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The Structure of Fixed Points Set of a Type of Generalized Nonexpansive Mappings in Convex Metric Spaces

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Abstract In this paper, we show that the fixed points set of self-mappings satisfying condition (C) on a nonempty closed convex subset of a convex metric space having property (D) is always closed and convex. Moreover, we prove that the fixed points set of such mappings on a nonempty bounded closed convex subset of a uniformly convex complete metric space is always nonempty, closed and convex. Our results improve and extend some results in the literature.

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1. Introduction

In 1970, Takahashi [1] introduced a notion of convexity in metric spaces and showed that all normed spaces and their convex subsets are convex metric spaces. He also presented some examples of the convex metric spaces which are not embedded in any normed/Banach spaces. Moreover, he gave some fixed point theorems for nonexpansive mappings in such spaces. Afterward Guay, Singh and Whitield [2], Shimizu and Takahashi [3], Beg [4, 5], Tian [6] and many other authors have given some fixed point theorems for nonexpansive mappings in convex metric spaces.

Recently, Suzuki [7] introduced a condition on mappings, called condition (C), which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Moreover, he got some interesting fixed point theorems and convergence theorems for such mappings in Banach spaces. In 2010, Nanjaras, Panyanak and Phuengrattana [8] extended Suzuki's results on fixed point theorems and convegence theorems to a special kind of metric spaces, namely CAT(0) spaces.

The purpose of this paper is to give some fixed point theorems for mappings satisfying condition (C) in convex metric spaces which are more general than Banach spaces.

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Moreover, we establish that the fixed points set of such mappings defined on a nonempty convex subset of a convex metric space having property (D) is always closed and convex. Particularly, we show that if T is a self-mapping on a nonempty bounded closed convex subset of a uniformly convex complete metric space which satisfies condition (C), then T has at least a fixed point and its fixed points set is closed and convex. Our results improve and extend the following results in [5, 7].

Theorem 1.1. ([5],Theorem 2.1) *Let X be a convex complete metric space having property (B) and K be a closed convex and bounded subset of X. If $T : K \to K$ is a nonexpansive mapping, then* $\inf\{d(x, Tx) : x \in K\} = 0$.

Theorem 1.2. ([5], Theorem 2.3) *Let X be a uniformly convex complete metric space having property (B) and K be a closed convex and bounded subset of X. If $T : K \to K$ is a nonexpansive mapping, then T has a fixed point in K.*

Theorem 1.3. ([5], Theorem 2.6) *Let $T : K \to K$ be a nonexpansive mapping on a closed, convex and bounded subset set K of a uniformly convex complete metric space X having property (B). Then the set of fixed points $F(T)$ of T is nonempty, closed and convex.*

Lemma 1.4. ([7], Lemma 4) *Let T be a mapping on a closed subset K of a Banach space X. Assume that T satisfies condition (C). Then $F(T)$ is closed. Moreover, if X is strictly convex and K is convex, then $F(T)$ is also convex.*

2. Preliminaries

In this section, we review some preliminaries. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers, respectively. In what follows, (X, d) is a metric space, K is a nonempty subset of X, and T is a self-mapping of K. We denote the diameter of K, the closure of K and the fixed points set of T, i.e., $\{x \in K : Tx = x\}$ by $diam(K)$, $cl(K)$ and $F(T)$, respectively. The mapping T is called nonexpansive if $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in K$. It is called quasi-nonexpansive if its fixed points set is nonempty and $d(Tx, u) \leqslant d(x, u)$ for all $x \in K$ and $u \in F(T)$. The mapping T is said to satisfy condition (C) [7] if $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$ implies $d(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in K$. It is easy to show that condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Definition 2.1. ([9]) A sequence $\{x_n\}$ in K is called an approximate fixed point sequence for T if $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$.

Definition 2.2. ([1]) Let (X, d) be a metric space and let $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X, if

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in [0, 1]$. The metric space (X, d) together with a convex structure W is called a convex metric space and denoted by (X, d, W). A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

Let X be a convex metric space. Takahashi [1] proved that the open balls and the closed balls are convex subsets of X. If $\{C_\alpha\}_{\alpha\in J}$ is a family of convex subsets of X, then $\bigcap\limits_{\alpha\in J} C_\alpha$ is a convex subset of X (see [1, 9] for more details). All normed spaces and their convex subsets are convex metric spaces, but there are some examples of convex metric spaces which are not embedded in any normed space (see [1]).

