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The Modification of Generalized Mixed Equilibrium Problems for Convergence Theorem of Variational Inequality Problems and Fixed Point Problems

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Abstract The purpose of this research, we modify generalized mixed equilibrium problems and prove a strong convergence theorem for approximating a common element of the set of such a problem and variational inequality problems and the set of fixed points of infinite family of strictly pseudo contractive mappings. Utilizing our main result, we also prove a strong convergence theorem involving generalized equilibrium problems and variational inequality problems.

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1. Introduction

Throughout this article, we assume that H is a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$.

Definition 1.1. Let $T : C \to C$ be a nonlinear mapping, then

(1) T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C,$$

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(2) T is said to be *quasi-nonexpansive* if

$$\|Tx-p\| \leq \|x-p\|, \forall x \in C \text{ and } \forall p \in Fix(T),$$

(3) T is said to be κ*-strictly pseudo-contractive* if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \kappa \|(I-T)x-(I-T)y\|^2, \forall x,y \in C.$$

Note that the class of strictly pseudo-contractive mappings includes the class of nonexpansive mappings. The mapping T is nonexpansive if and only if T is 0-strictly pseudo contractive.

A mapping $A : C \to H$ is called α*-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax-Ay, x-y\rangle \geq \alpha \|Ax-Ay\|^2,$$

for all $x,y \in C$.

A mapping A is said to be ρ*-strongly monotone* if there exists a positive real number ρ such that

$$\langle Ax-Ay, x-y\rangle \geq \rho \|x-y\|^2,$$

for all $x,y \in C$.

The *variational inequality problem* is to find a point $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C,A)$. The application of the variational inequality problem has been expanded to problems from economics, finance, optimization and game theory. Many authors have studied the variational inequality problem, see for instance [1] and [2].

Let $F : C \times C \to \mathbb{R}$ be a bifunction, $A : C \to H$ be a nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function. The *generalized mixed equilibrium problem* (see [3]), is to find $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x\rangle \geq 0, \tag{1.2}$$

for all $y \in C$. The set of solution of (1.2) is denoted by

$$GMEP(F,\varphi,A) = \{x \in C : F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y-x\rangle \geq 0, \forall y \in C\}.$$

Generalized mixed equilibrium problem has been studied by many authors, see for example [4], [5], [6] and [7]. If $\varphi = 0$, then (1.2) reduces to *the generalized equilibrium problem*, that is,

$$EP(F,A) = \{x \in C : F(x,y) + \langle Ax, y-x\rangle \geq 0, \forall y \in C\}. \tag{1.3}$$

If $A = 0$, then problem (1.3) reduces to *the equilibrium problem*, that is,

$$EP(F) = \{x \in C : F(x,y) \geq 0, \forall y \in C\}. \tag{1.4}$$

Optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem can be applied with the equilibrium problem. Many authors have introduced iterative algorithms in order to solve the equilibrium problem, see for instance [8], [9] and [10].

In 2005, Combettes and Hirstoaga [10] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. By using the viscosity approximation method, Takahashi and Takahashi [8] introduced an iteration for finding a common element of the set $EP(A)$ and $Fix(T)$ and proved a strong convergence theorem in a Hilbert space. In 2008, Takahashi and Takahashi [11] introduced another iterative scheme for finding the common element of the set $EP(F,A)$ and $Fix(T)$.

Recently, Kangtunyakarn [12] modified the set of solutions of generalized equilibrium problem as follows:

$$\begin{aligned} &EP(F,aA+(1-a)B) \\ &\quad = \{x\in C: F(x,y)+\langle (aA+(1-a)B)x, y-x\rangle \geq 0, \forall y\in C, a\in(0,1)\}. \end{aligned} \tag{1.5}$$

He introduced an iterative scheme for finding a common element of the set of fixed points of κ-strictly pseudo-contractive mapping and the set of solution of (1.5) as follows:

$$\begin{aligned} &F(u_n,y)+\langle (aA+(1-a)B)x_n, y-x_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y\in C, \\ &x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I-\gamma(I-T))u_n, \forall n\geq 1, \end{aligned} \tag{1.6}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions.

Let $D_1, D_2 : C \to H$ be two nonlinear mappings. Motivated by (1.2) and (1.5), we modify the set of solution of generalized mixed equilibrium problem as follows:

$$\begin{aligned} GMEP(F,\varphi,aD_1+(1-a)D_2) = \{&x\in C: F(x,y)+\varphi(y)-\varphi(x) \\ &+\langle (aD_1+(1-a)D_2)x, y-x\rangle \geq 0\}, \end{aligned} \tag{1.7}$$

for all $y\in C$ and $a\in(0,1)$. If $D_1 = D2$, then $GMEP(F,\varphi,aD_1+(1-a)D_2)$ is reduced to (1.2).

In this research, we modify generalized mixed equilibrium problems and prove the strong convergence theorem for approximating a common element of the set of such a problem and variational inequality problem and the set of fixed points of infinite family of a strictly pseudo contractive mappings. Based on main result, we prove a strong convergence theorem involving generalized equilibrium problems and variational inequality problems.

2. Preliminaries

In this paper, we denote weak and strong convergence by the notations " $\rightharpoonup$ " and " $\to$ ", respectively. In a real Hilbert space H, recall that the (nearest point) projection P_C from H onto C assigns to each $x\in H$ the unique point P_Cx satisfying the property

$$\|x-P_Cx\| = \min_{y\in C}\|x-y\|.$$

For a proof of the main theorem, we will use the following lemmas.

Lemma 2.1. [13] *Given $x\in H$ and $y\in C$, then $P_Cx=y$ if and only if we have the inequality*

$$\langle x-y, y-z\rangle \geq 0, \forall z\in C.$$

Lemma 2.2. [14] *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \forall n \geq 0$$

where α_n *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1): $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2): $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$..

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *Let* H *be a real Hilbert space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.4. [13] *Let* H *be a Hilbert space, let* C *be a nonempty closed convex subset of* H *and let* A *be a mapping of* C *into* H. *Let* $u \in C$. *Then, for* $\lambda > 0$,

$$u = P_C(I-\lambda A)u \Leftrightarrow u \in VI(C,A),$$

where P_C *is the metric projection of* H *onto* C.

