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Investigation of Fractional and Ordinary Differential Equations via Fixed Point Theory

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Abstract In this paper, we introduce the concept of generalized α-h-ϕ-contraction mapping and investigate the existence of the fixed point for such mappings in complete metric spaces. The presented results generalize many well known results in the literature. Moreover, we discuss the application of our results to the differential equations theory.

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1. Introduction and Preliminaries

Banach contraction principle [1] is not only the first result in the direction of metric fixed point theory, but also the most elegant, comprehensive, and original result. Following the result of Banach, several authors have reported generalization and extension of this famous results. Among them, the results of Michael A. Geraghty [2] is one of the most interesting extension of the Banach contraction principle and it has been improved and generalized in several direction by a number of authors, e.g. ([3]–[6]).

In this paper, we introduce a new class of mappings, called generalized α-h-ϕ-contraction mapping, which contains Geraghty-contraction type mapping and some of its extensions as a subclass. Moreover, we show that some weakly contractive type mappings are generalized α-h-ϕ-contraction mapping.

For the sake of completeness, we recall and recollect some basic definitions and remarkable results.
Let Φ be a family of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) ϕ is continuous and non-decreasing;
(2) $\phi(t) = 0$ if and only if $t = 0$.

We state the result of Dutta and Choudhury in the following.

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Theorem 1.1 ([7]). *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping such that for each $x, y \in X$ satisfying*

$$\phi(d(Tx,Ty)) \leq \phi(d(x,y)) - \psi(d(x,y)),$$

where $\phi, \psi \in \Phi$. Then T has a unique fixed point.

Let $\mathcal{F}$ be the class of those functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \implies t_n \to 0.$$

One of interesting extensions of the Banach contraction principle was given by Michael A. Geraghty as follows.

Theorem 1.2 ([2]). *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping such that for each $x, y \in X$ satisfying*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y),$$

where $\beta \in \mathcal{F}$. Then T has a unique fixed point.

Definition 1.3 ([8]). Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to X$ is said to be a generalized α-ϕ-Geraghty contraction if there exist two functions $\beta \in \mathcal{F}$ and $\phi \in \Phi$ such that for all $x, y \in X$

$$\alpha(x,y)\phi(d(Tx,Ty)) \leq \beta(\phi(M(x,y)))\phi(M(x,y)),$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Definition 1.4 ([9]). Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. The mapping T is said to be $\alpha-$admissible if

$$\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1.$$

An $\alpha-$admissible mapping T is said to be triangular $\alpha-$admissible if

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y,z) \geq 1 \implies \alpha(x,z) \geq 1.$$

Lemma 1.5 ([10]). *Let $T : X \to X$ be a triangular $\alpha-$admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 1.6 ([8]). Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a function. A sequence $\{x_n\}$ is said to be $\alpha-$regular if the following condition is satisfied:
If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

In [8], Karapınar proved the following theorem.

Theorem 1.7. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions are satisfied:*

(i) T is a generalized α-ϕ-Geraghty contraction type mapping;
(ii) T is triangular $\alpha-$admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iv) T is continuous or any sequence $\{x_n\}$ is regular.

Then T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*.

Popescu [11] suggested the concept of α-orbital admissible as a refinement of the α-admissible notion, defined in [9, 12].

Definition 1.8 ([11])**.** Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

If the additional condition,

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1,$$

is fulfilled, then the α-admissible mapping T is called triangular α-orbital admissible.

Notice that each α-admissible mapping is an α-orbital admissible.

2. Main Results

In this section, we introduce a new class of mappings which contain Geraghty-contraction type mapping and some of its extensions and some of weakly contractive type mappings as a subclass. Also we obtain some known and some new results in fixed point theory via the generalized α-h-ϕ-contraction mappings.
Let $\mathcal{H}(X)$ be a family of functions $h : X \times X \to [0, 1)$ satisfying the following condition:

$$\lim_{n\to\infty} h(x_n, y_n) = 1 \implies \lim_{n\to\infty} d(x_n, y_n) = 0,$$

for all sequences $\{x_n\}$ and $\{y_n\}$ in X that the sequence $\{d(x_n, y_n)\}$ is decreasing and convergent.

Example 2.1. Let $h_i : \mathbb{R} \times \mathbb{R} \to [0, 1)$, for $i = 1, 2$ defined by

(i) $h_1(x, y) = \dfrac{t}{t + x^2 + y^2}$, for some $t \in (0, \infty)$.

(ii) $h_2(x, y) = k$, for some $k \in (0, 1)$.

Then $h_1, h_2 \in \mathcal{H}(\mathbb{R})$.

Example 2.2. Let (X, d) be a metric space and $\beta \in \mathcal{F}$. Define $h : X \times X \to [0, 1)$, by

$$h(x, y) = \beta(d(x, y)).$$

If $\{x_n\}, \{y_n\}$ be sequences in X such that $\lim_{n\to\infty} h(x_n, y_n) = 1$, then $\lim_{n\to\infty} \beta(d(x_n, y_n)) = 1$. Thus

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

This implies that $h \in \mathcal{H}(X)$.