Definition 2.3. Let X be a convex metric space. A real-valued function f on X is called convex if it satisfies

$$f(W(x,y,\lambda)) \leqslant \lambda f(x) + (1-\lambda)f(y)$$

for each $x, y \in X$ and $\lambda \in I$.

Definition 2.4. ([3]) A convex metric space (X, d, W) is said to have property (C) if every decreasing sequence of nonempty bounded closed convex subsets of X has nonempty intersection.

Definition 2.5. ([5]) A convex metric space X is said to have property (B) if

$$d(W(a,x,\lambda), W(a,y,\lambda)) \leqslant (1-\lambda)d(x,y)$$

for all $a, x, y \in X$ and $\lambda \in I$.

Definition 2.6. A convex metric space (X, d, W) is said to have property (D) if $d(x,z)+d(y,z) = d(x,y)$, then there exists $\lambda \in I$ such that $z = W(x,y,\lambda)$, whenever $x, y, z \in X$.

Definition 2.7. A convex metric space (X, d, W) is called strictly convex if

$$d(a, W(x,y,\frac{1}{2})) < r$$

for all $x, y, a \in X$ with $d(x,a) = d(y,a) = r$ and $x \neq y$.

Definition 2.8. ([3]) A convex metric space (X, d, W) is said to be uniformly convex if for any $\epsilon > 0$, there exists $\alpha = \alpha(\epsilon) > 0$ such that for any $r > 0$ and for all $x, y, z \in X$ with $d(z,x) \leqslant r,\ d(z,y) \leqslant r$ and $d(x,y) \geqslant r\epsilon$, we have

$$d(z, W(x,y,\frac{1}{2})) < r(1-\alpha).$$

It is obvious that uniform convexity implies strict convexity.

The following example can be found in [3].

Example 2.9. (i) Uniformly convex Banach spaces are uniformly convex metric spaces.
(ii) Let H be a Hilbert space and X be a nonempty closed subset of the unit sphere of H such that if $x, y \in X, \lambda \in [0,1]$, then

$$(\lambda x + (1-\lambda)y)/\left\|\lambda x + (1-\lambda)y\right\| \in X \text{ and } diam(X) \leqslant \sqrt{2}/2.$$

Put $d(x,y) = \cos^{-1}(< x, y >)$ for each $x, y \in X$, where $<,>$ is the inner product in H. The mapping W defined by

$$W(x,y,\lambda) = \frac{\lambda x + (1-\lambda)y}{\|\lambda x + (1-\lambda)y\|}$$

for all $(x,y,\lambda) \in X \times X \times I$, is a structure on X, and (X, d, W) is a uniformly convex complete metric space (see [10] for more details).

We will need the following lemma and theorem.

Lemma 2.10. ([1]) *Let X be a convex metric space. Then for all $x, y \in X$, and $\lambda \in I$, the following statements hold:*
(i) $d(x,y) = d(x, W(x,y,\lambda)) + d(y, W(x,y,\lambda))$.
(ii) $d(x, W(x,y,\lambda)) = (1-\lambda)d(x,y)$ *and* $d(y, W(x,y,\lambda)) = \lambda d(x,y)$.

Theorem 2.11. ([3]) *Let X be a uniformly convex complete metric space. Then X has property (C).*

In the sequel, we gather some properties on condition (C).
The following propositions are very easy to verify.

Proposition 2.12. *Let K be a nonempty subset of a metric space X. Assume that $T : K \to K$ is a nonexpansive mapping, then T satisfies condition (C).*

The converse of the above theorem is not true in general (see Example 3.10).