Definition 2.5. [15] Let C be a nonempty convex subset of a real Hilbert space. Let $T_i, i = 1, 2, ...$ be mappings of C into itself. For each $j = 1, 2, ...$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$. Define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,n+1} &= I \\
U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I \\
U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I \\
&\vdots \\
U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I \\
U_{n,k} &= \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I \\
&\vdots \\
U_{n,2} &= \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I \\
S_n &= U_{n,1} = \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I.
\end{aligned}$$

Such mapping is called *S-mapping* generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$.

Lemma 2.6. [16] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : C \to C$ *be a self-mapping of* C. *If* S *is a* κ*-strict pseudo-contractive mapping, then* T *satisfies the Lipschitz condition;*

$$\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x-y\|, \forall x, y \in C.$$

For finding solutions of the equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a lower semicontinuous and convex function satisfies the following conditions:
$(A1)$ $F(x,x) = 0$ for all $x \in C$;
$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
$(A3)$ for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F\left(tz + (1-t)x, y\right) \leq F(x,y);$$

$(A4)$ for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous;
$(B1)$ for each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\left\langle y_x - z, z - x\right\rangle < 0;$$

$(B2)$ C is a bounded set.

Lemma 2.7. [17] *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfies $(A1)-(A4)$, $A : C \to H$ be a continuous monotone mapping, and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$ then there exists $z \in C$ such that*

$$F(z,y) + \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle.$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{z \in C : F(z,y) + \langle Ay, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\} \tag{2.1}$$

for all $x \in H$. Then the following conclusions hold:

(1) For each $x \in H, T_r \neq \emptyset$;
(2) T_r is single-valued;
(3) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) $Fix(T_r) = GMEP(F, \varphi, A)$
(5) $GMEP(F, \varphi, A)$is closed and convex.

Lemma 2.8. [15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Then, for every $x \in C$ and $k \in \mathbb{N}, \lim_{n\to\infty} U_{n,k}x$ exists.*

For every $k \in \mathbb{N}$ and $x \in C$. [15] defined mapping $U_{\infty,k}$ and $S : C \to C$ as follows:

$$\lim_{n\to\infty} U_{n,k}x = U_{\infty,k}x \tag{2.2}$$

and

$$\lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x = Sx. \tag{2.3}$$

Such a mapping S is called *S-mapping* generated by $T_n, T_{n-1}, ...$ and $\alpha_n, \alpha_{n-1}, ...$.

Remark 2.9. [15] For every $n \in \mathbb{N}$, S_n is nonexpansive and $\lim_{n\to\infty} sup_{x\in D}\|S_n x - Sx\| = 0$, for every bounded subset D of C.

Lemma 2.10. [15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mappings of C into itself with $\bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I\times I\times I$ where $I = [0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and $T_n, T_{n-1}, ...$ and $\alpha_n, \alpha_{n-1}, ...$, respectively. Then $Fix(S) = \bigcap_{i=1}^{\infty} Fix(T_i)$.*

Lemma 2.11. [18] *Let C be a nonempty closed convex subset of a real Hilbert space H.Let A, B be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $VI(C, A)\bigcap VI(C,B) \neq \emptyset$. Then*

$$VI(C, aA + (1-a)B) = VI(C, A)\bigcap VI(C, B), \forall a \in (0, 1). \tag{2.4}$$

Furthermore if $0 \leq \gamma \leq min\{2\alpha, 2\beta\}$, we have $I - \gamma(aA + (1-a)B)$ is nonexpansive mapping.

Remark 2.12. From Lemma (2.4) and Lemma (2.11), we have

$$VI(C, aA+(1-a)B) = VI(C, A)\bigcap VI(C, B) = Fix(P_C(I-\gamma(aA+(1-a)B))),$$

for all $a \in (0, 1)$ and $\gamma > 0$.

From (1.7), we have the following result.

Lemma 2.13. *Let C be a nonempty closed convex subset of a real Hilbert space H and F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfy $A1) - A4)$ and $F(x, z) \leq F(x, y) + F(y, z)$ for all $x, y, z \in C$. Let A, B be α, β-inverse strongly monotone, respectively, with $\alpha, \beta > 0$ and $GMEP(F, \varphi, A)\bigcap GMEP(F, \varphi, B) \neq \emptyset$. Then*

$$GMEP(F, \varphi, aA + (1-a)B) = GMEP(F, \varphi, A)\bigcap GMEP(F, \varphi, B), \forall a \in (0, 1).$$

Proof. It is obvious that $GMEP(F, \varphi, A)\bigcap GMEP(F, \varphi, B) \subseteq GMEP(F, \varphi, aA + (1-a)B)$. Next, we will show that $GMEP(F, \varphi, aA + (1-a)B) \subseteq GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$. Let $x_0 \in GMEP(F, \varphi, aA + (1-a)B)$ and $x^* \in GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$, we have

$$F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, y - x_0\rangle \geq 0, \forall y \in C, \tag{2.5}$$

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \forall y \in C \tag{2.6}$$

and

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Bx^*, y - x^*\rangle \geq 0, \forall y \in C. \tag{2.7}$$

For every $a \in (0, 1)$, we have

$$aF(x^*, y) + a\varphi(y) - a\varphi(x^*)\langle aAx^*, y - x^*\rangle \geq 0, \forall y \in C$$

and

$$(1-a)F(x^*, y)+(1-a)\varphi(y)-(1-a)\varphi(x^*)+\langle(1-a)Bx^*, y - x^*\rangle \geq 0, \forall y \in C.$$

By the monotonicity of B and $x^*, x_0 \in C$, we have

$$\begin{aligned}
&aF(x_0, x^*) + a\varphi(x^*) - a\varphi(x_0) + \langle aAx_0, x^* - x_0\rangle \\
&\quad = aF(x_0, x^*) + a\varphi(x^*) - a\varphi(x_0) + (1-a)\varphi(x^*) - (1-a)\varphi(x^*) \\
&\qquad + (1-a)\varphi(x_0) - (1-a)\varphi(x_0) + (1-a)F(x_0, x^*) - (1-a)F(x_0, x^*) \\
&\qquad + \langle aAx_0 + (1-a)Bx_0 - (1-a)Bx_0, x^* - x_0\rangle \\
&\quad = F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \\
&\qquad - (1-a)F(x_0, x^*) - (1-a)\varphi(x^*) + (1-a)\varphi(x_0) - \langle (1-a)Bx_0, x^* - x_0\rangle \\
&\quad \geq (1-a)F(x^*, x_0) + (1-a)\varphi(x_0) - (1-a)\varphi(x^*) + (1-a)\langle Bx_0, x_0 - x^*\rangle \\
&\quad = (1-a)(F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle Bx^*, x_0 - x^*\rangle + \langle Bx_0 - Bx^*, x_0 - x^*\rangle) \\
&\quad \geq 0. \qquad (2.8)
\end{aligned}$$

