Thai Journal of Mathematics
Volume 19 Number 2 (2021)
Pages 325–339

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

An Accelerated Forward-Backward Algorithm with Applications to Image Restoration Problems

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Abstract In this research, we introduce an accelerated fixed point algorithm for solving a common fixed point of a countable family of nonexpansive operators and analyze convergence behavior of the proposed method. We prove weak convergence of the proposed algorithm under some suitable conditions. We also apply our main result for solving a convex minimization problem. As an application, we apply our main result to solve image restoration problems and compare its convergence behavior with the existing well-known algorithms. We find that our algorithm outperforms than the others in the literature.

MSC: 47H09; 47H10; 47J25

Keywords: convex minimization problem; Hilbert space; weak convergence; image restoration problem; forward-backward algorithms

Submission date: 20.05.2020 / Acceptance date: 24.08.2020

1. Introduction

A recently emerging technique used in signal and image processing is compressive sensing (CS). An important brance of image/signal processing is image restoration which is one of the most popular classical inverse problems. Such problem has been extensively studied in various applications such as image debluring, astronomical imaging, remote sensing, radar imaging, digital photography, microscopic imaging. The image restoration problem can be explained in one dimensional vector by the following model:

$$Ax = b + w, \tag{1.1}$$

where $x \in \mathbb{R}^{n\times 1}$ is an original image, $b \in \mathbb{R}^{m\times 1}$ is the observed image, w is additive noise and $A \in \mathbb{R}^{m\times n}$ is the blurring operation. In order to solve problem (1.1), we aim

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Published by The Mathematical Association of Thailand.

to approximate the original image, vector x, by minimizing the additive noise, which is known as the least squares (LS) problem, by the following model:

$$\min_{x} \|Ax - b\|_2^2, \tag{1.2}$$

where $\|\cdot\|_2$ is l_2-norm defined by $\|x\|_2 = \sum_{i=1}^{n} |x_i|^2$. The solution of (1.2) can be estimated by many iterations such as Richardson iteration, see [1] for more detail. However, the number of unknown variables is much more than the observations which causes (1.2) to be ill-posed problem because of a huge norm result which is thus meaningless, see [2] and [3]. Therefore, in order to improve ill-conditioned least squares problem, several regularization methods were introduced. One of the most popular regularization methods is Tikhonov regularization suggested by Tikhonov, see [4]. It is defined to solve the following minimization problem:

$$\min_{x} \|Ax - b\|_2^2 + \lambda\|Lx\|_2^2, \tag{1.3}$$

where $\lambda > 0$, is called regularization parameter, and $L \in \mathbb{R}^{m\times n}$, is called Tikhonov matrix. In the standard form, L is set to be the identity. In statistics, (1.3) is known as ridge regression. For improving the original LS (1.2) and classical regularization such as subset selection and ridge regression (1.3), a new method for estimation a solution of (1.1) called least absolute shrinkage and selection operator (LASSO), was proposed and discussed by Tibshirani [5] as follows:

$$\min_{x} \|Ax - b\|_2^2 + \lambda\|x\|_1, \tag{1.4}$$

where $\|\cdot\|_1$ is l_1-norm defined by $\|x_1\| = \sum_{i=1}^{n} |x_i|$. Moreover, the LASSO can be applied to regression problems [5], image restoration problems [6], etc. In general,(1.2)-(1.4) can be formulated in a general form by estimating the minimizer of sum of two functions as follows:

$$\min_{x} F(x) := f(x) + g(x), \tag{1.5}$$

where g is a convex smooth (or possible non-smooth) function and f is a smooth convex loss function with gradient having Lipschitz constant L. By using Fermats rule, Theorem 16.3 of [7], the solution of (1.5) can be characterized as follows: x minimizing $(f + g)$ if and only if $0 \in \partial g(\bar{x}) + \nabla f(\bar{x})$ where $\partial g(\bar{x})$ and $\nabla f(\bar{x})$ refer to the subdifferential and gradient of g and f respectively. Moreover, Parikh and Boyd [8] showed that problem (1.5) can also be interpreted as a fixed point problem: $\bar{x}$ minimizing $(f + g)$ if and only if

$$\bar{x} = prox_{cg}(I - c\nabla f)(\bar{x}) = J_{c\partial g}(I - c\nabla f)(\bar{x}), \tag{1.6}$$

where $prox_g(x) = argmin_{y\in H}(g(y) + \frac{1}{2}\|x - y\|^2)$, $c > 0$, I is an identity operator, $prox_{cg}$ is the proximity operator of cg ,and $J_{\partial g}$ is the resolvent of ∂g defined by $J_{\partial g} = (I+\partial g)^{-1}$. For convenience, (1.6) can be rewritten as:

$$\bar{x} = T\bar{x}, \tag{1.7}$$

where $T := prox_{cg}(I - c\nabla f)$ which is called forward-backward operator. It is observed that a solution of (1.7) is a fixed point of T and T is a nonexpansive mapping when $c \in (0, \frac{2}{L})$. The existence of a fixed point of nonexpansive mappings was guaranteed by Browders theorem, see [9] for more detail. In order to find a point $\bar{x}$ satisfying (1.7), many researchers proposed various methods for finding the approximate solution. One of most popular iterative methods, called *Picard iteration process*, was defined by:

$$x_{n+1} = Tx_n, \tag{1.8}$$

where initial point x_1 is chosen randomly. In addition, other iterative methods for improving picard iteration process have been studied extensively such as follows.
Mann iteration process [10] is defined by:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n,\ n \geq 1, \tag{1.9}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$ is a sequence in $[0,1]$.
Ishikawa iteration process [11] is defined by:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n,\ n \geq 1, \end{cases} \tag{1.10}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.
S-iteration process [12] is defined by:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n,\ n \geq 1, \end{cases} \tag{1.11}$$

where initial point x_1 is chosen randomly and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. In 2017, Agqrwal, ORegan and Sahu [9] proved that this iteration process is independent of Mann and Ishikawa iteration process and converges faster than both of them. However, the processes mentioned above have a badly convergence rate. Thus, to give a better convergence behavior and improve speed, the technique enhanced with inertial step was introduced firstly by Polyak [13]. The following classical iterative method for finding a zero of sum of two operators, i.e. find $x^* \in H$ such that $x^* \in zer(\nabla f + \partial g)$ can be viewed as Mann interation and it is known as the *forward-backward algorithm* (FBA) is defined by:

$$\begin{cases} y_n = x_n - \gamma \nabla f x_n, \\ x_{n+1} = x_n + \alpha_n (J_{\gamma\partial g} y_n - x_n),\ n \geq 1, \end{cases} \tag{1.12}$$

where $x_0 \in H$, L is a Lipschitz constant of ∇f, $\gamma \in (0, \frac{2}{L})$, $\delta = 2 - \frac{\gamma L}{2}$ and a sequence $\{\alpha_n\}$ in $[0,\delta]$ such that $\sum_{n\in\mathbb{N}} \alpha_n(\delta - \alpha_n) = +\infty$. The following iterative method with inertial step can be used for improving performance of Forward-backward algorithm.
A fast iterative shrinkage-thresholding algorithm (FISTA) [6] is defined by:

$$\begin{cases} y_n = Tx_n, \\ t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}, \\ \theta_n = \frac{t_n - 1}{t_{n+1}}, \\ x_{n+1} = y_n + \theta_n (y_n - y_{n-1}),\ n \geq 1, \end{cases} \tag{1.13}$$

where $x_1 = y_0 \in \mathbb{R}^n$, $t_1 = 1$, $T := prox_{\frac{1}{L}g}(I - \frac{1}{L}\nabla f)$ and θ_n is called inertial step size. FISTA was suggested by Beck and Teboulle. They proved that rate of convergence of FISTA is better than that of iterative shrinkage-thresholding algorithms (ISTA) and applied FISTA to image deblurring problems [6]. The inertial step size θ_n of FISTA was firstly introduced by Nesterov [14]. A *new accelerated proximal gradient algorithm* (NAGA) [15] was defined by:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = T_n[(1-\alpha_n)y_n + \alpha_n T_n y_n], \; n \geq 1, \end{cases} \tag{1.14}$$

where $\{\theta_n\}, \{\alpha_n\}$ are sequences in $(0,1)$ and $\frac{\|x_n - x_{n-1}\|_2}{\theta_n} \to 0$. The NAGA was suggested by Verma and Shukla [15]. They proved a convergence theorem of NAGA and applied this method for solving the non-smooth convex minimization problem with sparsityinducing regularizers for the multitask learning framework. There are also recent works for modified forward-backward algorithms, see [16–20] for instance.

Motivated by the previous works mentioned above, we aim to introduce a new accelerated fixed point algorithm for finding a common fixed point of a countable family of nonexpansive mapping in a real Hilbert space. Then we analyze and compare convergence behavior of our method with the other for deblurring the image.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and C be a nonempty closed convex subset of H.