Definition 2.3. Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A mapping $T : X \to X$ is said to be generalized α-h-ϕ-contraction if there exist $h \in \mathcal{H}(X)$ and $\phi \in \Phi$ such that for all $x, y \in X$,

$$\alpha(x, y)\phi(d(Tx, Ty)) \leq h(x, y)\phi(M_a(x, y)),$$

where

$$M_a(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\big\}.$$

Now, we prove the following theorem that extend and generalize some known fixed point results.

Theorem 2.4. *Let (X, d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Suppose that $T : X \to X$ is self mapping satisfying the following conditions:*

(i) T is a generalized α-h-ϕ-contraction type mapping;
(ii) T is triangular α-orbital admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iv) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*.

Proof. From condition (iii), there exists $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) \geq 1.$$

Define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Suppose that for some positive integer k, we have $x_k = x_{k+1}$. This implies that $Tx_k = x_{k+1} = x_k$, that is, x_k is a fixed point of T. So, we can assume that $x_n \neq x_{n+1}$, $n = 0, 1, 2, \cdots$.
Due to Lemma 1.5, since T is a generalized α-h-ϕ-contraction type mapping, for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &\leq \alpha(x_{n-1}, x_n)\phi(d(x_n, x_{n+1})) \\ &= \alpha(x_{n-1}, x_n)\phi(d(Tx_{n-1}, Tx_n)) \\ &\leq h(x_{n-1}, x_n)\phi(M_a(x_{n-1}, x_n)) \\ &< \phi(M_a(x_{n-1}, x_n)). \end{aligned} \tag{2.1}$$

On the other hand

$$\begin{aligned} & M_a(x_{n-1}, x_n) \\ = \ & \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2}\right\} \\ = \ & \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\} \\ = \ & \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\ \leq \ & \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\ = \ & \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $M_a(x_{n-1}, x_n) = d(x_n, x_{n+1})$, applying (2.1), we deduce that

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &< \phi(M_a(x_{n-1}, x_n)) \\ &= \phi(d(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction. Thus, we conclude that $M_a(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ for each $n \in \mathbb{N}$. Now, from (2.1), we get that

$$\phi(d(x_n, x_{n+1})) < \phi(d(x_{n-1}, x_n)).$$

Monotony of ϕ, implies that for all $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Now, we deduce that the sequence $\{d(x_n, x_{n+1})\}$ is non-negative and decreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. In the sequel, we prove that $r = 0$. On the contrary, Suppose that $r > 0$. Then from (2.1), we have

$$0 < \frac{\phi(d(x_n, x_{n+1}))}{\phi(d(x_{n-1}, x_n))} \leq h(x_{n-1}, x_n),$$

which implies that $\lim_{n\to\infty} h(x_{n-1}, x_n) = 1$. Since $h \in \mathcal{H}$,

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

This implies that $r = 0$, which is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). On the contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, we can find $m_k \geq n_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \epsilon.$$

Also, choosing m_k as small as possible, it may be assumed that

$$d(x_{n_k}, x_{m_k-1}) < \epsilon.$$

Hence for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\epsilon \leq d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \\ &\leq \epsilon + d(x_{m_k-1}, x_{m_k}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon.$$

Note that, for any $k \in \mathbb{N}$,

$$\begin{aligned}\phi(d(x_{n_k+1}, x_{m_k+1})) &\leq \alpha(x_{n_k}, x_{m_k})\phi(d(x_{n_k+1}, x_{m_k+1})) \\ &= \alpha(x_{n_k}, x_{m_k})\phi(d(Tx_{n_k}, Tx_{m_k})) \\ &\leq h(x_{n_k}, x_{m_k})\phi(M_a(x_{n_k}, x_{m_k})). \end{aligned} \tag{2.2}$$

Also, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}&M_a(x_{n_k}, x_{m_k}) \\ &= \max\left\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{m_k}), \frac{d(x_{n_k}, Tx_{m_k}) + d(x_{m_k}, Tx_{n_k})}{2}\right\} \\ &= \max\left\{d(x_k, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\right\} \\ &\leq \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \frac{d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1})}{2} \\ &\quad + \frac{d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})}{2}\Big\}.\end{aligned}$$

Since $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0$, we deduce that

$$\lim_{k\to\infty} M_a(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}). \tag{2.3}$$

By using the triangular inequality and taking the limit as $k \to \infty$, we derive

$$\begin{aligned}\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) &\leq \lim_{k\to\infty} (d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})) \\ &= \lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}). \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) with the continuity of ϕ, we get

$$\lim_{k\to\infty} \phi(d(x_{n_k}, x_{m_k})) \leq \lim_{k\to\infty} h(x_{n_k}, x_{m_k}) \lim_{k\to\infty} \phi(d(x_{n_k}, x_{m_k})).$$

Since $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon > 0$, we deduce that

$$\lim_{k\to\infty} h(x_{n_k}, x_{m_k}) = 1.$$

Since $h \in \mathcal{H}(X)$, therefore

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = 0,$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence and so there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Since T is a continuous function, therefore

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

Thus $Tx^* = x^*$. ■

In the following theorem, we omit the continuity condition of the mapping T in Theorem 2.4.