Since $GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B) \subseteq GMEP(F, \varphi, aA + (1-a)B)$ and $x^* \in GMEP(F, \varphi, A) \bigcap GMEP(F, \varphi, B)$, we have

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle aAx^* + (1-a)Bx^*, y - x^*\rangle \geq 0. \forall y \in C. \qquad (2.9)$$

Since $x^* \in C$ and (2.5), we have

$$F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1-a)Bx_0, x^* - x_0\rangle \geq 0. \qquad (2.10)$$

From (2.9) and $x_0 \in C$, we have

$$F(x^*, x_0) + \varphi(x_0) - \varphi(x^*) + \langle aAx^* + (1-a)Bx^*, x_0 - x^*\rangle \geq 0. \qquad (2.11)$$

Summing up (2.10), (2.11) and (A2), we have

$$\langle a(Ax^* - Ax_0) + (1-a)(Bx^* - x_0), x_0 - x^*\rangle \geq 0. \qquad (2.12)$$

Since A, B are α, β-inverse strongly monotone, respectively, and (2.12), we have

$$\begin{aligned}
0 &\leq \langle a(Ax^* - Ax_0) + (1-a)(Bx^* - Bx_0), x_0 - x^*\rangle \\
&= \langle a(Ax^* - Ax_0), x_0 - x^*\rangle + \langle (1-a)(Bx^* - Bx_0), x_0 - x^*\rangle \\
&= a\langle Ax^* - Ax_0, x_0 - x^*\rangle + (1-a)\langle Bx^* - Bx_0, x_0 - x^*\rangle \\
&\leq -a\alpha\|Ax^* - Ax_0\|^2 - (1-a)\beta\|Bx^* - Bx_0\|^2.
\end{aligned}$$

This implies that

$$0 \leq -a\alpha\|Ax^* - Ax_0\|^2.$$

It follows that

$$Ax^* = Ax_0. \qquad (2.13)$$

By using the same method as (2.13), we obtain

$$Bx^* = Bx_0. \qquad (2.14)$$

For every $y \in C$. From (2.6), (2.8), (2.13) and $x^* \in GMEP(F, \varphi, A)$, we have

$$\begin{aligned}
&F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Ax_0, y - x_0\rangle \\
&\quad = F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Ax_0, y - x^* + x^* - x_0\rangle \\
&\quad = F(x_0, y) + \varphi(y) - \varphi(x_0) + \varphi(x^*) - \varphi(x^*) + F(x^*, y) - F(x^*, y) \\
&\qquad + \langle Ax_0, y - x^*\rangle + \langle Ax_0, x^* - x_0\rangle \\
&\quad = F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + F(x^*, y) + \varphi(y) - \varphi(x^*) \\
&\qquad + \langle Ax^*, y - x^*\rangle + \langle Ax_0, x^* - x_0\rangle \\
&\quad \geq F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0\rangle \\
&\quad \geq F(x_0, y) + F(y, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0\rangle \\
&\quad \geq F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Ax_0, x^* - x_0\rangle \\
&\quad \geq 0.
\end{aligned}$$

Then

$$x_0 \in GMEP(F, \varphi, A). \tag{2.15}$$

Since $x^*, x_0 \in C$ and (2.5), (2.13), we have

$$\begin{aligned}
&(1 - a)F(x_0, x^*) + (1 - a)\varphi(x^*) - (1 - a)\varphi(x_0) + \langle (1 - a)Bx_0, x^* - x_0\rangle \\
&\quad = (1-a)F(x_0, x^*) + (1-a)\varphi(x^*) - (1-a)\varphi(x_0) + aF(x_0, x^*) \\
&\qquad - aF(x_0, x^*) + \langle (1 - a)Bx_0 + aAx_0 - aAx_0, x^* - x_0\rangle \\
&\quad = F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle aAx_0 + (1 - a)Bx_0, x^* - x_0\rangle \\
&\qquad - aF(x_0, x^*) + a\varphi(x_0) - a\varphi(x^*) - \langle aAx_0, x^* - x_0\rangle \\
&\quad \geq aF(x^*, x_0) + a\varphi(x_0) - a\varphi(x^*) + \langle aAx_0, x_0 - x^*\rangle \\
&\quad = aF(x^*, x_0) + a\varphi(x_0) - a\varphi(x^*) + a\langle Ax^*, x_0 - x^*\rangle \\
&\quad \geq 0.
\end{aligned} \tag{2.16}$$

For every $y \in C$, from (2.7), (2.14), (2.16) and $x^* \in GMEP(F, \varphi, B)$, we have

$$\begin{aligned}
&F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Bx_0, y - x_0\rangle \\
&\quad = F(x_0, y) + \varphi(y) - \varphi(x_0) + \langle Bx_0, y - x^*\rangle + \langle Bx_0, x^* - x_0\rangle \\
&\quad = F(x_0, y) + \varphi(y) - \varphi(x_0) + \varphi(x^*) - \varphi(x^*) + F(x^*, y) - F(x^*, y) \\
&\qquad + \langle Bx_0, y - x^*\rangle + \langle Bx_0, x^* - x_0\rangle \\
&\quad = F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + F(x^*, y) + \varphi(y) - \varphi(x^*) \\
&\qquad + \langle Bx^*, y - x^*\rangle + \langle Bx_0, x^* - x_0\rangle \\
&\quad \geq F(x_0, y) - F(x^*, y) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0\rangle \\
&\quad \geq F(x_0, y) + F(y, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0\rangle \\
&\quad \geq F(x_0, x^*) + \varphi(x^*) - \varphi(x_0) + \langle Bx_0, x^* - x_0\rangle \\
&\quad \geq 0.
\end{aligned}$$