Definition 2.1. A mapping $T : C \to C$ is said to be

(i) *Lipschitzian* if there exists $\tau \geq 0$ such that

$\|Tx - Ty\| \leq \tau\|x - y\|, \; \forall x, y \in C;$

(ii) *contraction* if T is Lipschitzian with the coefficient $\tau \in [0, 1)$;

(iii) *nonexpansive* if T is Lipschitzian with the coefficient $\tau = 1$.

Let $T : C \to C$ be a mapping. We say that an element $x \in C$ is a *fixed point* of T if $x = Tx$. The set of all fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$ and is called the *fixed point set* of T. Let $\{T_n\}$ and Ω be families of nonexpansive operators of C into C such that $\emptyset \neq F(\Omega) \subset \Gamma := \bigcap_{n=1}^{\infty} F(T_n)$, where $F(\Omega)$ is the set of all common fixed points of each $T \in \Omega$, and let $\omega_w(x_n)$ denote the set of all weak-cluster point of a bounded sequence $\{x_n\}$ in C. A sequence $\{T_n\}$ is said to satisfy the *NST-condition(I)* with Ω [21], if for every bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 \text{ implies } \lim_{n\to\infty} \|x_n - T x_n\| = 0 \; \forall T \in \Omega.$$

If Ω is singleton, i.e., $\Omega = \{T\}$, then $\{T_n\}$ is siad to satisfy the *NST-condition(I)* with T. After that, Nakajo et al. [22] presented the *NST*-condition* which is more general than that of *NST-condition.* A sequence $\{T_n\}$ is siad to satisfy the *NST*-condition* if for every bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0 \text{ implies } \omega_w(x_n) \subset \Gamma.$$

Lemma 2.2 ([23])**.** *For a real Hilbert speace H, let $g : H \to \mathbb{R} \cup \{\infty\}$ be proper convex and lower semi-continuous function, and $f : H \to \mathbb{R}$ be convex differentiable with gradient ∇f being L-Lipschitz constant for some $L > 0$. If $\{T_n\}$ is the forward-backward operator of f*

and h with respect to $c_n \in (0, 2/L)$ such that c_n converges to c, then $\{T_n\}$ satisfies NST-condition(I) with T, where T is the forward-backward operator of f and h with respect to $c \in (0, 2/L)$.

Lemma 2.3 ([24]). *Let H be a real Hilbert space. Then the following results hold:*

(i) for all $t \in [0,1]$ and $x, y \in H$,

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2;$$

(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2 \ \forall x, y \in H$.

Lemma 2.4 ([25]). *Let $\{a_n\}, \{b_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\gamma_n)a_n + b_n, \ \ n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.5 ([26] Opial's Lemma). *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H such that there exists a nonempty set $\Gamma \subset H$ satisfying*

(i) for every $p \in \Gamma$, $\lim_{n\to\infty} \|x_n - p\|$ exists;
(ii) each weak-cluster point of the sequence $\{x_n\}$ is in Γ.

Then, there exists $x^* \in \Gamma$ such that $\{x_n\}$ weakly converges to x^*.

Lemma 2.6 ([27]). *Let $\{a_n\}$ and $\{\theta_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\theta_n)a_n + \theta_n a_{n-1}, \ \ n \in \mathbb{N}.$$

Then the following holds

$$a_{n+1} \leq K \cdot \prod_{j=1}^{n}(1+2\theta_j),$$

where $K = \max\{a_1, a_2\}$. Moreover, if $\sum_{n=1}^{\infty} \theta_n < \infty$, then $\{a_n\}$ is bounded.

3. Main Result

In this section, we introduce a new accelerated fixed point algorithm for finding a common fixed point of a countable family of nonexpansive operators and then we prove a weak convergence result of proposed method under some suitable conditions. We also apply the obtained result to solving a convex optimization problem.

Theorem 3.1. *Let $\{T_n\}$ be a family of nonexpansive operators of H into itself such that $\{T_n\}$ satisfies NST*-condition. Supppose that $\emptyset \neq \Gamma = \cap_{n=1}^{\infty} F(T_n)$ and let $\{x_n\}$ be a sequence in H defined by Algorithm 1. Then the following hold:*

(i) $\|x_{n+1} - x^*\| \leq K \cdot \Pi_{j=1}^{n}(1+2\theta_j)$, where $K = \max\{\|x_1 - x^*\|, \|x_2 - x^*\|\}$ and $x^* \in \Gamma$.

(ii) $\{x_n\}$ converges weakly to a point in $\cap_{n=1}^{\infty} F(T_n)$.