Theorem 2.5. *Let (X, d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Suppose that $T : X \to X$ is self mapping satisfying the following conditions:*

(i) T is a generalized α-h-ϕ-contraction mapping and for all sequences $\{x_n\}, \{y_n\} \subseteq X$ that $\alpha(x_n, y_n) \neq 0$, $\forall n \in \mathbb{N}$, the following condition is satisfied

$$\lim_{n\to\infty} h(x_n, y_n) = 1 \Longrightarrow \lim_{n\to\infty} d(Tx_n, Ty_n) = 0;$$

(ii) T is triangular $\alpha-$admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iv) the sequence $\{T^n x_0\}$ is $\alpha-$regular.

Then T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*.

Proof. From condition (iii), there exists $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) \geq 1.$$

Define the sequence $\{x_n\}$ by $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. Following the proof of Theorem 2.4, know that the sequence $\{x_n\}$ is convergent to some $x^* \in X$ and and by applying Lemma 1.5, $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$. Since the sequence $\{x_n\}$ is α-regular, there exists subsequence $\{x_{n_k}\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$. Without loss of generality, we assume that for all $n \in \mathbb{N}$,

$$\alpha(x_n, x^*) \geq 1. \tag{2.5}$$

Applying (2.5), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} \phi(d(x_{n+1}, Tx^*)) &= \phi(d(Tx_n, Tx^*)) \\ &\leq \alpha(x_n, x^*)\phi(d(Tx_n, Tx^*)) \\ &\leq h(x_n, x^*)\phi(M_a(x_n, x^*)). \end{aligned} \tag{2.6}$$

Also, we have

$$\begin{aligned} M_a(x_n, x^*) &= \max\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\} \\ &= \max\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, x_{n+1})}{2}\} \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, x^*) = 0$, then $\lim_{n\to\infty} M_a(x_n, x^*) = d(x^*, Tx^*)$. Applying (2.6) and continuity of ϕ, we get $\lim_{n\to\infty} h(x_n, x^*) = 1$, and hence from (i) we have

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(Tx_n, Tx^*) = 0.$$

Therefore $Tx^* = x^*$. ■

Let $Fix(T) = \{x \in X : T(x) = x\}$. Then for the uniqueness of the fixed point of a generalized α-h-ϕ-contraction mapping, we will consider the following condition.
$(H1)$ For all $x, y \in Fix(T)$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$.

Theorem 2.6. *Adding condition $(H1)$ to the hypotheses of the Theorem 2.4 (resp. Theorem 2.5), we obtain that x^* is the unique fixed point of T.*

Proof. Let $x^*, y^* \in X$ be two fixed points of T. We show that $x^* = y^*$. on the contrary case, let $x^* \neq y^*$. Then by applying (H_1), there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \text{ and } \alpha(z, y^*) \geq 1.$$

Since T is triangular $\alpha-$admissible, we have $\alpha(x^*, y^*) \geq 1$. Therefore

$$\begin{aligned} \phi(d(x^*, y^*)) &= \phi(d(Tx^*, Ty^*)) \\ &\leq \alpha(x^*, y^*)\phi(d(Tx^*, Ty^*)) \\ &\leq h(x^*, y^*)\phi(M_a(x^*, y^*)) \\ &< \phi(M_a(x^*, y^*)). \end{aligned} \tag{2.7}$$

On the other hand, we have

$$\begin{aligned} M_a(x^*, y^*) &= \max\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \frac{d(x^*, Ty^*) + d(y^*, Tx^*)}{2}\} \\ &= d(x^*, y^*). \end{aligned} \tag{2.8}$$

Applying (2.7) and (2.8), we have $\phi(d(x^*, y^*)) < \phi(d(x^*, y^*))$, which is a contradiction. This implies that $x^* = y^*$, and so the fixed point of T is unique. ■

3. Examples

The following example illustrates our results.

Example 3.1. Consider $X = [0, \infty)$ with the usual metric. If $T : X \to X$ is the mapping defined as

$$T(x) = \ln(1 + 4x^2), \quad x \in X,$$

then T is not a contraction mapping. Also, define $\alpha : X \times X \to \mathbb{R}$ as follows

$$\alpha(x, y) = \begin{cases} 1 & x, y \in [0, \frac{1}{8}] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi(t) = t$, for all $t \geq 0$, and $h : X \times X \to [0, 1)$ be a function defined by

$$h(x,y) = \begin{cases} \dfrac{\arctan 4|x^2 - y^2|}{4|x^2 - y^2|} & x \neq y, \\ 0 & x = y. \end{cases}$$