Hence

$$x_0 \in GMEP(F, \varphi, B). \tag{2.17}$$

By (2.15) and (2.17), we have $x_0 \in GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B)$. Then

$$GMEP(F,\varphi,aA+(1-a)B) \subseteq GMEP(F,\varphi,A)\bigcap GMEP(F,\varphi,B)$$

■

3. Main Result

In this section, we prove a strong convergence theorem and for the set of fixed point of strictly pseudo contractive mappings and the sets of solution of generalized mixed equilibrium problems and variational inequality problems by using Lemma 2.13.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1,F_2 be bifunctions from $C\times C$ to $\mathbb{R}$ satisfy A1)-A4) and $F_i(x,z) \le F_i(x,y)+F_i(y,z)$ for all $x,y,z \in C$ and $i=1,2$. Let $\varphi_1,\varphi_2 : C \to \mathbb{R}\bigcup\{+\infty\}$ be proper lower semicontinuous and convex function. Let A,B be α,β-inverse strongly monotone, respectively, and let D,E be L_D,L_E-Lipschitz continuous and μ,ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i)\bigcap GMEP(F_1,\varphi_1,A)\bigcap GMEP(F_1,\varphi_1,B)\bigcap GMEP(F_2,\varphi_2,A)$ $\bigcap GMEP(F_2,\varphi_2,B)\bigcap VI(C,D)\bigcap VI(C,E) \ne \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j,\alpha_2^j,\alpha_3^j)$ $\in I\times I\times I$ where $I=[0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$, $\alpha_1^j+\alpha_2^j \le b<1$ and $\alpha_1^j,\alpha_2^j,\alpha_3^j \in (\kappa,1)$ for all $j=1,2,\dots$. For every $n\in\mathbb{N}$, let S_n be S-mapping generated by $T_n,T_{n-1},\dots,T_1$ and $\alpha_n,\alpha_{n-1},\dots,\alpha_1$. Assume the either B_1) or B_2) holds. Let the sequence $\{x_n\}$ generated by $x_1,u\in C$ and*

$$\begin{cases} F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle a_nAx_n+(1-a_n)Bx_n, y-u_n\rangle \\ +\dfrac{1}{r_n^1}\langle y-u_n,u_n-x_n\rangle \ge 0, \\ F_2(v_n,y)+\varphi_2(y)-\varphi_2(v_n)+\langle a_nAx_n+(1-a_n)Bx_n, y-v_n\rangle \\ +\dfrac{1}{r_n^2}\langle y-v_n,v_n-x_n\rangle \ge 0, \forall y\in C, \\ y_n=\delta_nu_n+(1-\delta_n)v_n, \\ x_{n+1}=\alpha_nu+\beta_nx_n+\lambda_nS_nx_n+\eta_nP_C(I-\gamma_n(a_nD+(1-a_n)E))y_n, \end{cases} \tag{3.1}$$

for all $n\ge 1$, where the sequences $\{\alpha_n\},\{\beta_n\},\{\lambda_n\},\{\eta_n\},\{\delta_n\} \subseteq [0,1]$ with $\alpha_n+\beta_n+\lambda_n+\eta_n=1$ for all $n\in\mathbb{N}$, $\{a_n\}\subset(0,1)$ and $\{r_n^j\}\subseteq[b,c]\subset(0,2min\{\alpha,\beta\})$ for all $j=1,2$. Suppose the following conditions hold:

(i): $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\lim_{n\to\infty}\alpha_n=0$, $\{\beta_n\}\subseteq[d,e]\subset(0,1)$;

(ii): $0<\gamma_n\le min\{\frac{2\mu}{L_D^2},\frac{2\rho}{L_E^2}\}$;

(iii): $\lim_{n\to\infty}\delta_n=\delta\in(0,1)$, $\sum_{n=1}^{\infty}\alpha_1^n<\infty$;

$$\textbf{(iv):} \sum_{n=1}^{\infty}\left|r_{n+1}^{j}-r_{n}^{j}\right|<\infty \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_{n}|<\infty, \sum_{n=1}^{\infty}|\gamma_{n+1}-\gamma_{n}|<\infty,$$
$$\sum_{n=1}^{\infty}|\delta_{n+1}-\delta_{n}|<\infty, \sum_{n=1}^{\infty}|a_{n+1}-a_{n}|<\infty, \sum_{n=1}^{\infty}|\beta_{n+1}-\beta_{n}|<\infty,$$
$$\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_{n}|<\infty, \sum_{n=1}^{\infty}|\eta_{n+1}-\eta_{n}|<\infty \textit{ for all } j=1,2.$$

Then the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}}u$.

Proof. First, we show that D is $\frac{\mu}{L_D^2}$-inverse strongly monotone mapping. Let $x, y \in C$, we have

$$\begin{aligned}\langle x-y, Dx-Dy\rangle &\geq \mu\|x-y\|^2\\ &\geq \frac{\mu}{L_D^2}\|Dx-Dy\|^2.\end{aligned}$$

Similarly, we get E is $\frac{\rho}{L_E^2}$-inverse-strongly monotone mapping.

Next, we show that $I-\gamma_n D$ and $I-\gamma_n E$ are nonexpansive mappings. For every $x, y \in C$, we have

$$\begin{aligned}\|(I-\gamma_n D)x-(I-\gamma_n D)y\|^2 &= \|x-y\|^2+\gamma_n^2\|Dx-Dy\|^2-2\gamma_n\langle x-y, Dx-Dy\rangle\\ &\leq \|x-y\|^2+\gamma_n^2\|Dx-Dy\|^2-\frac{2\gamma_n\mu}{L_D^2}\|Dx-Dy\|^2\\ &= \|x-y\|^2+\gamma_n\left(\gamma_n-\frac{2\mu}{L_D^2}\right)\|Dx-Dy\|^2\\ &\leq \|x-y\|^2.\end{aligned}$$

Then we obtain $I-\gamma_n D$ is a nonexpansive mapping. Similarly, we can show that $I-\gamma_n E$ is also a nonexpansive mapping.