Proposition 2.13. *Let K be a nonempty subset of a metric space X, and let $T : K \to K$ be a mapping satisfying condition (C) and has a fixed point. Then it is a quasi-nonexpansive mapping.*

Lemma 2.14. *Let K be a nonempty subset of a metric space X. If $T : K \to K$ is a mapping satisfying condition (C), then for each $x, y \in K$, the following statements hold:*
(i) $d(Tx, T^2x) \leqslant d(x, Tx)$.
(ii) Either $\frac{1}{2}d(x, Tx) \leqslant d(x, y)$ or $\frac{1}{2}d(Tx, T^2x) \leqslant d(Tx, y)$.
(iii) Either $d(Tx, Ty) \leqslant d(x, y)$ or $d(T^2x, Ty) \leqslant d(Tx, y)$.

Proof. (i) Since T satisfies condition (C), (i) follows from $\frac{1}{2}d(x, Tx) \leqslant d(x, Tx)$.
(ii) Suppose, for contradiction, that

$$\tfrac{1}{2}d(x, Tx) > d(x, y) \text{ and } \tfrac{1}{2}d(Tx, T^2x) > d(Tx, y).$$

This and (i) imply

$$\begin{aligned} d(x, Tx) &\leqslant d(x, y) + d(y, Tx) \\ &< \frac{1}{2}d(x, Tx) + \frac{1}{2}d(Tx, T^2x) \\ &\leqslant d(x, Tx), \end{aligned}$$

which is a contradiction. Thus (ii) is true. (iii) Follows from (ii).
Therefore, this completes the proof of the lemma. ■

Lemma 2.15. *Let K be a nonempty subset of a metric space X, and let $T : K \to X$ be a mapping satisfying condition (C). Then*

$$d(x, Ty) \leqslant 3d(x, Tx) + d(x, y)$$

holds for each $x, y \in K$.

Proof. Let $x, y \in K$. By Lemma 2.14(iii),

$$\text{either } d(Tx, Ty) \leqslant d(x, y) \text{ or } d(T^2x, Ty) \leqslant d(Tx, y)$$

holds. In the first case, we have

$$\begin{aligned} d(x, Ty) &\leqslant d(x, Tx) + d(Tx, Ty) \\ &\leqslant d(x, Tx) + d(x, y) \\ &\leqslant 3d(x, Tx) + d(x, y). \end{aligned}$$

In the second case, from Lemma 2.14(i), we get

$$\begin{aligned} d(x,Ty) &\leqslant d(x,Tx)+d(Tx,T^2x)+d(T^2x,Ty)\\ &\leqslant d(x,Tx)+d(x,Tx)+d(Tx,y)\\ &\leqslant 3d(x,Tx)+d(x,y). \end{aligned}$$

Therefore, we obtain the desired result in both cases. ■

3. Main Results

In this section, we suppose that X is a convex metric space together with a metric d and a structure convex W.

The following lemma plays an important role in this section whose proof is similar to the proof of Lemma 2.2 in [11].

Lemma 3.1. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a convex metric space X such that*

$$x_{n+1}=W(y_n,x_n,\lambda) \text{ and } d(y_n,y_{n+1})\leqslant d(x_n,x_{n+1})$$

for all $n\in\mathbb{N}$, where $\lambda\in(0,1)$. Then $\lim_{n\to\infty} d(x_n,y_n)=0$.

The following lemma plays a basic role to prove Theorems 3.8 and 3.9 as well as improves and extends Theorem 1.1.

Lemma 3.2. *Let T be a self-mapping on a nonempty bounded convex subset K of a convex metric space X. Assume that T satisfies condition (C). Define a sequence $\{x_n\}$ in K by $x_1\in K$ and*

$$x_{n+1}=W(Tx_n,x_n,\lambda)$$

for all $n\in\mathbb{N}$, where $\lambda\in[\frac{1}{2},1)$. Then $\{x_n\}$ is an approximate fixed point sequence for T.