In the sequel, we show that T is a generalized α-h-ϕ-contraction type mapping. It is easy to see that $h \in \mathcal{H}(\mathcal{R})$ and $\phi \in \Phi$. Since for all $t \in [0, 1]$,

$$\ln(1 + t) \leq \arctan(t), \tag{3.1}$$

thus for all $x, y \in [0, \frac{1}{8}]$, we get

$$\begin{aligned} \alpha(x,y)\phi(d(Tx,Ty)) &= |\ln(\frac{1+4x^2}{1+4y^2})| \\ &\leq \ln(1 + 4|x^2 - y^2|) \\ &\leq \arctan(4|x^2 - y^2|) \\ &= \frac{\arctan(4|x^2 - y^2|)}{4|x^2 - y^2|} 4|x^2 - y^2| \\ &\leq \frac{\arctan(4|x^2 - y^2|)}{4|x^2 - y^2|} |x - y| \\ &= h(x,y)\phi(d(x,y)) \\ &\leq h(x,y)\phi(M_a(x,y)). \end{aligned}$$

If $x > \frac{1}{8}$ or $y > \frac{1}{8}$, then $\alpha(x, y) = 0$, and so $\alpha(x, y)\phi(d(Tx, Ty)) \leq h(x, y)\phi(M_a(x, y))$. Hence T is a generalized α-h-ϕ-contraction type mapping. Obviously, other hypothesis of Theorem 2.4 are satisfied. Therefore T has a fixed point. Note that $x^* = 0$ is the fixed point of T.

Example 3.2. Consider $X = [0, \infty)$ with the usual metric. If $T : X \to X$ is the mapping defined as

$$T(x) = \begin{cases} \dfrac{1}{2 + 2x^2} & x \in [0, 2], \\ \dfrac{1}{1 + x^2} & x \in (2, \infty). \end{cases}$$

Then T is not a contraction mapping. Also, define $\alpha : X \times X \to \mathbb{R}$ as follows

$$\alpha(x,y) = \begin{cases} 1 & x, y \in [0, 2], \\ \dfrac{1}{x + y} & \text{otherwise}. \end{cases}$$

Let $\phi(t) = t$, for all $t \geq 0$, and $h : X \times X \to [0, 1)$ be a function defined by

$$h(x,y) = \begin{cases} 0 & x = y = 0, \\ \dfrac{1}{1 + x^2 + y^2} & \text{otherwise}. \end{cases}$$

In the sequel, we show that T is a generalized α-h-ϕ-contraction type mapping. It is easy to see that $h \in \mathcal{H}(\mathcal{R})$ and $\phi \in \Phi$.
Case 1: Let $x, y \in [0,2]$, we get

$$\begin{aligned}
\alpha(x,y)\phi(d(Tx,Ty)) &= |\frac{1}{2+2x^2} - \frac{1}{2+2y^2}| \\
&= \frac{1}{4}\frac{|y^2-x^2|}{(1+x^2)(1+y^2)} \\
&= \frac{1}{4}\frac{|y+x|}{(1+x^2)(1+y^2)}|y-x| \\
&\leq \frac{1}{1+x^2+y^2}|y-x| \\
&= h(x,y)d(x,y) \\
&\leq h(x,y)\phi(M_a(x,y))
\end{aligned}$$

Case 2: Let $x \in (2,\infty)$ or $y \in (2,\infty)$. Then

$$\begin{aligned}
\alpha(x,y)\phi(d(Tx,Ty)) &= \frac{1}{x+y}|\frac{1}{1+x^2} - \frac{1}{1+y^2}| \\
&= \frac{1}{x+y}\frac{|y^2-x^2|}{(1+x^2)(1+y^2)} \\
&= \frac{1}{x+y}\frac{(x+y)}{(1+x^2)(1+y^2)}|y-x| \\
&\leq \frac{1}{1+x^2+y^2}|y-x| \\
&= h(x,y)d(x,y) \\
&\leq h(x,y)\phi(M_a(x,y))
\end{aligned}$$

Hence T is a generalized α-h-ϕ-contraction type mapping. Obviously, other hypothesis of Theorem 2.5 are satisfied. Therefore T has a fixed point.

4. Particular Cases

Now, we consider some special cases, where in our result deduce several well-known fixed point theorems of the existing literature.

Corollary 4.1. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a mapping. Suppose that the following conditions are satisfied:*

(i) There exist $\phi \in \Phi$ and $\beta \in \mathcal{F}$ such that for each $x, y \in X$

$$\alpha(x,y)\phi(d(Tx,Ty)) \leq \beta(\phi(M_a(x,y)))\phi(M_a(x,y)); \tag{4.1}$$

(ii) T is triangular α-orbital admissible;
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iv) T is continuous or any sequence $\{x_n\}$ is α-regular.

Then T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*.