The proof of Theorem 3.1 will be divided into five steps:

Step 1. We show that the sequence $\{x_n\}$ is bounded.
From (3.1) and Lemma 2.7, we have $u_n = T_{r_n^1}(I-r_n^1(a_nA+(1-a_n)B))x_n$ and $v_n = T_{r_n^2}(I-r_n^2(a_nA+(1-a_n)B))x_n$.
From Lemma 2.7 and Lemma 2.13, we have

$$\begin{aligned}F\left(T_{r_n^1}\left(I-r_n^1\left(a_nA+(1-a_n)B\right)\right)\right) &= GMEP(F_1,\varphi_1,a_nA+(1-a_n)B)\\ &= GMEP(F_1,\varphi_1,A)\bigcap GMEP(F_1,\varphi_1,B)\end{aligned}$$

and

$$\begin{aligned}F\left(T_{r_n^2}\left(I-r_n^2\left(a_nA+(1-a_n)B\right)\right)\right) &= GMEP(F_2,\varphi_2,a_nA+(1-a_n)B)\\ &= GMEP(F_2,\varphi_2,A)\bigcap GMEP(F_2,\varphi_2,B).\end{aligned}$$

Let $z \in \mathcal{F}$. From Lemma 2.4 and Lemma 2.11, we have

$$z \in VI(C, a_nD+(1-a_n)E) = Fix(P_C(I-\gamma_n(a_nD+(1-a_n)E))).$$

From the nonexpansiveness of $T_{r_n^1}$, $T_{r_n^2}$ and Lemma 2.11, we have

$$\begin{aligned}
\|x_{n+1}-z\| &\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\lambda_n\|S_nx_n-z\|\\
&\quad+\eta_n\|P_C(I-\gamma_n(a_nDx+(1-a_n)E))y_n-z\|\\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\lambda_n\|x_n-z\|+\eta_n\|y_n-z\|\\
&= \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\lambda_n\|x_n-z\|\\
&\quad+\eta_n\|\delta_n(u_n-z)+(1-\delta_n)(v_n-z)\|\\
&= \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\lambda_n\|x_n-z\|\\
&\quad+\eta_n\|\delta_n(T_{r_n^1}(I-r_n^1(a_nA+(1-a_n)B))x_n-z)\\
&\quad+(1-\delta_n)(T_{r_n^2}(I-r_n^2(a_nA+(1-a_n)B))x_n-z)\|\\
&\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\lambda_n\|x_n-z\|\\
&\quad+\eta_n(\delta_n\|x_n-z\|+(1-\delta_n)\|x_n-z\|)\\
&= \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\lambda_n\|x_n-z\|+\eta_n\|x_n-z\|\\
&= \alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|.
\end{aligned}\tag{3.2}$$

Put $M=max\{\|u-z\|,\|x_1-z\|\}$. From induction, we can show that $\|x_n-z\|\le M$, for all $n\in\mathbb{N}$. Therefore $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. For every $n\in\mathbb{N}$, put $J_n=a_nD+(1-a_n)E$ and $G_n=a_nA+(1-a_n)B$. From the definition of x_n and the nonexpansivenss of $Pc(I-\gamma_nJ_n)$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|\\
&\quad+\lambda_n\|S_nx_n-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\|\\
&\quad+\|\eta_nP_C(I-\gamma_nJ_n)y_n-\eta_{n-1}P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|\\
&\quad+\lambda_n\|S_nx_n-S_nx_{n-1}\|+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|\\
&\quad+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\|+\eta_n\|P_C(I-\gamma_nJ_n)y_n-P_C(I-\gamma_nJ_n)y_{n-1}\|\\
&\quad+\eta_n\|P_C(I-\gamma_nJ_n)y_{n-1}-P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\
&\quad+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\lambda_n\|x_n-x_{n-1}\|\\
&\quad+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\|\\
&\quad+\eta_n\|y_n-y_{n-1}\|+\eta_n\|P_C(I-\gamma_nJ_n)y_{n-1}-P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\
&\quad+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|.
\end{aligned}\tag{3.3}$$

Since $y_n=\delta_nu_n+(1-\delta_n)v_n$, we have

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|\delta_nu_n+(1-\delta_n)v_n-\delta_{n-1}u_{n-1}-(1-\delta_{n-1})v_{n-1}\|\\
&= \|\delta_n(u_n-u_{n-1})+(\delta_n-\delta_{n-1})u_{n-1}+(1-\delta_n)(v_n-v_{n-1})\\
&\quad+(\delta_{n-1}-\delta_n)v_{n-1}\|\\
&\le \delta_n\|u_n-u_{n-1}\|+|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\|v_n-v_{n-1}\|\\
&\quad+|\delta_n-\delta_{n-1}|\|v_{n-1}\|.
\end{aligned}\tag{3.4}$$

From the nonexpansiveness of P_C, we have

$$\begin{aligned}\|P_C(I-\gamma_n J_n)y_{n-1}-&P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\ &\leq \|(I-\gamma_n J_n)y_{n-1}-(I-\gamma_{n-1}J_{n-1})y_{n-1}\|\\ &= \|\gamma_n J_n y_{n-1}-\gamma_{n-1}J_{n-1}y_{n-1}\|\\ &\leq \gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\|+a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|\\ &\quad+\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\|+(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\|.\end{aligned} \tag{3.5}$$

Substitute (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \leq\ & |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|\\ &+\lambda_n\|x_n-x_{n-1}\|\\ &+\lambda_n\|S_n x_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\|\\ &+\eta_n\delta_n\|u_n-u_{n-1}\|+\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n\|v_n-v_{n-1}\|\\ &+\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\|\\ &+\eta_n a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\|\\ &+\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\|\\ &+|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\|.\end{aligned} \tag{3.6}$$

By the same method as Theorem 3.1 in [18], we have

$$\|S_n x_{n-1}-S_{n-1}x_{n-1}\| \leq \alpha_1^n\frac{2}{1-\kappa}\|x_{n-1}-z\|. \tag{3.7}$$