Proof. From Lemma 2.10, it follows that

$$\frac{1}{2}d(x_n,Tx_n)\leqslant \lambda d(x_n,Tx_n)=d(x_n,W(Tx_n,x_n,\lambda))=d(x_n,x_{n+1})$$

for all $n\in\mathbb{N}$. From condition (C), we conclude that

$$d(Tx_n,Tx_{n+1})\leqslant d(x_n,x_{n+1})$$

for all $n\in\mathbb{N}$. Lemma 3.1 implies $\lim_{n\to\infty} d(x_n,Tx_n)=0$. This completes the proof of the lemma. ■

Lemma 3.3. *Let K be a nonempty convex subset of a convex metric space X, and let $T:K\to K$ be a mapping satisfying condition (C) whose fixed points set is nonempty. Define a sequence $\{x_n\}$ in K by $x_1\in K$ and*

$$x_{n+1}=W(Tx_n,x_n,\lambda)$$

for all $n\in\mathbb{N}$, where $\lambda\in(0,1)$. Then $\{d(x_n,u)\}$ is a convergent sequence for all $u\in F(T)$.

Proof. Given $u \in F(T)$, by Proposition 2.13, we have

$$\begin{aligned} d(x_{n+1}, u) &= d(W(Tx_n, x_n, \lambda), u) \\ &\leqslant \lambda d(Tx_n, u) + (1-\lambda) d(x_n, u) \\ &\leqslant \lambda d(x_n, u) + (1-\lambda) d(x_n, u) \\ &= d(x_n, u) \end{aligned}$$

for all $n \in \mathbb{N}$. Thus $\{d(x_n, u)\}$ is a bounded decreasing sequence. Therefore, the proof is complete. ■

Proposition 3.4. *Let X be a strictly convex metric space. Then X is having property (D).*

Proof. Arguing by contradiction, we assume that

$$d(x, z) + d(y, z) = d(x, y)$$

holds for some $x, y, z \in X$ with $x \neq y$, but $z \neq W(x, y, \lambda)$ for all $\lambda \in I$.
Set $\lambda = \frac{d(z,y)}{d(x,y)}$. By Lemma 2.10, we have

$$d(x, W(x, y, \lambda)) = d(x, z) \text{ and } d(y, W(x, y, \lambda)) = d(y, z).$$

Since $z \neq W(x, y, \lambda)$ and X is strictly convex, we obtain

$$\begin{aligned} &d(x, W(z, W(x, y, \lambda), \frac{1}{2})) < d(x, z), \\ &d(y, W(z, W(x, y, \lambda), \frac{1}{2})) < d(y, z). \end{aligned}$$

The above inequalities imply

$$\begin{aligned} d(x, y) &\leqslant d(x, W(z, W(x, y, \lambda), \frac{1}{2})) + d(y, W(z, W(x, y, \lambda), \frac{1}{2})) \\ &< d(x, z) + d(y, z) \\ &= d(x, y). \end{aligned}$$

This is a contradiction. Therefore, we obtain the desired result. ■

Using the above proposition, we can get the following corollary.

Corollary 3.5. *Let X be a uniformly convex metric space. Then X has property (D).*

The following proposition extends and improves Lemma 1.4.

Proposition 3.6. *Let K be a nonempty closed subset of a convex metric space X, and let $T : K \to K$ be a mapping satisfying condition (C). Then $F(T)$ is closed. Furthermore, if K is convex and X has property (D), then $F(T)$ is also convex.*

Proof. Suppose, for contradiction, that $F(T)$ is not closed. Therefore, there is an element x of $cl(F(T))$ such that $x \notin F(T)$. So $Tx \neq x$. Put $r = \frac{d(x,Tx)}{3}$. $x \in cl(F(T))$ implies $B(x, r) \cap F(T)$ is nonempty, where $B(x, r)$ is the open ball with center x and radius r.

Let $z \in B(x,r) \cap F(T)$. Thus $d(x,z) < r$ and $Tz = z$. By Proposition 2.13, we have

$$\begin{aligned} d(x,Tx) &\leqslant d(x,z) + d(Tx,z) \\ &\leqslant 2d(x,z) \\ &< 2r \\ &= \frac{2d(x,Tx)}{3}, \end{aligned}$$

which is a contradiction. Hence $F(T)$ is closed. Next, we suppose that X has property (D) and K is convex. Let $\lambda \in (0,1)$ and $x, y \in F(T)$ with $x \neq y$. We show that $W(x,y,\lambda)$ is an element of $F(T)$. Set $u = W(x,y,\lambda)$. By Lemma 2.10 and Proposition 2.13, we have

$$d(u,x) = (1-\lambda)d(x,y), \; d(u,y) = \lambda d(x,y)$$

and

$$d(Tu,x) \leqslant d(u,x), \; d(Tu,y) \leqslant d(u,y). \tag{3.1}$$

From the above relations, we obtain

$$\begin{aligned} d(x,y) &\leqslant d(Tu,x) + d(Tu,y) \\ &\leqslant d(u,x) + d(u,y) \\ &= d(x,y). \end{aligned}$$