Proof. Define $h : X \times X \to [0, \infty)$ by

$$h(x, y) = \beta(\phi(M_a(x, y)), \quad x, y \in X.$$

Suppose that $\{x_n\}, \{y_n\} \subseteq X$ are such that $\lim_{n\to\infty} h(x_n, y_n) = 1$. Then

$$\lim_{n\to\infty} \phi(M_a(x_n, y_n) = 0.$$

Since ϕ is continuous and $\phi^{-1}\{0\} = 0$, then $\lim_{n\to\infty} M_a(x_n, y_n) = 0$. This implies that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(y_n, Ty_n) = 0. \tag{4.2}$$

Hence $h \in \mathcal{H}(X)$, and by (4.1), we have

$$\alpha(x, y)\phi(d(Tx, Ty)) \leq h(x, y)\phi(M_a(x, y)),$$

for each $x, y \in X$. Therefore T is a generalized α-h-ϕ-contraction type mapping. Also applying (4.2) and triangular inequality implies that $\lim_{n\to\infty} d(Tx_n, Ty_n) = 0$. Hence all hypotheses of Theorems 2.4 and 2.5 are satisfied. Thus T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*. ∎

If $\alpha(x, y) = 1$, for all $x, y \in X$, then by applying Theorem 2.4, we have the following corollary:

Corollary 4.2. *Let (X, d) be a complete metric space. Suppose $T : X \to X$ is a self mapping and there exists $\phi \in \Phi$ and $h \in \mathcal{H}(X)$, such that for each $x, y \in X$*

$$\phi(d(Tx, Ty)) \leq h(x, y)\phi(M_a(x, y)).$$

Then there exists a unique $x^* \in X$ such that $Tx^* = x^*$.

In the following corollaries we obtain some known results in fixed point theory via the generalized α-h-ϕ-contraction type mapping.

Corollary 4.3 ([8])**.** *Let (X, d) be a complete metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Suppose that $T : X \to X$ is a self mapping and there exists $\phi \in \Phi$ and $h \in \mathcal{H}(X)$, such that*

- *(i) T is a generalized α-ϕ-Geraghty contraction type mapping;*
- *(ii) T is triangular α-orbital admissible;*
- *(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- *(iv) T is continuous or any sequence $\{x_n\}$ is regular.*

Then T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*.

Proof. From (i) and using Definition 1.3, there exists $\beta \in \mathcal{F}$ and $\phi \in \Phi$ such that for each $x, y \in X$

$$\alpha(x, y)\phi(d(Tx, Ty)) \leq \beta(\phi(M(x, y)))\phi(M(x, y)),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Now, define $h : X \times X \to [0, \infty)$ by

$$h(x, y) = \beta(\phi(M(x, y)), \quad x, y \in X.$$

Similarly to Corollary 4.1, we can show that $h \in \mathcal{H}(X)$ and for any sequences $\{x_n\}, \{y_n\} \subseteq X$,

$$\lim_{n\to\infty} h(x_n, y_n) = 1 \Longrightarrow \lim_{n\to\infty} d(Tx_n, Ty_n) = 0.$$

Since ϕ is a non-decreasing function, then for each $x, y \in X$, we have

$$\alpha(x,y)\phi(d(Tx,Ty)) \leq h(x,y)\phi(M(x,y)) \leq h(x,y)\phi(M_a(x,y)).$$

Therefore T is a generalized α-h-ϕ-contraction mapping. Hence all hypotheses of Theorem 2.4 and Theorem 2.5 are satisfied, which implies that T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*. ∎

Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$, for all $t \in [0,\infty)$. Similarly to Theorem 1.7, we can prove that generalized α-Geraghty contraction mappings are as a subclass of generalized α-h-ϕ-contraction type mappings. Applying Theorem 2.4 and Theorem 2.5, we have the following corollaries:

Corollary 4.4 ([13])**.** *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : X \to X$ be a mapping. Suppose that the following conditions are satisfied:*

(i) T is a generalized α-Geraghty contraction type mapping, that is

$$\alpha(x,y)d(Tx,Ty) \leq \beta((M(x,y)))M(x,y),$$

for each $x, y \in X$;

(ii) T is triangular α-orbital admissible;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iv) T is continuous or any sequence $\{x_n\}$ is regular.

Then T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*.

Corollary 4.5 (Geraghty fixed point theorem [2])**.** *Let (X,d) be a complete metric space and let T be a mapping on X. Suppose that there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y).$$

Then T has a unique fixed point x^* and $\{T^n x\}$ converges to x^* for each $x \in X$.

Proof. Define $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) = t$ and $h : X \times X \to [0,\infty)$, with $h(x,y) = \beta(d(x,y))$, for each $x, y \in X$. Applying Example 2.2, $h \in H$, then T is a continuous α-h-ϕ-contraction mapping, with $\alpha = 1$. Obviously the hypotheses of Corollary 4.2 are satisfied. Therefore T has a fixed point $x^* \in X$, and $\{T^n x_0\}$ converges to x^*. ∎

Corollary 4.6 (ϕ-ψ-Weakly contractive fixed point theorem [7])**.** *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping that satisfies the following condition:*

$$\phi(d(Tx,Ty)) \leq \phi(d(x,y)) - \psi(d(x,y)), \tag{4.3}$$

where $\phi, \psi \in \Phi$. Then T has a unique fixed point in X.