Since $u_n=T_{r_n^1}(I-r_n^1G_n)x_n$ where $G_n=a_nA+(1-a_n)B$. From the definition of T_{r_n}, we have

$$F_1(u_n,y)+\varphi_1(y)-\varphi_1(u_n)+\langle G_n x_n, y-u_n\rangle+\frac{1}{r_n^1}\langle y-u_n, u_n-x_n\rangle\geq 0 \tag{3.8}$$

and

$$\begin{aligned}F_1(u_{n+1},y)+\varphi_1(y)-\varphi_1(u_{n+1})&+\langle G_{n+1}x_{n+1}, y-u_{n+1}\rangle\\ &+\frac{1}{r_{n+1}^1}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle\geq 0,\end{aligned} \tag{3.9}$$

for all $y\in C$.
From (3.8) and (3.9), we have

$$\begin{aligned}F_1(u_n,u_{n+1})+\varphi_1(u_{n+1})-\varphi_1(u_n)&+\langle G_n x_n, u_{n+1}-u_n\rangle\\ &+\frac{1}{r_n^1}\langle u_{n+1}-u_n, u_n-x_n\rangle\geq 0.\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}F_1(u_{n+1},u_n)+\varphi_1(u_n)-\varphi_1(u_{n+1})&+\langle G_{n+1}x_{n+1}, u_n-u_{n+1}\rangle\\ &+\frac{1}{r_{n+1}^1}\langle u_n-u_{n+1}, u_{n+1}-x_{n+1}\rangle\geq 0.\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$F_1(u_n, u_{n+1}) + \varphi_1(u_{n+1}) - \varphi_1(u_n) + \frac{1}{r_n^1}\langle u_{n+1} - u_n, u_n - x_n + r_n^1 G_n x_n\rangle \geq 0 \tag{3.12}$$

and

$$\begin{aligned} &F_1(u_{n+1}, u_n) + \varphi_1(u_n) - \varphi_1(u_{n+1}) \\ &\qquad + \frac{1}{r_{n+1}^1}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} + r_{n+1}^1 G_{n+1} x_{n+1}\rangle \geq 0. \end{aligned} \tag{3.13}$$

Summing up (3.12) and (3.13), we have

$$\begin{aligned} &\frac{1}{r_n^1}\langle u_{n+1} - u_n u_n - x_n + r_n^1 G_n x_n\rangle \\ &\qquad + \frac{1}{r_{n+1}^1}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1} + r_{n+1}^1 G_{n+1} x_{n+1}\rangle \geq 0. \end{aligned}$$

It follows that

$$\langle u_{n+1} - u_n, \frac{u_n - (I - r_n^1 G_n)x_n}{r_n^1} - \frac{u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}}{r_{n+1}^1}\rangle \geq 0.$$

This implies that

$$\begin{aligned} 0 &\leq \langle u_{n+1} - u_n, u_n - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle \\ &= \langle u_{n+1} - u_n, u_n - u_{n+1}\rangle \\ &\quad + \langle u_{n+1} - u_n, u_{n+1} - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, u_{n+1} - (I - r_n^1 G_n)x_n - \frac{r_n^1}{r_{n+1}^1}(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle \\ &= \langle u_{n+1} - u_n, (I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n \\ &\quad + \left(1 - \frac{r_n^1}{r_{n+1}^1}\right)(u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\big(\|(I - r_{n+1}^1 G_{n+1})x_{n+1} - (I - r_n^1 G_n)x_n\| \\ &\quad + \left|1 - \frac{r_n^1}{r_{n+1}^1}\right| \|u_{n+1} - (I - r_{n+1}^1 G_{n+1})x_{n+1}\|\big). \end{aligned}$$

Then

$$\begin{aligned}
\|u_{n+1}-u_n\| &\leq \|(I-r_{n+1}^1G_{n+1})x_{n+1}-(I-r_n^1G_n)x_n\| \\
&\quad +\frac{1}{r_{n+1}^1}\left|r_{n+1}^1-r_n^1\right|\|u_{n+1}-(I-r_{n+1}^1G_{n+1})x_{n+1}\| \\
&\leq \|(I-r_{n+1}^1G_{n+1})x_{n+1}-(I-r_{n+1}^1G_{n+1})x_n\| \\
&\quad +\|(I-r_{n+1}^1G_{n+1})x_n-(I-r_n^1G_n)x_n\| \\
&\quad +\frac{1}{r_{n+1}^1}\left|r_{n+1}^1-r_n^1\right|\|u_{n+1}-(I-r_{n+1}^1G_{n+1})x_{n+1}\| \\
&\leq \|x_{n+1}-x_n\|+\|r_{n+1}^1G_{n+1}x_n-r_n^1G_nx_n\| \\
&\quad +\frac{1}{r_{n+1}^1}\left|r_{n+1}^1-r_n^1\right|\|u_{n+1}-(I-r_{n+1}^1G_{n+1})x_{n+1}\| \\
&\leq \|x_{n+1}-x_n\| \\
&\quad +\|r_{n+1}^1(a_{n+1}A+(1-a_{n+1})B)x_n-r_{n+1}^1(a_nA+(1-a_n)B)x_n\| \\
&\quad +|r_{n+1}^1-r_n^1|\|G_nx_n\|+\frac{1}{r_{n+1}^1}\left|r_{n+1}^1-r_n^1\right|\|u_{n+1}-(I-r_{n+1}^1G_{n+1})x_{n+1}\| \\
&= \|x_{n+1}-x_n\| \\
&\quad +\|r_{n+1}^1(a_{n+1}-a_n)Ax_n+r_{n+1}^1((1-a_{n+1})-(1-a_n))Bx_n\| \\
&\quad +|r_{n+1}^1-r_n^1|\|G_nx_n\|+\frac{1}{r_{n+1}^1}\left|r_{n+1}^1-r_n^1\right|\|u_{n+1}-(I-r_{n+1}^1G_{n+1})x_{n+1}\| \\
&\leq \|x_{n+1}-x_n\|+r_{n+1}^1|a_{n+1}-a_n|\|Ax_n\|+r_{n+1}^1|a_{n+1}-a_n|\|Bx_n \\
&\quad +|r_{n+1}^1-r_n^1|\|G_nx_n\|+\frac{1}{b}\left|r_{n+1}^1-r_n^1\right|\|u_{n+1}-(I-r_{n+1}^1G_{n+1})x_{n+1}\|.
\end{aligned} \tag{3.14}$$

From (3.14), we have

$$\begin{aligned}
\|u_n-u_{n-1}\| &\leq \|x_n-x_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Bx_{n-1}\| \\
&\quad +|r_n^1-r_{n-1}^1|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|\|u_n-(I-r_n^1G_n)x_n\|.
\end{aligned} \tag{3.15}$$