Algorithm 1

1: **Initialization.** Take x_0, $x_1 \in H$ are arbitary and $n = 1$, $\gamma_n \in [a_1, b_1] \subset (0, 1)$, $\beta_n \in [0, 1]$, $\alpha_n \in [0, b_2] \subset [0, 1)$, $\theta_n \geq 0$ and $\sum_{n=1}^{\infty} \theta_n < \infty$.

2: **Iterative Step.** Compute ω_n, z_n, y_n and x_{n+1} using

$$\begin{cases} \omega_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1-\gamma_n)\omega_n + \gamma_n T_n \omega_n, \\ y_n = (1-\beta_n) z_n + \beta_n T_n z_n, \\ x_{n+1} = (1-\alpha_n) T_n z_n + \alpha_n T_n y_n, \; n \geq 1. \end{cases}$$

Then update $n := n + 1$ and go to Iterative Step.

Proof. Let $x^* \in \cap_{n=1}^{\infty} F(T_n)$. From Algorithm 1, we have

$$\|\omega_n - x^*\| \leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|, \tag{3.1}$$

$$\begin{aligned} \|z_n - x^*\| &\leq (1-\gamma_n)\|\omega_n - x^*\| + \gamma_n \|T_n \omega_n - x^*\| \\ &= (1-\gamma_n)\|\omega_n - x^*\| + \gamma_n \|T_n \gamma_n - T_n x^*\| \\ &\leq \|\omega_n - x^*\| \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\beta_n) z_n + \beta_n T_n z_n - x^*\| \\ &\leq (1-\beta_n)\|z_n - x^*\| + \beta_n \|T_n z_n - x^*\| \\ &\leq (1-\beta_n)\|z_n - x^*\| + \beta_n \|z_n - x^*\| \\ &\leq \|z_n - x^*\| \\ &\leq \|\omega_n - x^*\|. \end{aligned} \tag{3.3}$$

These imply that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)(T_n z_n - x^*) + \alpha(T_n y_n - x^*)\| \\ &\leq (1-\alpha_n)\|T_n z_n - x^*\| + \alpha_n \|T_n y_n - T_n x^*\| \\ &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha \|y_n - x^*\| \\ &\leq \|\omega_n - x^*\|. \end{aligned} \tag{3.4}$$

From Algorithm 1 and (3.4), we get

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + \theta_n \|x_n - x_{n-1}\|. \tag{3.5}$$

This impiles

$$\|x_{n+1} - x^*\| \leq (1+\theta_n)\|x_n - x^*\| + \theta_n \|x_{n-1} - x^*\|. \tag{3.6}$$

Apply Lemma 2.6, we get $\|x_{n+1} - x^*\| \leq K \cdot \Pi_{j=1}^{n}(1 + 2\theta_j)$, where

$$K = \max\{\|x_1 - x^*\|, \|x_2 - x^*\|\}.$$

So (i) is obtained.

Since $\sum_{n=1}^{\infty} \theta_n < \infty$, we obtain $\{x_n\}$ is bounded. Thus

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty. \tag{3.7}$$

By (3.6) and Lemma 2.4, we get $\lim_{n\to\infty} \|x_n - x^*\|$ exists. By Lemma 2.3(ii), we obtain

$$\|\omega_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\|. \tag{3.8}$$

By Lemma 2.3(i), we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|(1-\gamma_n)(\omega_n - x^*) + \gamma_n (T_n \omega_n - x^*)\| \\ &= (1-\gamma_n)\|\omega_n - x^*\|^2 + \gamma_n \|T_n \gamma_n - x^*\|^2 - \gamma_n (1-\gamma_n)\|\omega_n - T_n \omega_n\|^2 \\ &\leq \|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_n\omega_n\|^2. \end{aligned} \tag{3.9}$$

Using Lemma 2.3(i) again together with (3.3), (3.8) and (3.9), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(T_n z_n - x^*) + \alpha_n (T_n y_n - x^*)\|^2 \\ &= (1-\alpha_n)\|T_n z_n - x^*\|^2 + \alpha_n ||T_n y_n - x^*\|^2 \\ &\quad - \alpha_n (1-\alpha_n)\|T_n z_n - T_n y_n\|^2 \\ &\leq (1-\alpha_n)\|T_n z_n - x^*\|^2 + \alpha_n ||T_n y_n - x^*\|^2 \\ &\leq (1-\alpha_n)\|z_n - x^*\|^2 + \alpha_n ||y_n - x^*\|^2 \\ &\leq (1-\alpha_n)\big(\|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_n\omega_n\|^2\big) \\ &\quad + \alpha_n \|\omega_n - x^*\|^2 \\ &= (1-\alpha_n)\|\omega_n - x^*\|^2 - (1-\alpha_n)\gamma_n(1-\gamma_n)\|\omega_n - T_n\omega_n\|^2 \\ &\quad + \alpha_n \|\omega_n - x^*\|^2 \\ &= \|\omega_n - x^*\|^2 - (1-\alpha_n)\gamma_n(1-\gamma_n)\|\omega_n - T_n\omega_n\|^2 \\ &\leq \|x_n - x^*\|^2 + \theta_n^2 \|x_n - x_{n-1}\| + 2\theta_n \|x_n - x^*\| \|x_n - x_{n-1}\| \\ &\quad - (1-\alpha_n)\gamma_n(1-\gamma_n)\|\omega_n - T_n\omega_n\|^2. \end{aligned} \tag{3.10}$$