Proof. Define $h : X \times X \to [0,1)$, by

$$h(x,y) = \begin{cases} \dfrac{\phi(d(x,y)) - \psi(d(x,y))}{\phi(d(x,y))} & if\ x \neq y, \\ 0 & if\ x = y. \end{cases} \tag{4.4}$$

Let $\{x_n\}, \{y_n\} \subseteq X$ be such that sequence $\{d(x_n,y_n)\}$ is decreasing and $\lim_{n\to\infty} d(x_n,y_n) = r$. Suppose that $\lim_{n\to\infty} h(x_n,y_n) = 1$. We show that $\lim_{n\to\infty} d(x_n,y_n) = 0$. In the contrary case, let $\lim_{n\to\infty} d(x_n,y_n) = r > 0$. Since ϕ and ψ are continuous, thus

$$\lim_{n\to\infty} h(x_n,y_n) = \lim_{n\to\infty} \frac{\phi(d(x_n,y_n)) - \psi(d(x_n,y_n))}{\phi(d(x_n,y_n))} = \frac{\phi(r) - \psi(r)}{\phi(r)} = 1,$$

which implies that $\psi(r) = 0$, and so $r = 0$. This is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

This implies that $h \in \mathcal{H}(X)$. Applying (4.3) and (4.4) we conclude that for each $x, y \in X$,

$$\phi(d(Tx, Ty)) \le h(x, y)\phi(d(x, y)) \le h(x, y)\phi(M_a(x, y)).$$

The hypotheses of the Corollary 4.2 are satisfied. Hence the mapping T has a unique fixed point. ∎

Let Ψ be a class of all upper semi-continuous from the right functions $\psi : [0, \infty) \to [0, \infty)$ such that $\psi^{-1}\{0\} = \{0\}$ and $\psi(t) < t$, for all $t > 0$.

Corollary 4.7 (Boyd and Wong fixed point theorem [14]). *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that for every $x, y \in X$,*

$$d(Tx, Ty) \le \psi(d(x, y)), \tag{4.5}$$

where $\psi \in \Psi$. Then T has a unique fixed point.

Proof. Define $h : X \times X \to [0, 1)$, by

$$h(x, y) = \begin{cases} \dfrac{\psi(d(x, y))}{d(x, y)} & if\ x \ne y, \\ 0 & if\ x = y. \end{cases}$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that the sequence $\{d(x_n, y_n)\}$ is decreasing and convergent. Suppose that $\lim_{n\to\infty} h(x_n, y_n) = 1$, we prove that $\lim_{n\to\infty} d(x_n, y_n) = 0$. In the contrary case, let $\lim_{n\to\infty} d(x_n, y_n) = r > 0$. Since ψ is upper semi-continuous from the right, thus

$$1 = \lim_{n\to\infty} h(x_n, y_n) = \lim_{n\to\infty} \frac{\psi(d(x_n, y_n))}{d(x_n, y_n)} \le \frac{\psi(r)}{r},$$

which implies that $\psi(r) \ge r$. This is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

This implies that $h \in \mathcal{H}(X)$. Let $\phi(t) = t$, for all $t \in [0, \infty)$. From (4.5) we conclude that for each $x, y \in X$

$$\phi(d(Tx, Ty)) \le h(x, y)\phi(d(x, y)).$$

The hypotheses of the Corollary 4.2 are satisfied. Therefore the mapping T has a unique fixed point. ∎

5. Application to Nonlinear Fractional Differential Equations

In this section we discuss the application of our results in the research fields of differential equations.

Let β be a positive real number and Γ be a gamma function. For a continuous function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order β is defined as

$$c_{D^\beta}(g(t)) = \frac{1}{\Gamma(n - \beta)} \int_0^t (t - s)^{n-\beta-1} g^{(n)}(s) ds, \quad n = [\beta] + 1.$$

Consider the following nonlinear fractional differential equation:

$$c_{D^\beta}(u(t)) = f(t, u(t)), \tag{5.1}$$

where $t \in I$ and $1 < \beta \leq 2$. via the integral boundary condition

$$u(0) = 0,\ u(1) = \int_0^r u(s)ds, \quad r \in (0,1),$$

where $u \in C[0,1]$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We define the operator equation $T : C[0,1] \to C[0,1]$ as follow

$$\begin{aligned} T(u)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, u(s))ds \\ &- \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s, u(s))ds \\ &+ \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^r \left(\int_0^s (s-z)^{\beta-1} f(z, u(z))dz \right) ds, \quad t \in I. \end{aligned}$$

We know that $u \in C[0,1]$ is a solution of (5.1) if and only if $u \in C[0,1]$ be fixed point of the mapping T. Suppose the following conditions:

(H_1) there exist $\xi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi \in \Psi$ such that for all $t \in I$ and $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$,

$$|f(t,a), f(t,b)| \leq \frac{\Gamma(\beta+1)}{5} \psi(|a-b|);$$

(H_2) there exist $u_0 \in C[0,1]$ such that $\xi(u_0(t), T(u_0(t)) \geq 0$ for all $t \in I$;