This implies

$$\begin{aligned} d(x,y) &= d(Tu,x) + d(Tu,y) \\ &= d(u,x) + d(u,y). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have $d(Tu,x) = d(u,x)$. Property (D) and (3.2) imply that there exists $t \in I$ such that

$$Tu = W(x,y,t).$$

Since $d(Tu,x) = d(u,x)$, Lemma 2.10 implies $t = \lambda$, hence $Tu = u$ holds, and the proof is finished. ∎

Remark 3.7. In comparison with Lemma 1.4, Banach space and strict convexity have been replaced by convex metric space and property (D), respectively.

Theorem 3.8. *Let K be a nonempty compact convex subset of a convex metric space X, and let $T : K \to K$ be a mapping satisfying condition (C). Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and*

$$x_{n+1} = W(Tx_n, x_n, \lambda)$$

for all $n \in \mathbb{N}$, where $\lambda \in [\frac{1}{2}, 1)$. Then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof. Since K is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that converges to an element u of K. Lemma 3.2 implies $\lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$, and from Lemma 2.15, we obtain

$$d(x_{n_k}, Tu) \leqslant 3d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, u) \text{ for all } k \in \mathbb{N}.$$

Taking lim sup on both sides the above inequality, we get $\lim_{k\to\infty} d(x_{n_k}, Tu) = 0$. Since $\lim_{k\to\infty} d(x_{n_k}, u) = 0$, we have $Tu = u$. Thus u is a fixed point of T. Lemma 3.3 implies $\lim_{n\to\infty} x_n = u$. This completes the proof of the theorem. ∎

The following theorem improves and extends Theorems 1.2 and 1.3.

Theorem 3.9. *Let K be a nonempty bounded closed convex subset of a uniformly convex complete metric space X, and let $T : K \to K$ be a mapping satisfying condition (C). Then $F(T)$ is nonempty, closed and convex.*

Proof. Since X is uniformly convex, it is strictly convex. Corollary 3.5 implies that X has property (D). Now by Proposition 3.6, $F(T)$ is closed and convex. We next show that $F(T)$ is nonempty. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and

$$x_{n+1} = W(Tx_n, x_n, \tfrac{1}{2})$$

for all $n \in \mathbb{N}$. Lemma 3.2 implies $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Define a continuous convex function f from K to $[0, \infty)$ by

$$f(x) = \limsup_{n\to\infty} d(x_n, x)$$

for all $x \in K$. Put $r = \inf\{f(x) : x \in K\}$ and $C_n = \{x \in K : f(x) \leqslant r + \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Since K is closed and f is continuous, C_n is closed for all $n \in \mathbb{N}$. It is obvious that the sequence $\{C_n\}$ is decreasing and C_n is nonempty for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, $x, y \in C_n$ and $\lambda \in I$. As f is convex, we have

$$\begin{aligned} f(W(x, y, \lambda)) &\leqslant \lambda f(x) + (1-\lambda)f(y) \\ &\leqslant r + \frac{1}{n}. \end{aligned}$$

This implies that C_n is convex. Also by the definition of f, there exists a point $k \in \mathbb{N}$ such that

$$d(x_k, x) < r + \tfrac{2}{n} \text{ and } d(x_k, y) < r + \tfrac{2}{n}.$$

The above inequalities imply that C_n is bounded. As $\{C_n\}$ is a bounded decreasing sequence of nonempty closed convex subsets of K, by Theorem 2.11, $L = \bigcap_{n=1}^{\infty} C_n$ is nonempty. We next show that L is single-element. Let $x, y \in L$. Either $r = 0$ or $r > 0$. In the first case, that is, $r = 0$, we have

$$d(x, y) \leqslant diam(C_n) < \tfrac{4}{n} \text{ for all } n \in \mathbb{N}.$$