Since (3.7) and $\lim_{n\to\infty} \|x_n - x^*\|$ exists, we obtain $\lim_{n\to\infty} \|\omega_n - T_n\omega_n\| = 0$. Note that

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - \omega_n\| + \|\omega_n - T_n\omega_n\| + \|T_n\omega_n - T_n x_n\| \\ &\leq 2\|x_n - \omega_n\| + \|\omega_n - T_n\omega_n\|. \end{aligned} \tag{3.11}$$

From $\|x_n - \omega_n\| = \theta_n \|x_n - x_{n-1}\| \to 0$, it follows from above inequality that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Consider

$$\begin{aligned} \|y_n - z_n\| &\leq \|y_n - \omega_n\| + \|\omega_n - z_n\| \\ &= \|(1-\beta_n)z_n + \beta_n T_n z_n - \omega_n\| + \|\omega_n - z_n\| \\ &\leq \|z_n - \omega_n\| + \beta_n \|T_n z_n - z_n\| + \|\omega_n - z_n\| \\ &\leq 2\|\omega_n - z_n\| + \beta_n(\|T_n z_n - T_n\omega_n\| + \|T_n\omega_n - \omega_n\| + \|\omega_n - z_n\|) \\ &\leq 2\|\omega_n - z_n\| + \beta_n(2\|\omega_n - z_n\| + \|T_n\omega_n - \omega_n\|). \end{aligned} \tag{3.12}$$

These imply by Algorithm 1 that $\lim_{x\to\infty}\|x_n - T_n x_n\| = 0$ and $\lim_{x\to\infty}\|y_n - z_n\| = 0$. By the nonexpansiveness of T_n, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|T_n z_n - x_n\| + \alpha_n\|T_n y_n - T_n z_n\| \\ &\leq \|T_n z_n - T_n x_n\| + \|T_n x_n - x_n\| + \alpha_n\|y_n - z_n\| \\ &\leq \|z_n - x_n\| + \|T_n x_n - x_n\| + \alpha_n\|y_n - z_n\| \\ &\leq \|z_n - \omega_n\| + \|\omega_n - x_n\| + \|T_n x_n - x_n\| + \alpha_n\|y_n - z_n\|. \end{aligned} \tag{3.13}$$

From (3.13) and $\|z_n - \omega_n\| = \gamma_n\|T_n\omega_n - \omega_n\| \to 0$, we get $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$. Since $\{T_n\}$ satisfies the *NST*-condition*, we obtain $\omega_w(x_n) \subset \Gamma := \cap_{n=1}^{\infty}F(T_n)$. By Lemma 2.5, we can conclude that $\{x_n\}$ converges weakly to a point in $\cap_{n=1}^{\infty}F(T_n)$. This completes the proof. ∎

4. Application on Convex Minimization Problems

Let $f, g : \mathbb{R}^n \to (-\infty, \infty]$. Consider the following problem:
Find $x^* \in \mathbb{R}^n$ such that

$$x^* \in ArgminF(x) : f(x) + g(x), \tag{4.1}$$

where g is a convex smooth (or possible non-smooth) function and f is a smooth convex loss function with gradient having Lipschitz constant L.

Note that the subdifferential operator ∂g is a maximal monotone (see [28] for more details) and the solution of (4.1) is a fixed point of the following operator:

$$x^* \in Argmin(f+g) \Leftrightarrow x^* = prox_{cg}(I - c\nabla f)(x^*), \tag{4.2}$$

where $prox_g(x) = Argmin_{y\in H}(g(y) + \frac{1}{2}\|x-y\|^2)$, $c > 0$. For convenience, (4.2) can be rewritten as:

$$x^* = Tx^*, \tag{4.3}$$

where $T := prox_{cg}(I - c\nabla f)$ which is called forward-backward operator. It is observed that a solution of (4.3) is a fixed point of T and T is a nonexpansive mapping when $c \in (0, \frac{2}{L})$.