(H_3) for all $t \in I$ and $u, v \in C[0,1]$, if $\xi(u(t), v(t)) \geq 0$, then

$$\xi(T(u(t)), T(V(t))) \geq 0;$$

(H_4) Let $\{u_n\}$ be a sequence in $C[0,1]$ such that $u_n \to u$ in $C[0,1]$ and for all $t \in I$,

$$\xi(u_n(t), u_{n+1}(t)) \geq 0,\ \forall n \in \mathbb{N} \Longrightarrow \quad \xi(u_n(t), u(t)) \geq 0.$$

Theorem 5.1. *Suppose that conditions $(H_1) - (H_4)$ are satisfied. Then T has at least one solution $u^* \in C[0,1]$.*

Proof. We prove that T is a generalized α-h-ϕ-contraction mapping. Now, let $u, v \in C[0,1]$ such than for all $t \in I$, $\xi(u(t), v(t)) \geq 0$. Applying H_1,

$$
\begin{aligned}
& |T(u)(t) - T(v)(t)| \\
= & \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,u(s))ds \right. \\
- & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,u(s))ds \\
+ & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^r \left(\int_0^s (s-z)^{\beta-1} f(z,u(z))dz \right) ds \\
- & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,v(s))ds \\
+ & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,v(s))ds \\
- & \left. \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^r \left(\int_0^s (s-z)^{\beta-1} f(z,v(z))dz \right) ds \right| \\
\leq & \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} |f(s,u(s)) - f(s,v(s))| ds \\
+ & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s,u(s)) - f(s,v(s))| ds \\
+ & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^r \left| \int_0^s (s-z)^{\beta-1} |f(z,u(z)) - f(z,v(z))| dz \right| ds \\
\leq & \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} \frac{\Gamma(\beta+1)}{5} \psi(|v(s)-u(s)|) ds \\
+ & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1} \frac{\Gamma(\beta+1)}{5} \psi(|v(s)-u(s)|) ds \\
+ & \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^r \left(\int_0^s |s-z|^{\beta-1} \frac{\Gamma(\beta+1)}{5} \psi(|v(z)-u(z)|) dz \right) ds \\
\leq & \frac{\Gamma(\beta+1)}{5} \psi(||v-u||_\infty) \times \sup_{t\in(0,1)} \left(\frac{1}{\Gamma(\beta)} \int_0^1 |t-s|^{\beta-1} ds \right. \\
+ & \left. \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^1 |1-s|^{\beta-1} ds + \frac{2t}{(2-r^2)\Gamma(\beta)} \int_0^r \int_0^s |s-z|^{\beta-1} dz ds \right) \\
\leq & \psi(||v-u||_\infty) = \psi(d(u,v)).
\end{aligned}
$$

We define $\alpha : C[0,1] \times C[0,1] \to [0,\infty)$ by

$$
\alpha(u,v) = \begin{cases} 1 & \xi(u(t),v(t)) \geq 0 \ \forall t \in I, \\ 0 & otherwise. \end{cases}
$$

and

$$
h(u,v) = \begin{cases} \dfrac{\psi(d(u,v))}{d(u,v)} & if\ u \neq v, \\ 0 & if\ u = v. \end{cases}
$$

Then, for all $u, v \in C[0,1]$, we have

$$\begin{aligned}\alpha(u,v)d(Tu,Tv) \leq \psi(d(u,v)) &= \frac{\psi(d(u,v)}{d(u,v)}d(u,v) \\ &= h(u,v)d(u,v) \\ &\leq h(u,v)M_a(u,v).\end{aligned}$$

Let ϕ be a constant mapping. Then T is a generalized α-h-ϕ-contraction type mapping. One can prove that all the hypotheses of Theorem 2.5 are satisfied. Therefore there exists $u^* \in C[0,1]$ such that $Tu^* = u^*$. ∎

6. Application to Ordinary Differential Equations

Let $X = C[0,1]$ be the space of all continuous functions defined on $I = [0,1]$ and $u \in X$. Consider the following two-point boundary value problem of a second-order differential equation:

$$\begin{cases} -u''(t) - f(t,u(t)) = 0; \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{6.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. It is known that $u = u(t) \in C[0,1]$ is a solution of (6.1) if and only if $u \in C[0,1]$ is a solution of the integral equation

$$u(t) = \int_0^1 k(t,s)f(s,u(s))ds,$$

where $k(t,s)$ is defined as following

$$k(t,s) = \begin{cases} t(1-s) & 0 \leq t \leq s \leq 1, \\ s(1-t) & 0 \leq s \leq t \leq 1. \end{cases}$$

Theorem 6.1. *Suppose that $\psi \in \Psi$ and the following conditions are satisfied:*

