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# The Endospectrum of $(n-3)$-Regular Graphs of Order $n$

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**Abstract** Let $G$ be a graph, the endomorphism spectrum or simply called for short by endospectrum of $G$ is the 6-tuple of 6 types of endomorphisms on $G$ the following cardinalities

$$\text{Endspec } G = (|End(G)|, |HEnd(G)|, |LEnd(G)|, |QEnd(G)|, |SEnd(G)|, |Aut(G)|).$$

In this paper, we find the endospectrum of an $(n-3)$-regular graph of order $n$.

**MSC:** 05C25; 05C60

**Keywords:** endomorphism spectrum; $(n-3)$-regular graph of order $n$; join product; complement of cycle

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## 1. Introduction and Preliminaries

The endomorphism spectrum and the endomorphism type of graph was defined by Knauer and Böttcher. In 1992 and resp. 2003 [1, 2], Knauer and Böttcher, proved that there is graph $G$ with Endotype $G = x$, when given $x; x \in \{0, \ldots, 31\} \setminus \{1, 17\}$. Furthermore, on the sufficiency, who found the results of endospectrum and endotype of some family of graph as following. In 2001 [3], Fan found the results on bipartite graph with diameter 3 and girth 6. In 2008 [4], Hou, Luo and Cheng found the results on complement of path. In 2009 [5], Hou, Fan and Luo found the results on generalized polygons. In 2011 [6], Wang and Hou found the results on $n$-prism graph. In 2014 [7], Pipattanajinda found the endotype of $(n-3)$-regular graphs of order $n$. The results for the number of endomorphisms of path, cycle, cycle complement, generalized wheel graphs, and the number of locally strong endomorphisms of paths, see [8–12] and [13], respectively. Further, see [14], the result of the endomorphisms monoids of graphs of order $n$ with a minimum degree $n-3$. The results of the endomorphisms monoids and the endotype in [7, 14, 15], gives the interesting to make the absoluteness of the results of endospectrum on the $(n-3)$-regular graphs of order $n$.

Consider finite simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \to V(G)$ be a mapping. We recall the 6 types of endomorphisms of a graph $G$.

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First, the mapping $f$ is said to be an *endomorphism* if $f$ preserves edges, i.e. $\{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(G)$. Further, the endomorphism $f$ is said to be a *half strong endomorphism* if $\{f(u), f(v)\} \in E(G)$ implies that there exists $x \in f^{-1}(f(u))$, the preimages of $f(u)$, and $y \in f^{-1}(f(v))$, the preimages of $f(v)$, such that $\{x, y\} \in E(G)$. The endomorphism $f$ is said to be a *locally strong endomorphism* if $\{f(u), f(v)\} \in E(G)$ implies for each $x \in f^{-1}(f(u))$ that there exists $y \in f^{-1}(f(v))$ such that $\{x, y\} \in E(G)$, and analogously for each $y \in f^{-1}(f(v))$. The endomorphism $f$ is said to be a *quasi strong endomorphism* if $\{f(u), f(v)\} \in E(G)$ implies that there exists $x \in f^{-1}(f(u))$ such that $\{x, y\} \in E(G)$ for all $y \in f^{-1}(f(v))$, and analogously for preimages of $f(v)$. The endomorphism $f$ is said to be a *strong endomorphism* if $\{f(u), f(v)\} \in E(G)$ implies $\{x, y\} \in E(G)$, for all $x \in f^{-1}(f(u))$ and $y \in f^{-1}(f(v))$. Finally, the endomorphism $f$ is said to be an *automorphism* if $f$ is bijective and $f^{-1}$ is an endomorphism.

In this paper we use the following notations:

- $End(G)$, the set of all endomorphisms of $G$,
- $HEnd(G)$, the set of all half strong endomorphisms of $G$,
- $LEnd(G)$, the set of all locally strong endomorphisms of $G$,
- $QEnd(G)$, the set of all quasi strong endomorphisms of $G$,
- $SEnd(G)$, the set of all strong endomorphisms of $G$, and
- $Aut(G)$, the set of all automorphisms of $G$.

It is clear that, $End(G) \supseteq HEnd(G) \supseteq LEnd(G) \supseteq QEnd(G) \supseteq SEnd(G) \supseteq Aut(G)$. With this sequence, we associate the sequence of respective cardinalities by

$$\text{Endspec } G = (|End(G)|, |HEnd(G)|, |LEnd(G)|, |QEnd(G)|, |SEnd(G)|, |Aut(G)|)$$

and call this 6-tuple the *endomorphism spectrum* or *endospectrum* of $G$.