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by:*

$$\begin{aligned}\omega_n &= x_n + \theta_n(x_n - x_{n-1}) \\ z_n &= (1-\gamma_n)\omega_n + \gamma_n prox_{c_n g}(I - c_n\nabla f)\omega_n \\ y_n &= (1-\beta_n)z_n + \beta_n prox_{c_n g}(I - c_n\nabla f)z_n \\ x_{n+1} &= (1-\alpha_n)prox_{c_n g}(I - c_n\nabla f)z_n + \alpha_n prox_{c_n g}(I - c_n\nabla f)y_n,\ n \geq 1,\end{aligned}$$

where x_0, $x_1 \in \mathbb{R}^n$, γ_n, β_n, α_n ,θ_n are the same as in Theorem 3.1, and $c_n \in (0, 2/L)$ such that $\{c_n\}$ converges to c and $f, g : \mathbb{R}^n \to (-\infty, +\infty]$ are such that g is a convex function and f is smooth convex function with gradient having Lipschitz constant L. Then the following hold:

(i) $\|x_{n+1} - x^*\| \leq K \cdot \Pi_{j=1}^{n}(1 + 2\theta_j)$, where $K = max\{\|x_1 - x^*\|, \|x_2 - x^*\|\}$ and $x^* \in Argmin(f+g)$.

(ii) $\{x_n\}$ converges weakly to a point in $Argmin(f+g)$.

Proof. Let T be the forward-backward operator of f and g with respect to c, and T_n be the forward-backward operator of f and g with respect to c_n, that is $T := prox_{cg}(I - \nabla f)$ and $T_n := prox_{c_n g}(I - c_n \nabla f)$. Then T and $\{T_n\}$ are nonexpansive operators for all n and $F(T) = \cap_{n=1}^{\infty} F(T_n) = Argmin(f + g)$; see Proposition 26.1 in [7]. By Lemma 2.2, we have that $\{T_n\}$ satisfies the *NST*-condition*. Therefore, we obtain the required result directly by Theorem 3.1. ■

5. Simulated Results for the Image Restoration Problem

In this section, we apply Algorithm 1 to solving the image restoration problem (1.4) and compare the deblurring efficiency of the Algorithm 1 with FISTA [6] and NAGA [15]. Our programs were written in Matlab and all algorithms ran on a laptop, Intel core i5, 4.00 GB RAM . All algorithms were applied to solving problem (1.4), where $f(y) = \|Ay - a\|_2^2, g(y) = \lambda\|y\|_1$, A is the blurring operator, a is the observed image and λ is the regularization parameter. In this experiment, two gray-scale images, Lenna and Cameraman of size 256^2 are considered the original images. The images went through a Gaussian blur of size 9^2 and standard deviation $\sigma = 4$. We use the peak signal-to-noise ratio (PSNR) [29] to measure the performance our the algorithms where PSNR(x_n) is defined by:

$$PSNR(x_n) = 10log_{10}\left(\frac{255^2}{MSE}\right),$$

where $MSE = \frac{1}{M}\|x_n - \bar{x}\|_2^2$ M is the number of image samples and $\bar{x}$ is the original image. For these experiments, the regularization parameter was chosen to be $\lambda = 5 \times 10^{-5}$, and the initial image was the blurred image. The Lipschitz constant L, was computed by the maximum eigenvalues of the matrix $A^T A$. We set parameters as follows:

$$c_n = \frac{n}{L(n+1)} \quad \text{and} \quad c = \frac{1}{L},$$

for NAGA, $\theta_n = 0.99$ and for FISTA, $\theta_n = \frac{t_n - 1}{t_{n+1}}$, $1 \leq n < N$,

where t_n is a sequence defined by $t_1 = 1$ and $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$, and N is a number of iterations that we use to stop, for Algorithm 1, $\alpha_n = 0.99$

$$\theta_n = \begin{cases} 0.99, & 1 \leq n < N \\ \frac{1}{2^n}, & \text{otherwise.} \end{cases}$$

The results of deblurring image of Cameraman and Lenna with 500^{th} iteration of the studied algorithms are shown in Tables 1, 2, 3 and Figures 1, 2, 3, 4.

Table 1. Comparison of image restorations of the studied methods.

	Lenna	**Cameraman**
Algorithms	**PSNR**	**PSNR**
Algorithm 1	36.523148	34.202795
FISTA	34.326150	32.007629
NAGA	29.640613	27.353732

TABLE 2. The values of PSNR at $x_1, x_5, x_{10}, x_{25}, x_{50}, x_{100}, x_{250}, x_{500}$ (Lenna).