(H_1) *there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$ and for all $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$, we have*

$$|f(t,a) - f(t,b)| \leq 8\psi(|a-b|);$$

(H_2) *there exists $u_1 \in C[0,1]$ such that for all $t \in I$,*

$$\xi\left(u_1(t), \int_0^1 k(t,s)f(s,u_1(s))ds\right) \geq 0;$$

(H_3) *for all $t \in I$ and $u, v \in C[0,1]$, if $\xi(u(t), v(t)) \geq 0$, then*

$$\xi\left(\int_0^1 k(t,s)f(s,u(s))ds, \int_0^1 k(t,s)f(s,v(s))ds\right) \geq 0;$$

(H_4) *Let $\{u_n\}$ be a sequence in $C[0,1]$ such that $u_n \to u$ in $C[0,1]$. Let for all $t \in I$ and $n \in \mathbb{N}$, if $\xi(u_n(t), u_{n+1}(t)) \geq 0$, then*

$$\xi(u_n(t), u(t)) \geq 0.$$

Then the boundary value problem (6.1) has a solution.

Proof. We define $T : C[0,1] \to C[0,1]$ by

$$T(u(t)) = \int_0^1 k(t,s)f(s,u(s))ds, \quad \forall t \in I.$$

Thus, a solution of problem (6.1) corresponds with a fixed point of T. Now our purpose is to prove that integral operator T is generalized $\alpha - h - \phi-$contraction.
Let $u, v \in C[0,1]$ such that $\xi(u(t), v(t)) \geq 0$ for all $t \in I$. Applying H_1,

$$\begin{aligned} |Tu(t) - Tv(t)| &= \left| \int_0^1 k(t,s)\left(f(s,u(s)) - f(s,y(s))\right) ds \right| \\ &\leq \int_0^1 k(t,s)\left|f(s,u(s)) - f(s,v(s))\right| ds \\ &\leq \int_0^1 k(t,s)\left(8\psi(|u(s) - v(s)|)\right) ds \\ &\leq 8 \sup_{t\in I} \int_0^1 k(t,s)ds\psi(d(u,v)) \\ &= \psi(d(u,v)) \end{aligned}$$

We define $\alpha : C[0,1] \times C[0,1] \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} 1 & \text{if } \xi(u(t), v(t)) \geq 0 \text{ for all } t \in I, \\ 0 & otherwise. \end{cases}$$

and

$$h(u,v) = \begin{cases} \dfrac{\psi(d(u,v))}{d(u,v)} & if\ u \neq v, \\ 0 & if\ u = v, \end{cases}$$

Then, for all $u, v \in C[0,1]$, we have

$$\begin{aligned} \alpha(u,v)d(Tu,Tv) \leq \psi(d(u,v)) &= \frac{\psi(d(u,v))}{d(u,v)} d(u,v) \\ &= h(u,v)d(u,v) \\ &\leq h(u,v)M_a(u,v). \end{aligned}$$

Let ϕ be a constant mapping. Then T is a generalized $\alpha - h - \phi-$contraction type mapping.
Let $\{u_n\}, \{v_n\}$ be sequences in $C[0,1]$ such that $\lim_{n\to\infty} h(u_n, v_n) = 1$ and for all $n \in \mathbb{N}$, $\alpha(u_n, v_n) \neq 0$. By definition of α, For all $n \in \mathbb{N}$ and $t \in [0,1]$ we have $\xi(u_n(t), v_n(t)) \geq 0$ and so $d(Tu_n(t), Tv_n(t)) \leq \psi(d(u_n, v_n))$ which implies that for all $n \in \mathbb{N}$, $d(Tu_n, Tv_n) \leq \psi(d(u_n, v_n))$. Since $\lim_{n\to\infty} d(u_n, v_n) = 0$ therefore $\lim_{n\to\infty} d(Tu_n, Tv_n) = 0$. This implies that the condition (i) of Theorem 2.5 is satisfied. Applying conditions $H_2 - H_4$, all hypotheses of Theorem 2.5 are satisfied. Therefore there exists $u^* \in C[0,1]$ such that $Tu^* = u^*$. ■

7. Conclusion

In this paper, we aim to combine and unify several existing results in our main results by using the auxiliary functions. Especially, the function α plays crucial roles for our purpose. Our main theorems combine the following cases:

(1) a fixed point theorem in the setting of standard metric space,
(2) corresponding fixed point theorem in the context of partially ordered set endowed with a metric,
(3) corresponding fixed point theorems in the frame of cyclic construction.

Accordingly, each consequence of our result yields three different results on the topic in the literature. More precisely, we derive standard metric fixed point results by taking $\alpha : X \times X \to [0, \infty)$ as $\alpha(x, y) = 1$ for all x, y in a metric space (X, d).

We get the related fixed point theorem in the context of partially ordered set endowed with a metric, if we define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

Similarly, we derive the corresponding fixed point theorems in the frame of cyclic construction if define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise,} \end{cases}$$

where A_1 and A_2 are closed subsets of the complete metric space (X, d), and $Y = X \times X$.

So, by regarding the discussion above, we can list a number of consequence of our main results. Thus, our results covers several papers on the topic in the literature.

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