By using the same method as (3.15), we have

$$\begin{aligned}
\|v_n-v_{n-1}\| &\leq \|x_n-x_{n-1}\|+r_n^2|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^2|a_n-a_{n-1}|\|Bx_{n-1}\| \\
&\quad +|r_n^2-r_{n-1}^2|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|\|v_n-(I-r_n^2G_n)x_n\|.
\end{aligned} \tag{3.16}$$

Substitute (3.7), (3.15) and (3.16) into (3.6), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad +\lambda_n\|x_n-x_{n-1}\|+\lambda_n\|S_nx_{n-1}-S_{n-1}x_{n-1}\|+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\quad +\eta_n\delta_n\|u_n-u_{n-1}\|+\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n\|v_n-v_{n-1}\| \\
&\quad +\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\| \\
&\quad +\eta_n a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\| \\
&\quad +\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\| \\
&\quad +|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\lambda_n\|x_n-x_{n-1}\| \\
&\quad +\lambda_n\left(\alpha_1^n\frac{2}{1-\kappa}\|x_{n-1}-z\|\right)+|\lambda_n-\lambda_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\quad +\eta_n\delta_n(\|x_n-x_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^1|a_n-a_{n-1}|\|Bx_{n-1}\| \\
&\quad +|r_n^1-r_{n-1}^1|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|\|u_n-(I-r_n^1G_n)x_n\|) \\
&\quad +\eta_n|\delta_n-\delta_{n-1}|\|u_{n-1}\|+(1-\delta_n)\eta_n(\|x_n-x_{n-1}\| \\
&\quad +r_n^2|a_n-a_{n-1}|\|Ax_{n-1}\|+r_n^2|a_n-a_{n-1}|\|Bx_{n-1}\| \\
&\quad +|r_n^2-r_{n-1}^2|\|G_{n-1}x_{n-1}\|+\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|\|v_n-(I-r_n^2G_n)x_n\|) \\
&\quad +\eta_n|\delta_n-\delta_{n-1}|\|v_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Dy_{n-1}\| \\
&\quad +\eta_n a_{n-1}|\gamma_n-\gamma_{n-1}|\|Dy_{n-1}\|+\eta_n\gamma_n|a_n-a_{n-1}|\|Ey_{n-1}\| \\
&\quad +\eta_n(1-a_{n-1})|\gamma_n-\gamma_{n-1}|\|Ey_{n-1}\| \\
&\quad +|\eta_n-\eta_{n-1}|\|P_C(I-\gamma_{n-1}J_{n-1})y_{n-1}\| \\
&\le (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+|\beta_n-\beta_{n-1}|M_1 \\
&\quad +\alpha_1^n\frac{2}{1-\kappa}M_1+|\lambda_n-\lambda_{n-1}|M_1 \\
&\quad +c|a_n-a_{n-1}|M_1+c|a_n-a_{n-1}|M_1+|r_n^1-r_{n-1}^1|M_1+\frac{1}{b}\left|r_n^1-r_{n-1}^1\right|M_1 \\
&\quad +|\delta_n-\delta_{n-1}|M_1+c|a_n-a_{n-1}|M_1+c|a_n-a_{n-1}|\|M_1+|r_n^2-r_{n-1}^2|M_1 \\
&\quad +\frac{1}{b}\left|r_n^2-r_{n-1}^2\right|M_1+|\delta_n-\delta_{n-1}|M_1+|a_n-a_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_1 \\
&\quad +|a_n-a_{n-1}|M_1+|\gamma_n-\gamma_{n-1}|M_1+|\eta_n-\eta_{n-1}|M_1,
\end{aligned}$$

where $M_1 := max_{n\in\mathbb{N}}\{\|u\|,\|x_n\|,\|x_n-z\|,\|S_nx_n\|,\|Ax_n\|,\|Bx_n\|,\|G_nx_n\|,$ $\|\|u_n-(I-r_n^1G_n)x_n\|,\|u_n\|,\|v_n-(I-r_n^2G_n)x_n\|,\|v_n\|,\|Dy_n\|,\|Ey_n\|,$ $\|P_C(I-\gamma_nJ_n)y_n\|\}$. From the conditions (ii), (iv) and Lemma 2.2, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.17}$$

Step 3. We show that $\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|v_n-x_n\|=\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|P_C(I-\gamma_nJ_n)y_n-x_n\|=0$. Let $z\in\mathcal{F}$. Since $u_n=T_{r_n^1}(I-r_n^1G_n)x_n$, $v_n=$

$T_{r_n^2}(I - r_n^2 G_n)x_n$ and T_{r_n} is a firmly nonexpensive mapping, we have

$$\begin{aligned}
\|T_{r_n^1}(I - r_n^1 G_n)x_n)) - z\|^2 &= \|T_{r_n^1}(I - r_n^1 G_n)x_n - T_{r_n^1}(I - r_n^1 G_n)z\|^2 \\
&\leq \langle (I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z, u_n - z\rangle \\
&= \frac{1}{2}(\|(I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z\|^2 + \|u_n - z\|^2 \\
&\quad - \|(I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z - u_n + z\|^2) \\
&\leq \frac{1}{2}(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - r_n^1(G_n x_n - G_n z)\|^2) \\
&= \frac{1}{2}(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n)\|^2 - (r_n^1)^2\|G_n x_n - G_n z\|^2 \\
&\quad + 2r_n^1\langle x_n - T_{r_n^1}(I - r_n^1 G_n)x_n, G_n x_n - G_n z\rangle) \\
&\leq \frac{1}{2}(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 - (r_n^1)^2\|G_n x_n - G_n z\|^2 \\
&\quad + 2r_n^1\|x_n - T_{r_n^1}(I - r_n^1 G_n)x_n\|\|G_n x_n - G_n z\|).
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - u_n\|^2 - (r_n^1)^2\|G_n x_n - G_n z\|^2 \\
+ 2r_n^1\|x_n - T_{r_n^1}(I - r_n^1 G_n)x_n\|\|G_n x_n - G_n z\|.
\end{aligned} \tag{3.18}$$

Applying (3.18) and $v_n = T_{r_n^2}(I - r_n^2 G_n)x_n$, we have

$$\begin{aligned}
\|v_n - z\|^2 \leq \|x_n - z\|^2 - \|x_n - v_n\|^2 - (r_n^2)^2\|G_n x_n - G_n z\|^2 \\
+ 2r_n^2\|x_n - T_{r_n^1}(I - r_n^2 G_n)x_n\|\|G_n x_n - G_n z\|.
\end{aligned} \tag{3.19}$$