No. Iterations	Algorithm 1	FISTA algorithm	NAGA algorithm
1	24.483376	24.151658	24.427685
5	26.076036	25.144785	25.344547
10	27.293600	25.830097	25.828445
25	28.405658	27.286514	26.632544
50	29.715964	28.547210	27.336996
100	31.524497	29.879153	28.039350
250	34.443226	32.151290	28.938490
500	36.523148	34.326150	29.640613

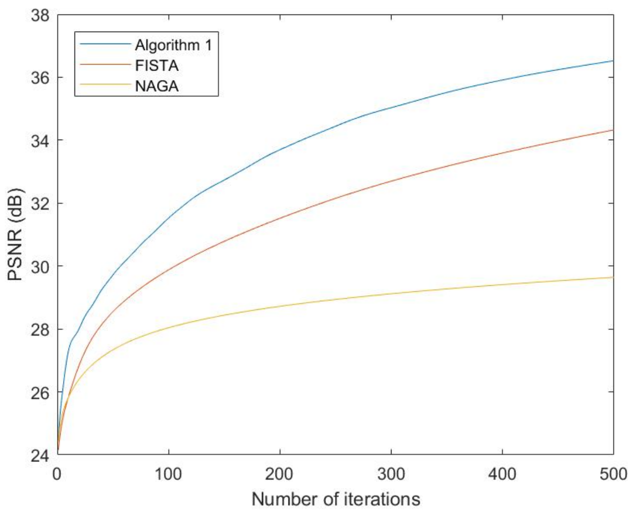

FIGURE 1. The graphs of peak signal-to-noise ratio (PSNR) for Lenna.

TABLE 3. The values of PSNR at $x_1, x_5, x_{10}, x_{25}, x_{50}, x_{100}, x_{250}, x_{500}$ (Cameraman).

No. Iterations	Algorithm 1	FISTA algorithm	NAGA algorithm
1	21.789949	21.56719	21.751774
5	23.221470	22.271305	22.445619
10	24.701898	22.916905	22.920312
25	25.832694	24.620232	23.828038
50	27.157767	26.155117	24.677890
100	29.066243	27.603217	25.537534
250	32.079542	29.873896	26.592485
500	34.202795	32.007629	27.353732

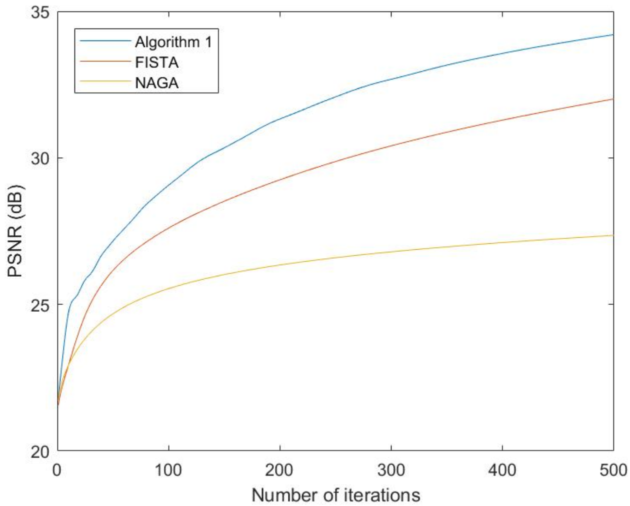

FIGURE 2. The graphs of peak signal-to-noise ratio (PSNR) for Cameraman.