We associate with the above sequence a 5-tuple $(s_1, s_2, s_3, s_4, s_5)$ with $s_i \in \{0, 1\}, i = 1, 2, 3, 4, 5$, where 1 stands for $\neq$ and 0 stands for $=$ at the respective position in the above sequence, i.e. $s_1 = 1$ indicates that $End(G) \neq HEnd(G)$ etc. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is called the *endomorphism type* or *endotype* of graph $G$ and is denoted by Endotype $G$.

Let $G$ be a graph. The number of vertices of $G$ is often called the *order* of $G$. The *degree* of a vertex $u$ in a graph $G$ is the number of vertices adjacent to $u$ and is denoted by $d_G(u)$ or simply by $d(u)$ if the graph $G$ is clear from the context. If $d(u) = r$ for every vertex $u$ of $G$, where $0 \leq r \leq n-1$, then $G$ is called a $r$*-regular*. The *complement (graph)* $\overline{G}$ of $G$ is a graph such that $V(\overline{G}) = V(G)$ and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$ for any $u, v \in V(G), u \neq v$. A subgraph $H$ of $G$ is called an *induced subgraph*, if for any $u, v \in V(H), \{u, v\} \in E(G)$ implies $\{u, v\} \in E(H)$. Let $G$ and $H$ be two graphs. The *join* of $G$ and $H$, denoted by $G + H$, is a graph such that $V(G+H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} | u \in V(G), v \in V(H)\}$. The graph with vertex set $\{1, \ldots, n\}$, such that $n \geq 3$, and edge set $\{\{i, i+1\} | i = 1, \ldots, n\} \cup \{1, n\}$ is called a *cycle* $C_n$.

Some results on the $(n-3)$-regular graph of order $n$.

In [15], N. Pipattanajinda, U. Knauer, B. Gyurov and S. Panma investigated $(n-3)$-regular graph of order $n$.

**Lemma 1.1** ([15]). *Let $G$ be a graph of order $n \geq 3$. Then $G$ is an $(n-3)$-regular graph if and only if $G = \overset{s}{\underset{i=1}{+}} \overline{C}_{n_i}$ where $n = n_1 + \ldots + n_s$ and $s \geq 1$. In particular $s > 1$ implies $n \geq 6$.*

Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_s)_s}$ be an $(n-3)$-regular graph of order $n$. Sets $O_i = \{1_i, 3_i, \ldots, (2m_i - 1)_i\}$ and $E_i = \{2_i, 4_i, \ldots, (2m_i)_i\}$. Denote by $S_{X_1,X_2,\cdots,X_s} = X_1 \cup X_2 \cup \ldots \cup X_s$ where $X_i \in \{O_i, E_i\}$, $1 \leq i \leq s$. Further, let $f \in End(G)$ and $G_1$ is an induced subgraph of $G$. Denote the set of all elements $f(x)$ where $x \in V(G_1)$ by $f(G_1)$, and the restriction of $f$ on $G_1$ by $f|_{G_1}$.

**Lemma 1.2** ([15]). *Let $G = \overline{C}_{(2m_1)_1} + \ldots + \overline{C}_{(2m_s)_s}$ and $f : V(G) \to V(G)$. Then $f \in End(G)$ if and only if $f$ satisfies:*

*(1) If $f(x_i) = f(y_i)$ for some two different elements $x_i, y_i \in V(\overline{C}_{n_i})$, then $y_i = (x-1)_i$ or $y_i = (x+1)_i$.*

*(2) $f(S_{O_1,\cdots,O_s}) = X_1 \cup \ldots \cup X_s$ and $f(S_{E_1,\cdots,E_s}) = Y_1 \cup \ldots \cup Y_s$ where $X_i, Y_i \in \{O_i, E_i\}$, $1 \leq i \leq s$.*

*(3) If $O_i, E_i \in f(G)$ for some $1 \leq i \leq s$, then $f(\overline{C}_{(2m_j)_j}) = \overline{C}_{(2m_i)_i}$ and $m_j = m_i$ for some $1 \leq j \leq s$ such that $f|_{\overline{C}_{(2m_j)_j}}$ is an isomorphism from $\overline{C}_{(2m_j)_j}$ to $\overline{C}_{(2m_i)_i}$.*

*(4) If $f(x_i) = f((x+1)_i)$ for some $x_i \in V(\overline{C}_{2m_i})$, then*

*(4.1) $f(1_i) = f(2_i), f(3_i) = f(4_i), \ldots, f((2m_i - 1)_i) = f((2m_i)_i)$, if $x$ is odd, or*

*(4.2) $f((2m_i)_i) = f(1_i), f(2_i) = f(3_i), \ldots, f((2m_i - 2)_i) = f((2m_i - 1)_i)$, if $x$ is even.*