This implies $x = y$. In the second case, that is, $r > 0$, we suppose that $x \neq y$. Since $x, y \in L$, we have $f(x) = f(y) = r$. Set $\epsilon = \frac{d(x,y)}{r+1}$. Since X is uniformly convex, there exists a point $\alpha = \alpha(\epsilon) \in (0, 1)$ such that if β is an element of $(0, \min\{1, \frac{r\alpha}{1-\alpha}\})$, then there exists a point $k \in \mathbb{N}$ such that

$$\sup_{n \geqslant k} d(x_n, x) < r + \beta, \ \sup_{n \geqslant k} d(x_n, y) < r + \beta \text{ and } d(x, y) \geqslant (r + \beta)\epsilon.$$

Also we have

$$d(x_n, W(x, y, \tfrac{1}{2})) < (r + \beta)(1 - \alpha)$$

for all $n \geqslant k$. Taking lim sup in the above inequality, we obtain

$$f(W(x, y, \lambda)) \leqslant (r + \beta)(1 - \alpha).$$

Convexity of K and the definition of r imply

$$\begin{aligned} r &\leqslant f(W(x, y, \lambda)) \\ &\leqslant (r + \beta)(1 - \alpha) \\ &< r, \end{aligned}$$

which is a contradiction. So L is single-element in both cases. Assume $\bigcap_{n=1}^{\infty} C_n = \{u\}$. By Lemma 2.15, we have

$$d(x_n, Tu) \leqslant 3d(x_n, Tx_n) + d(x_n, u) \text{ for all } n \in \mathbb{N}.$$

Taking lim sup on both sides in the above inequality, we get $f(Tu) \leqslant f(u)$. This implies $Tu \in \bigcap_{n=1}^{\infty} C_n$. Therefore, $Tu = u$. Hence u is a fixed point of T. By Corollary 3.5 and Proposition 3.6, $F(T)$ is closed and convex. This completes the proof of the theorem. ∎

Example 3.10. Let $X = \mathbb{R}$ and $K = [0, 2]$. Define a mapping T on K as follows:

$$Tx = \begin{cases} 0 & \text{if } x \in [0,1] \cup \{2\} \\ x - 1 & \text{if } x \in (1,2) \end{cases}$$

for all $x \in K$. Then T satisfies condition (C), but it is not nonexpansive. We also have $F(T) = \{0\}$ which is closed and convex.

Theorem 3.11. *Let K be a nonempty bounded closed convex subset of a uniformly convex complete metric space X, and let $\{T_n\}_{n=1}^{\infty}$ be a family of commuting self-mappings on K satisfying condition (C). Then $\{T_n\}_{n=1}^{\infty}$ has a common fixed point, that is, $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. In addition, $\bigcap_{n=1}^{\infty} F(T_n)$ is closed and convex.*

Proof. Given $n \in \mathbb{N}$ and set $K_n = \bigcap_{i=1}^{n} F(T_i)$. By induction, we show that K_n is nonempty. Theorem 3.9 implies that $K_1 = F(T_1)$ is nonempty. We suppose that K_{n-1} is nonempty, whit $n > 1$. Let $1 \leqslant i \leqslant n-1$ and $x \in K_{n-1}$. Since $\{T_j\}_{j=1}^{\infty}$ is commuting, we have

$$T_i(T_n x) = T_n(T_i x) = T_n x.$$

This implies that K_{n-1} is T_n-invariant, namely $T_n(K_{n-1}) \subseteq K_{n-1}$. Since K_{n-1} is nonempty, bounded, closed and convex, by Theorem 3.9, there is an element x_0 of K_{n-1} such that $x_0 \in F(T_n)$. Thus $x_0 \in K_n$. By induction, K_n is nonempty for all $n \in \mathbb{N}$. As $\{K_n\}_{n=1}^{\infty}$ is a bounded decreasing sequence of nonempty closed convex subsets of K, by Theorem 2.11, $\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. $\bigcap_{n=1}^{\infty} F(T_n)$ is closed and convex, because $F(T_n)$ is closed and convex for all $n \in \mathbb{N}$. Therefore, we get the desired result. ∎

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