$ as $k \to \infty$.

Now, by Lemma (2.2), Lemma (2.3) and condition (v)

$$\begin{aligned} p(z, F_2v) \quad \leq \quad & d(z, Jx_{2k+1}) + D(F_1x_{2k}, F_2v) \\ & \leq d(z, Jx_{2k+1}) + g(d(Ix_{2k}, Jv), p(Ix_{2k}, F_1x_{2k}), p(Jv, F_2v), p(Ix_{2k}, F_2v), p(Jv, F_1x_{2k})) \\ & \leq d(z, Jx_{2k+1}) + g(d(Ix_{2k}, z), p(y_{2k}, y_{2k+1}), p(z, F_2v), p(Ix_{2k}, F_2v), \\ & d(z, y_{2k+1})) \end{aligned} \quad (3.4)$$

letting $k \to \infty$ it implies

$$p(z, F_2v) \leq g(0, 0, p(z, F_2v), p(z, F_2v), 0).$$

which, by (ii), yields that $p(z, F_2v) = 0$. So by Lemma (11.2.1) we get $\{z\} \subseteq F_2v$ i.e. $Jv \in \{F_2v\}_1$.

Since by (iv)(b), $\{F_2(X)\}_1 \subseteq I(X)$ and $Jv \in \{F_2v\}_1$ therefore there is a point $u \in X$ such that

$$Iu = Jv = z \in \{F_2v\}_1.$$

Now, by Lemma 2.3, we have

$$\begin{aligned} p(Iu, F_1u) &= p(F_1u, Iu) \leq D_1(F_1u, F_2v) \leq D(F_1u, F_2v) \\ &\leq g(d(Iu, Jv), p(Iu, F_1u), p(Jv, F_2v), p(Iu, F_2v), p(Jv, F_1u)) \end{aligned} \quad (3.5)$$

yielding thereby

$$p(Iu, F_1u) \leq g(0, p(Iu, F_1u), 0, 0, p(Iu, F_1u))$$

which, by (ii), gives $p(Iu, F_1u) = 0$. Thus, by Lemma (2.1), $Iu \subseteq F_1u$ i.e. $Iu \in \{F_1u\}_1$. Now, by R-weakly commutativity of pairs $\{F_1, I\}$ and $\{F_2, J\}$, we have

$$H(I\{F_1u\}_1, \{F_1Iu\}_1) \leq Rd(Iu, \{F_1u\}_1) = 0$$

$$H(J\{F_2v\}_1, \{F_2Jv\}_1) \leq Rd(Jv, \{F_2v\}_1) = 0$$

which gives $I\{F_1u\}_1 = \{F_1Iu\}_1 = \{F_1z\}$, and $J\{F_1v\}_1 = \{F_2Jv\}_1 = \{F_2z\}_1$ respectively. But $Iu \in \{F_1u\}_1$ and $Jv \in \{F_2v\}_1$ implies

$$Iz = IIu \in I\{F_1u\}_1 = \{F_1z\}_1.$$

$$Jz = JJv \in J\{F_2v\}_1 = \{F_2z\}_1.$$

Hence $Iz \subseteq F_1z$ and $Jz \subseteq F_2z$. Thus the theorem completes.

Remark 3.1 : If $J(X)$ is complete, then in Theorem (3.1), it is sufficient that (iv)(b) holds only for $\alpha = 1$. Similarly if $I(X)$ is complete then (iv)(a) for $\alpha = 1$ is sufficient to consider.

Corollary 3.1 : Theorem 3.1.

Proof : Taking $I = J =$ identity in Theorem 3.1.

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