**Lemma 1.3** ([15]). *Let $G$ be an $(n-3)$-regular graph of order $n$. Denote by $G_x$ and $G_E$ set of all induced subgraphs $\overline{C}_x$ and $\overline{C}_{2m}$ of $G$, respectively (note that $G_x = \emptyset$, if $G$ does not contain an induced subgraph $\overline{C}_x$). Then*

*(1) $|End(G)| = |End(G_E)| \times |End(G_3)| \times |End(G_5)| \times |End(G_7)| \times \ldots$, if $G_x \neq \emptyset$, for all $x = 3, 5, 7, \ldots$,*

*(2) $End(G_3) \cong S_{m_1} \times T_3$, where $|G_3| = m_1$, and*

*(3) for each odd integer $x \geq 5$, $End(G_x) = Aut(G_x) \cong S_{m_2} \times D_x$, where $|G_x| = m_2$.*

**Example 1.4.** Let $G = \overline{C}_{4_1} + \overline{C}_{4_2} + \overline{C}_{6_3}$, the 11-regular graph of order 14, and $f : V(G) \to V(G)$ such that

$$f = \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 1_2 & 2_2 & 3_2 & 4_2 & 1_3 & 2_3 & 3_3 & 4_3 & 5_3 & 6_3 \\ 3_2 & 3_2 & 4_3 & 4_3 & 3_1 & 2_1 & 1_1 & 4_1 & 6_3 & 1_2 & 1_2 & 2_3 & 2_3 & 6_3 \end{pmatrix}.$$

Since for each $x, y \in V(G), \{x, y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(G)$, $f$ is an endomorphism.

Furthermore, from Lemma 1.2, $f$ satisfies (1), since if $f(x) = f(y)$ then $y = x - 1$ or $y = x + 1$.

$f$ satisfies (2), because $S_{O_1,O_2,O_3} = \{1_1, 3_1, 1_2, 3_2, 1_3, 3_3, 5_3\}$ and $S_{E_1,E_2,E_3} = \{2_1, 4_1, 2_2, 4_2, 2_3, 4_3, 6_3\}$ such that $f(S_{O_1,O_2,O_3}) = \{3_2, 4_3, 3_1, 1_1, 6_3, 1_2, 2_3\} = O_1 \cup O_2 \cup E_3$ and $f(S_{E_1,E_2,E_3}) = \{3_2, 4_3, 2_1, 4_1, 1_2, 2_3, 6_3\} = E_1 \cup O_2 \cup E_3$.

$f$ satisfies (3), because $O_1, E_1 \in f(G)$ and there exists $\overline{C}_{4_2}$ such that $f(\overline{C}_{4_2}) = \overline{C}_{4_1}$ and $f|_{\overline{C}_{4_2}}$ is an isomorphism from $\overline{C}_{4_2}$ to $\overline{C}_{4_1}$.

Finally, $f$ satisfies (4), because $f(1_1) = f(2_1), f(3_1) = f(4_1)$ and $f(6_3) = f(1_3), f(2_3) = f(3_3), f(4_3) = f(5_3)$, when $f(1_1) = f(2_1)$ and $f(2_3) = f(3_3)$, respectively.

In [7], N. Pipattanajinda found the results of the endomorphism of an $(n-3)$-regular graph of order $n$ as following.

**Lemma 1.5** ([7]). *Let $G$ be an $(n-3)$-regular graph of order $n$ and $f \in End(G)$.*

*(1) If $x \in f(G)$, then $1 \le |f^{-1}(x)| \le 2$.*

*(2) If $x, y \in f(G)$ with $f^{-1}(x) = \{u\}$ and $f^{-1}(y) = \{v\}$, then $\{x, y\} \in E(G)$ if and only if $\{u, v\} \in E(G)$.*

*(3) If $x, y \in f(G)$ with $f^{-1}(x) = \{u_1, u_2\}$ and $f^{-1}(y) = \{v\}$, then $\{x, y\} \in E(G)$ if and only if $\{u_i, v\} \in E(G)$, for all $i = 1, 2$.*

**Lemma 1.6** ([7]). *Let $G$ be an $(n-3)$-regular graph of order $n$. Then the following statements are trues.*

*(1) $End(G) = LEnd(G)$.*

*(2) $End(G) \neq QEnd(G)$ if and only if $G$ contain induced subgraph $\overline{C}_4$.*

*(3) $QEnd(G) \neq SEnd(G)$ if and only if $G$ contain induced subgraph $\overline{C}_{2r}, r > 2$.*

*(4) $SEnd(G) \neq Aut(G)$ if and only if $G$ contain induced subgraph $\overline{C}_3$.*

*(5) The Endotype $G$ is division by 4.*

## 2. The Endospectrum of $(n-3)$-Regular Graphs of Order $n$

Denote $\bigoplus\limits_{s}\overline{C}_t$ by the joins of $s$ complement of cycles which length $t$. Let $G = \bigoplus\limits_{n_3}\overline{C}_3 + \bigoplus\limits_{n_4}\overline{C}_4 + \cdots + \bigoplus\limits_{n_{2k+1}}\overline{C}_{2k+1}$, the $(n-3)$-regular graph of order $n$ where $n = 3n_3 + 4n_4 + \cdots + (2k+1)n_{2k+1}$. From Lemma 1.6(1), $End(G) = LEnd(G)$. In [16], Knauer and Nieporte found the result of strong endomorphism of graph.