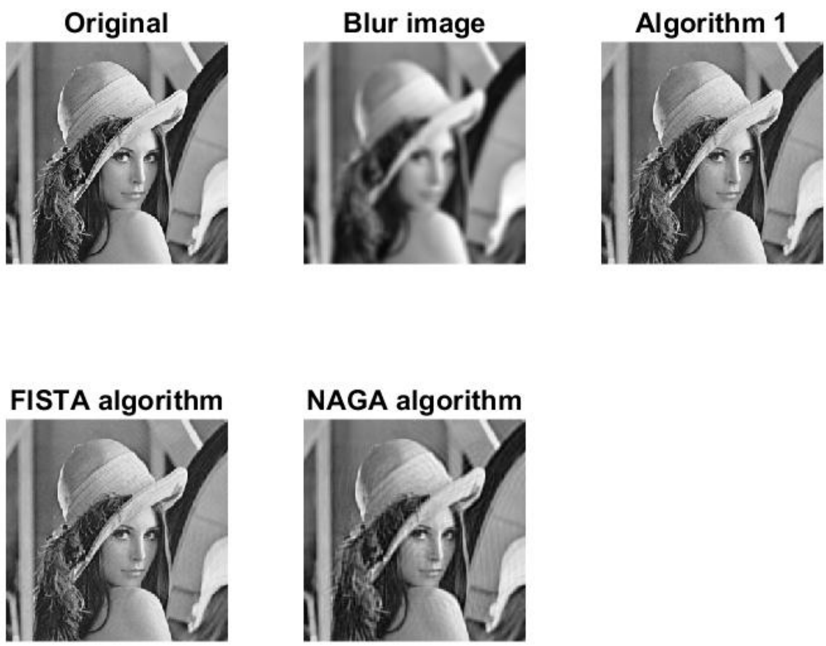

FIGURE 3. Results for deblurring of the Lenna.

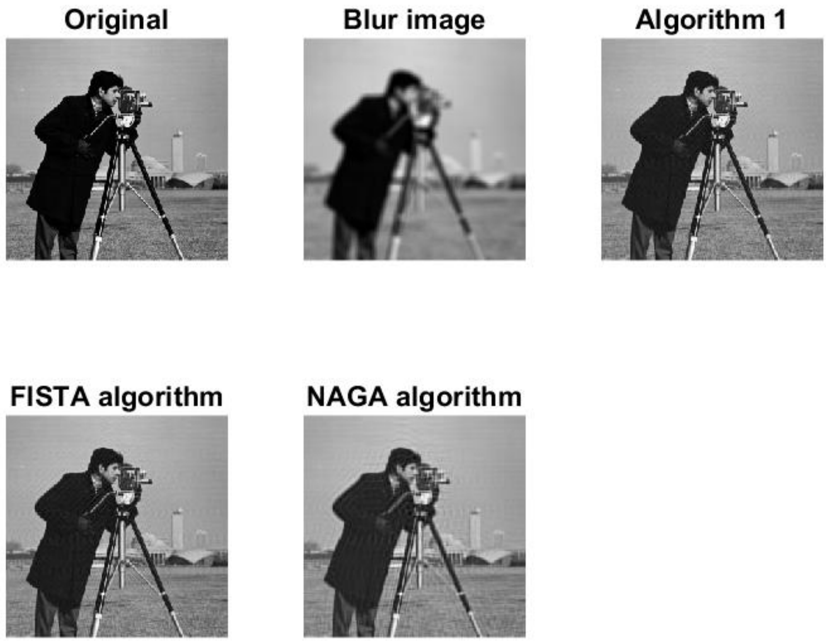

FIGURE 4. Results for deblurring of the Cameraman.

From Table 2, Table 3 and the graph of PSNR in Figure 1, Figure 2 , we see that Algorithm 1 gives a higher PSNR than the other algorithms, so the performance of the image restoration of Algorithm 1 is better than those of FISTA and NAGA. We also see that after 500 iterations, Algorithm 1 gives a better result of deblurring for Lenna and Cameraman, as shown in Figure 3 and Figure 4.

The results of deblurring image of Lenna and Cameraman for the 500th iteration of the Algorithm 1 under different parameters θ_n are shown in Table 4, where θ_n is defined by:

$$\theta_n = \begin{cases} \mu_n, & 1 \leq n < N \\ \frac{1}{2^n}, & \text{otherwise}, \end{cases}$$

where μ_n is a sequence of nonnegative real numbers and N is a number of iterations that we want to stop. We observe that the inertial parameter μ_n using by Algorithm 1 plays an important role in improving quality of deblurring image. It is noted that if $\{\theta_n\}$ is nondecreasing and tends to 1, the values of PSNR increase, as shown in Table 4. However, we can see the result of the deblurring image of Algorithm 1 with different inertial parameters θ_n (six cases), as shown in Table 4. We also observe from Table 4 that the parameter $\mu_n = \frac{n}{n+1}$ gives a higher PSNR than the others.

Table 4. Effective parameters of our method for image restoration.

		Lenna	Cameraman
case	**parameter**	**PSNR**	**PSNR**
1	$\mu_n = \frac{1}{2^n}$	29.803179	27.522892
2	$\mu_n = \frac{500}{n^2}$	29.934126	27.531330
3	$\mu_n = 0.5$	30.575956	28.31232
4	$\mu_n = 0.99$	36.523148	34.202795
5	$\mu_n = \frac{n}{n+1}$	36.849454	34.557969
6	$\mu_n = 1$	32.494240	30.138879

Acknowledgement

We would like to thank the referees for their comments and suggestions on the manuscript and Chiang Mai University for the financial support.

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