**Lemma 2.1** ([16]). *Let $G$ be a graph, $x_1, x_2 \in V(G)$. There exists a strong endomorphism $f \in SEnd(G)$ with $f(x_1) = f(x_2)$ if and only if $N(x_1) = N(x_2)$, by $N(x)$ for $x \in V(G)$ denote the neighborhood of $x \in G$.*

Then we get the result of strong endomorphism of the $(n-3)$-regular graph of order $n$.

**Lemma 2.2.** *Let $G = \bigoplus\limits_{n_3}\overline{C}_3 + \bigoplus\limits_{n_4}\overline{C}_4 + \cdots + \bigoplus\limits_{n_{2k+1}}\overline{C}_{2k+1}$ and $f \in End(G)$. Then $f$ is strong if and only if the mapping $f|_{\bigoplus\limits_{n_x}\overline{C}_x}$ is $1-1$, for all even integer $x; x \ge 4$.*

*Proof.* Let $x, y \in V(G)$. Then $N(x) = N(y)$ if and only if $x, y \in \overline{C}_3$. So, from Lemma 2.1, $f$ is a strong endomorphism with $f(x) = f(y)$ if and only if $x, y \in \overline{C}_3$. ∎

Next, the characterization of the quasi strong endomorphisms of $G$.

**Lemma 2.3.** *Let $G = \bigoplus\limits_{n_3}\overline{C}_3 + \bigoplus\limits_{n_4}\overline{C}_4 + \cdots + \bigoplus\limits_{n_{2k+1}}\overline{C}_{2k+1}$ and $f \in End(G)$. Then $f$ is quasi strong if and only if the mapping $f|_{\bigoplus\limits_{n_4}\overline{C}_4}$ is $1-1$.*

*Proof.* Necessity. Suppose that $f|_{\bigoplus\limits_{n_4}\overline{C}_4}$ is not $1-1$ mapping. Then from Lemma 1.2(4), there exists some subgraph $\overline{C}_4$ of $G$ such that $f(1) = f(2) = x$ and $f(3) = f(4) = y$ (or $f(4) = f(1) = x$ and $f(2) = f(3) = y$) with $\{x, y\} \in E(G)$, by Lemma 1.5(1), implies that $f^{-1}(x) = \{1, 2\}$ and $f^{-1}(y) = \{3, 4\}$. Since $\{1, 4\}, \{2, 3\} \notin E(G)$, $f$ is not quasi strong.

Sufficiency. Let $f|_{\underset{n_4}{\oplus}\overline{C}_4}$ is an $1-1$ mapping. Thus for each $x, y \in V(G), f(x) = f(y)$ implies $x = y+1$ (or $x = y-1$) and $x, y \in V(\overline{C}_{2m})$ for some $m > 2$.

Further, if $|f^{-1}(x)| = |f^{-1}(y)| = 1$ or $|f^{-1}(x)| = 2, |f^{-1}(y)| = 1$, then $\{u, v\} \in E(G)$, for all $u \in f^{-1}(x)$ and $v \in f^{-1}(y)$, by Lemma 1.5(2) and (3).

Let $f^{-1}(x) = \{u_1, u_2\}$ and $f^{-1}(y) = \{v_1, v_2\}$, s'pose that $u_1 < u_2 < v_1 < v_2, u_2 = u_1 + 1, v_2 = v_1 + 1$ and belong to same complement of cycle $\overline{C}_{2m}$. The mapping look like:

$$f = \begin{pmatrix} \dots & u_1 & u_2 & \dots & v_1 & v_2 & \dots \\ \dots & x & x & \dots & y & y & \dots \end{pmatrix}.$$

If ether $u_1 \neq 1$ nor $v_2 \neq 2m$, then $\{u_1, v_1\}, \{u_1, v_2\} \in E(G)$. If $u_1 = 1$ and $v_2 = 2m$, since $2m > 4$, $\{u_2, v_1\} \in E(G)$ implies that $\{u_2, v_1\}, \{u_2, v_2\} \in E(G)$. This is show that $f \in QEnd(G)$. ∎

Next, we will compute the number of endomorphisms of $(n-3)$-regular graphs of order $n$. From Lemma 1.3, as following:

**Lemma 2.4.** *Let $G = \underset{n_3}{\oplus}\overline{C}_3 + \underset{n_4}{\oplus}\overline{C}_4 + \cdots + \underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1}$, $n = 3n_3 + 4n_4 + \cdots + (2k+1)n_{2k+1}$, an $(n-3)$-regular graphs of order $n$. Then*

*(1) $|End(G)| = |End(\underset{n_4}{\oplus}\overline{C}_4 + \underset{n_6}{\oplus}\overline{C}_6 + \cdots + \underset{n_{2k}}{\oplus}\overline{C}_{2k})| \times |End(\underset{n_3}{\oplus}\overline{C}_3)| \times |End(\underset{n_5}{\oplus}\overline{C}_5)| \times |End(\underset{n_7}{\oplus}\overline{C}_7)| \times \ldots \times |End(\underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1})|$, where $|End(\underset{n_x}{\oplus}\overline{C}_x)| = 1$ if $n_x = 0$,*

*(2) $|End(\underset{n_3}{\oplus}\overline{C}_3)| = |S_{n_3} \times T_3| = 9n_3!$, if $n_3 \neq 0$,*

*(3) $|Aut(\underset{n_3}{\oplus}\overline{C}_3)| = |S_{n_3} \times D_3| = 6n_3!$, if $n_3 \neq 0$,*

*(4) for each odd integer $x \geq 5$ such that $n_x \neq 0$, $|End(\underset{n_x}{\oplus}\overline{C}_x)| = |Aut(\underset{n_x}{\oplus}\overline{C}_x)| = |S_{n_x} \times D_x| = 2xn_x!$, and*

*(5) for each even integer $x \geq 4$ such that $n_x \neq 0$, $|Aut(\underset{n_x}{\oplus}\overline{C}_x)| = |S_{n_x} \times D_x| = 2xn_x!$.*

**Lemma 2.5.** *Let $G = \underset{n_3}{\oplus}\overline{C}_3 + \underset{n_5}{\oplus}\overline{C}_5 + \underset{n_7}{\oplus}\overline{C}_7 + \ldots + \underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1}$. Then*

*(1) $|End(G)| = |SEnd(G)| = 3 \cdot 2^{k-1} \prod_{i=1}^{k} [(2i+1)n_{2i+1}!]$,*

*(2) $|Aut(G)| = 2^k \prod_{i=1}^{k} [(2i+1)n_{2i+1}!]$.*

*Proof.* (1) Clearly by Lemma 2.4(1)(2) and (4), the cardinality of $|End(\underset{n_3}{\oplus}\overline{C}_3 + \underset{n_5}{\oplus}\overline{C}_5 + \underset{n_7}{\oplus}\overline{C}_7 + \ldots + \underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1})| = |End(\underset{n_3}{\oplus}\overline{C}_3)| \times |End(\underset{n_5}{\oplus}\overline{C}_5)| \times |End(\underset{n_7}{\oplus}\overline{C}_7)| \times \ldots \times |End(\underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1})|$
$= 9n_3!(2)(5)n_5!(2)(7)n_7! \cdots (2)(2k+1)n_{2k+1}! = 3 \cdot 2^{k-1} \prod_{i=1}^{k} [(2i+1)n_{2i+1}!]$.

(2) Clearly by Lemma 2.4(1)(3) and (4), the cardinality of $|Aut(\underset{n_3}{\oplus}\overline{C}_3 + \underset{n_5}{\oplus}\overline{C}_5 + \underset{n_7}{\oplus}\overline{C}_7 + \ldots + \underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1})| = |Aut(\underset{n_3}{\oplus}\overline{C}_3)| \times |Aut(\underset{n_5}{\oplus}\overline{C}_5)| \times |Aut(\underset{n_7}{\oplus}\overline{C}_7)| \times \ldots \times |Aut(\underset{n_{2k+1}}{\oplus}\overline{C}_{2k+1})| = 6n_3!(2)(5)n_5!(2)(7)n_7! \cdots (2)(2k+1)n_{2k+1}! = 2^k \prod_{i=1}^{k} [(2i+1)n_{2i+1}!]$. ∎

Consider the cardinality $|End(\underset{n_4}{\oplus}\overline{C}_4 + \underset{n_6}{\oplus}\overline{C}_6 + \cdots + \underset{n_{2k}}{\oplus}\overline{C}_{2k})|$.

Let $F([n_4 : r_4], [n_6 : r_6], \ldots, [n_{2k} : r_{2k}])$, $0 \leq r_s \leq n_s$, be the set of all endomorphisms $f \in End(\underset{n_4}{\oplus}\overline{C}_4 + \underset{n_6}{\oplus}\overline{C}_6 + \cdots + \underset{n_{2k}}{\oplus}\overline{C}_{2k})$ such that $f|_{\underset{r_s}{\oplus}\overline{C}_s}$ is the $1-1$ mapping from $\underset{r_s}{\oplus}\overline{C}_s$ embed in $\underset{n_s}{\oplus}\overline{C}_s$. Denote $P(n,r) = \frac{n!}{(n-r)!}$, the permutations of $n$ elements $r$ at a time.

Let $G = \underset{n_4}{\oplus}\overline{C}_4 + \underset{n_6}{\oplus}\overline{C}_6 + \cdots + \underset{n_{2k}}{\oplus}\overline{C}_{2k}$ and $f \in F([n_4 : 0], [n_6 : 0], \ldots, [n_{2k} : 0])$, $f \in End(G)$ with $|f^{-1}(x)| = 2$ for all $x \in f(G)$. Using the same technique as in [15], we obtain Lemmas 2.6 and 2.7.

**Lemma 2.6.** *Let $\varrho_f$ be the congruence of the graph $G$ when defining $x\varrho_f y \Leftrightarrow f(x) = f(y)$ which here means $x, y$ are elements of same complement of cycle with $y = x+1$ or $y = x-1$. Denote by $G_{\varrho_f}$ the factor graph. Then for each induced subgraph $\overline{C}_{2m}$ of $G$ either $V((\overline{C}_{2m})_{\varrho_f}) = \{\{1,2\}, \ldots \{2m-1, 2m\}\}$ or $V((\overline{C}_{2m})_{\varrho_f}) = \{\{2m,1\}, \ldots \{2(m-1), 2m-1\}\}$.*

**Lemma 2.7.** *Let $\hat{f} : V(G_{\varrho_f}) \to V(G)$ be defined by $\hat{f}(x_{\varrho_f}) = f(x)$. Then for each induced subgraph $\overline{C}_{2m}$ of $G$, there exist subset $C$ of $V(G)$ either $\hat{f}(C_{\varrho_f}) = \{1, 3, \ldots, 2m-1\}$ or $\hat{f}(C_{\varrho_f}) = \{2, 4, \ldots, 2m\}$.*

From Lemmas 2.6 and 2.7, we can define the following classes of endomorphisms of $F([n_4 : 0], [n_6 : 0], \ldots, [n_{2k} : 0])$ on $G$ by $\varrho_f$ and $\hat{f}(C_{\varrho_f}) \subseteq V(\overline{C}_{2m})$.

(1) $S^{or}_m$, the class of all endomorphisms $f$ of $G$ where $\hat{f}(C_{\varrho_f})$ are the odd integers and $\{1,2\} \in \varrho_f$,
(2) $S^{er}_m$, the class of all endomorphisms $f$ of $G$ where $\hat{f}(C_{\varrho_f})$ are the even integers and $\{1,2\} \in \varrho_f$,
(3) $S^{ol}_m$, the class of all endomorphisms $f$ of $G$ where $\hat{f}(C_{\varrho_f})$ are the odd integers and $\{2m,1\} \in \varrho_f$, and
(4) $S^{el}_m$, the class of all endomorphisms $f$ of $G$ where $\hat{f}(C_{\varrho_f})$ are the even integers and $\{2m,1\} \in \varrho_f$.

**Example 2.8.** For the graph $\underset{2}{\oplus}\overline{C}_6 = \overline{C}_6 + \overline{C}_6$ with the set $F([n_4 : 0], [n_6 : 0], \ldots, [n_{2k} : 0])$ such that $n_6 = 2$ and $n_4 = n_8 = \cdots = n_{2k} = 0$, we choose following 16 $(= 4^2)$ notations at

$$S^{or}_{3_1} \times S^{or}_{3_2}, S^{or}_{3_1} \times S^{er}_{3_2}, S^{or}_{3_1} \times S^{ol}_{3_2}, S^{or}_{3_1} \times S^{el}_{3_2}, S^{er}_{3_1} \times S^{or}_{3_2}, S^{er}_{3_1} \times S^{er}_{3_2}, S^{er}_{3_1} \times S^{ol}_{3_2}, S^{er}_{3_1} \times S^{el}_{3_2},$$

$$S^{ol}_{3_1} \times S^{or}_{3_2}, S^{ol}_{3_1} \times S^{er}_{3_2}, S^{ol}_{3_1} \times S^{ol}_{3_2}, S^{ol}_{3_1} \times S^{el}_{3_2}, S^{el}_{3_1} \times S^{or}_{3_2}, S^{el}_{3_1} \times S^{er}_{3_2}, S^{el}_{3_1} \times S^{ol}_{3_2}, S^{el}_{3_1} \times S^{el}_{3_2},$$

and some elements as follow

$$\begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 1_1 & 3_1 & 3_1 & 5_1 & 5_1 & 1_2 & 1_2 & 3_2 & 3_2 & 5_2 & 5_2 \end{pmatrix}, \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 3_1 & 3_1 & 3_2 & 3_2 & 5_1 & 5_1 & 1_1 & 1_1 & 1_2 & 1_2 & 5_2 & 5_2 \end{pmatrix} \in S^{or}_{3_1} \times S^{or}_{3_2}$$

and

$$\begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 1_1 & 1_1 & 3_1 & 3_1 & 5_1 & 5_1 & 6_2 & 2_2 & 2_2 & 4_2 & 4_2 & 6_2 \end{pmatrix}, \begin{pmatrix} 1_1 & 2_1 & 3_1 & 4_1 & 5_1 & 6_1 & 1_2 & 2_2 & 3_2 & 4_2 & 5_2 & 6_2 \\ 6_2 & 6_2 & 1_1 & 1_1 & 4_2 & 4_2 & 3_1 & 2_2 & 2_2 & 5_1 & 5_1 & 3_1 \end{pmatrix} \in S^{or}_{3_1} \times S^{el}_{3_2}.$$

**Proposition 2.9.** *The sets* $S_2^{s_{2_1}} \times \cdots \times S_2^{s_{2_{n_4}}} \times S_3^{s_{3_1}} \times \cdots \times S_3^{s_{3_{n_6}}} \times \cdots \times S_k^{s_{k_1}} \times \cdots \times S_k^{s_{k_{n_{2k}}}}$, *with* $s_{x_y} \in \{or, er, ol, el\}$ *for all* $x = 2, 3, \ldots, k$ *and* $y = n_4, n_6, \ldots, n_{2k}$ *form groups isomorphic to* $S_m$*, where* $m = 2n_4 + 3n_6 + \cdots + kn_{2k} = \sum_{i=2}^{k} in_{2i}$.

**Theorem 2.10.** $|F([n_4:0],[n_6:0],\ldots,[n_{2k}:0])| = 4^{\sum_{i=2}^{k} n_{2i}} [\sum_{i=2}^{k} in_{2i}]!$.

*Proof.* It follows directly from Proposition 2.9. ∎

**Remark 2.11.** Since $F([n_4:0],[n_6:0],\ldots,[n_{2k}:0])$ form a (disjoint) union of groups, the $F([n_4:0],[n_6:0],\ldots,[n_{2k}:0])$ is a completely regular semigroup.

**Theorem 2.12.** $|F([n_4:r_4],[n_6:r_6],\ldots,[n_{2k}:r_{2k}])| =$

$$\frac{4^{\sum_{i=2}^{k}(n_{2i}-r_{2i})}[\sum_{i=2}^{k} i(n_{2i}-r_{2i})]! \prod_{i=2}^{k}[(4i)P(n_{2i},r_{2i})]}{\prod_{r_{2i}=0} 4i}, \text{ where } 0 \le r_s \le n_s.$$

*Proof.* For each $1 \le s \le k$, assume that $r_{2s} > 0$. This is certainly that the mapping from $\bigoplus_{r_{2s}} \overline{C}_{2s}$ is embed to $\bigoplus_{n_{2s}} \overline{C}_{2s}$ is possible to $P(n_s, r_s)$ pattern. Since for each mapping is same to the dihedral group $D_{2s}$, the mapping is possible to $2s$. That initiate to remainder is $n_{2s} - r_{2s}$, the mapping is same Theorem 2.10.

If $r_{2s} = 0$, implies $(4s)P(n_{2s}, 0) = 4s$. In this case, we need division $(4s)P(n_{2s}, 0)$ by $4s$. ∎

From Lemma 1.6, Lemma 2.2, Lemma 2.3, Lemma 2.5 and Theorem 2.12, the cardinality of endomorphisms, half strong, locally strong, quasi strong, strong endomorphisms and automorphisms of $(n-3)$-regular graph of order $n$ as following.

**Theorem 2.13.** *Let* $G = \bigoplus_{n_3} \overline{C}_3 + \bigoplus_{n_4} \overline{C}_4 + \cdots + \bigoplus_{n_{2k+1}} \overline{C}_{2k+1}$, $n = 3n_3 + 4n_4 + \cdots + (2k+1)n_{2k+1}$*, an* $(n-3)$*-regular graphs of order* $n$*. Then*

*(1)* $|End(G)| = |HEnd(G)| = |LEnd(G)| = 3 \cdot 2^{k-1} \prod_{i=1}^{k} (2i+1)n_{2i+1}!$
$$\times \sum_{r_4=0, r_6=0, \ldots, r_{2k}=0}^{n_4, n_6, \ldots, n_{2k}} |F([n_4:r_4],[n_6:r_6],\ldots,[n_{2k}:r_{2k}])|,$$

*(2)* $|QEnd(G)| = 3 \cdot 2^{k-1} \prod_{i=1}^{k} (2i+1)n_{2i+1}!$
$$\times \sum_{r_6=0, \ldots, r_{2k}=0}^{n_6, \ldots, n_{2k}} |F([n_4:n_4],[n_6:r_6],\ldots,[n_{2k}:r_{2k}])|,$$

*(3)* $|SEnd(G)| = 3 \cdot 2^{k-1} \prod_{i=1}^{k} (2i+1)n_{2i+1}!$
$\times |F([n_4:n_4],[n_6:n_6],\ldots,[n_{2k}:n_{2k}])|$*, and*

*(4)* $|Aut(G)| = 2^k \prod_{i=1}^{k} (2i+1)n_{2i+1}! \times |F([n_4:n_4],[n_6:n_6],\ldots,[n_{2k}:n_{2k}])|$.

**Proposition 2.14.** *The Endospectrum of graph $G$ as follow*

$$Endspec\ G = (x_1, x_1, x_1, x_2, x_3, x_4),$$

*where $x_i$ is the value of Theorem 2.13*(i).

**Remark 2.15.** From Theorem 2.12, we found the equation 2.1,

$$|F([n_4 : n_4], [n_6 : n_6], \dots, [n_{2k} : n_{2k}])| = 2^{k-1} \prod_{i=2}^{k} (2in_{2i}!). \tag{2.1}$$

Furthermore, from Theorem 2.13(3)-(4) with the equation 2.1, let $G = \bigoplus_{n_3} \overline{C}_3 + \bigoplus_{n_4} \overline{C}_4 + \dots + \bigoplus_{n_{2k+1}} \overline{C}_{2k+1}$, then the following to the number of strong endomorphisms and automorphisms of graph $G$,

(1) $|SEnd(G)| = 3 \cdot 2^{2(k-1)} (2k+1)! \prod_{i=2}^{2k+1} n_i!$, and

(2) $|Aut(G)| = 2^{2k-1} (2k+1)! \prod_{i=2}^{2k+1} n_i!$.

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## References

[1] M. Böttcher, U. Knauer, Endomorphism spectra of graphs, Discrete Math. 109 (1992) 45–57.

[2] M. BÖttcher, U. Knauer, Postscript: "Endomorphism spectra of graphs" [Discrete Mathematics 109 (1992) 45–57], Discrete Math. 270 (2003) 329–331.

[3] S. Fan, Endomorphism spectra of bipartite graphs with diameter three and girth six, Southeast Asian Bull. Math. 25 (2001) 217–221.

[4] H. Hou, Y. Luo, Z. Cheng, The endomorphism monoid of $\overline{P}_n$ European J. Combin. 29 (2008) 1173–1185.

[5] H. Hou, X. Fan, Y. Luo, Endomorphism types of generalized polygons, Southeast Asian Bull. Math. 33 (2009) 433–441.

[6] W. Wang, H. Hou, The endomorphism monoid of $N$-prism, International Math. Forum 6 (50) (2011) 2461–2471.

[7] N. Pipattanajinda, The endotype of $(n-3)$-regular graphs of order $n$, Southeast Asian Bull. Math. 38 (2014) 535–541.

[8] Sr. Arworn, An algorithm for the numbers of endomorphisms on paths, Discrete Math. 309 (2009) 94–103.

[9] Sr. Arworn, P. Wojtylak, An algorithm for the number of path homomorphisms, Discrete Math. 309 (18) (2009) 5569–5573.

[10] N. Pipattanajinda, Finding the Numbers of Cycle Homomorphisms, MS Thesis, Chiang Mai University, Thailand, 2006.

[11] W. Li, Graphs with regular monoid, Discrete Math. 265 (2003) 105–118.

[12] N. Pipattanajinda, U. Knauer, Sr. Arworn, Endo-regularity of generalized wheel graphs, Chamchuri J. Math. 3 (2011) 45–57.

[13] Sr. Arworn, U. Knauer, S. Leeratanavalee, Locally strong endomorphisms of paths, Discrete Math. 308 (12) (2008) 2525–2532.

[14] N. Pipattanajinda, U. Knauer, B. Gyurov, S. Panma, The endomorphisms monoids of graphs of order $n$ with a minimum degree $n-3$, Algebra Discrete Math. 18 (2) (2014) 274–294.

[15] N. Pipattanajinda, U. Knauer, B. Gyurov, S. Panma, The endomorphism monoids of $(n-3)$-regular graphs of order $n$, Algebra Discrete Math. 22 (2) (2016) 284–300.

[16] U. Knauer, M. Nieporte, Endomorphisms of graphs I. The monoid of strong endomorphisms, Arch. Math. 52 (1989) 607–614.