For every $x, y \in C$, we have

$$\begin{aligned}
\langle G_n x_n - G_n z, x_n - z\rangle &= \langle (a_n A + (1 - a_n)B)\, x_n - (a_n A + (1 - a_n)B)\, z, x_n - z\rangle \\
&= \langle a_n\,(A x_n - Az) + (1 - a_n)\,(B x_n - Bz)\,, x_n - z\rangle \\
&= a_n\,\langle A x_n - Az, x_n - z\rangle + (1 - a_n)\,\langle B x_n - Bz, x_n - z\rangle \\
&\geq a_n\alpha\,\|A x_n - Az\|^2 + (1 - a_n)\beta\,\|B x_n - Bz\|^2.
\end{aligned} \tag{3.20}$$

From the definition of u_n and (3.20), we have

$$\begin{aligned}
\|u_n - z\|^2 &= \|T_{r_n^1}(I - r_n^1 G_n)x_n - T_{r_n^1}(I - r_n^1 G_n)z\|^2 \\
&\leq \|(I - r_n^1 G_n)x_n - (I - r_n^1 G_n)z\|^2 \\
&= \|x_n - z\|^2 - 2r_n^1\langle x_n - z, G_n x_n - G_n z\rangle + (r_n^1)^2\|(G_n x_n - G_n z)\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n^1 a_n\alpha\,\|A x_n - Az\|^2 - 2r_n^1(1 - a_n)\beta\,\|B x_n - Bz\|^2 \\
&\quad + (r_n^1)^2\|a_n(A x_n - Az) + (1 - a_n)(B x_n - Bz)\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n^1 a_n\alpha\,\|A x_n - Az\|^2 - 2r_n^1(1 - a_n)\beta\,\|B x_n - Bz\|^2 \\
&\quad + (r_n^1)^2 a_n\|A x_n - Az\|^2 + (1 - a_n)(r_n^1)^2\|B x_n - B)\|^2 \\
&\leq \|x_n - z\|^2 - r_n^1 a_n(2\alpha - r_n^1)\,\|A x_n - Az\|^2 \\
&\quad - r_n^1(1 - a_n)(2\beta - r_n^1)\,\|B x_n - Bz\|^2.
\end{aligned} \tag{3.21}$$

Applying (3.21) and $v_n = T_{r_n^2}(I - r_n^2 G_n)x_n$, we have

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - r_n^2 a_n(2\alpha - r_n^2)\|Ax_n - Az\|^2 - r_n^2(1 - a_n)(2\beta - r_n^2)\|Bx_n - Bz\|^2. \tag{3.22}$$

From the definition of x_n, (3.21) and (3.22), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|S_n x_n - z\|^2 \\ &\quad + \eta_n\|P_C(I - \gamma_n(a_n D + (1 - a_n)E))y_n - z)\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 + \eta_n\|y_n - z\|^2 \\ &= \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n\|\delta_n(u_n - z) + (1 - \delta_n)(v_n - z)\|^2 \\ &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n(\delta_n\|u_n - z\|^2 + (1 - \delta_n)\|v_n - z\|^2)\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \lambda_n\|x_n - z\|^2 \\ &\quad + \eta_n(\delta_n(\|x_n - z\|^2 - r_n^1 a_n(2\alpha - r_n^1)\|Ax_n - Az\|^2 \\ &\quad - r_n^1(1 - a_n)(2\beta - r_n^1)\|Bx_n - Bz\|^2) + (1 - \delta_n)(\|x_n - z\|^2 \\ &\quad - r_n^2 a_n(2\alpha - r_n^2)\|Ax_n - Az\|^2 - r_n^2(1 - a_n)(2\beta - r_n^2)\|Bx_n - Bz\|^2)) \\ &\leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \eta_n a_n(r_n^1\delta_n(2\alpha - r_n^1) + r_n^2(1 - \delta_n)(2\alpha - r_n^2))\|Ax_n - Az\|^2 \\ &\quad - (1 - a_n)\eta_n(r_n^1\delta_n(2\beta - r_n^1) + r_n^2(1 - \delta_n)(2\beta - r_n^2))\|Bx_n - Bz\|^2.\end{aligned} \tag{3.24}$$

From (3.24), we have

$$\begin{aligned}\eta_n a_n(r_n^1\delta_n(2\alpha - r_n^1) + r_n^2(1 - \delta_n)(2\alpha - r_n^2))\|Ax_n - Az\|^2 \\ \leq \alpha_n\|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ \leq \alpha_n\|u - z\|^2 + (\|x_n - z\|^2 + \|x_{n+1} - z\|)(\|x_{n+1} - x_n\|).\end{aligned}$$

From the condition (i) and (3.17), we have

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.25}$$

By using the same method as (3.25), we have

$$\lim_{n\to\infty}\|Bx_n - Bz\| = 0. \tag{3.26}$$

Since $G_n = a_n A + (1 - a_n)B$, we obtain

$$\|G_n x_n - G_n z\| \leq a_n\alpha\|Ax_n - Az\|^2 + (1 - a_n)\beta\|Bx_n - Bz\|^2.$$

From (3.25) and (3.26), we have

$$\lim_{n\to\infty}\|G_n x_n - G_n z\| = 0. \tag{3.27}$$

From (3.21), (3.22), (3.23) and the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2\\
&\quad+\eta_n(\delta_n\|u_n-z\|^2+(1-\delta_n)\|v_n-z\|^2)\\
&\leq \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2\\
&\quad+\eta_n(\delta_n(\|x_n-z\|^2-\|x_n-u_n\|^2-(r_n^1)^2\|G_nx_n-G_nz\|^2\\
&\quad+2r_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|)\\
&\quad+(1-\delta_n)(\|x_n-z\|^2-\|x_n-v_n\|^2-(r_n^2)^2\|G_nx_n-G_nz\|^2\\
&\quad+2r_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|))\\
&\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\eta_n\delta_n\|x_n-u_n\|^2-(1-\delta_n)\eta_n\|x_n-v_n\|^2\\
&\quad-\eta_n(\delta_n(r_n^1)^2+(1-\delta_n)(r_n^2)^2\|G_nx_n-G_nz\|^2\\
&\quad+2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|\\
&\quad+2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\eta_n\delta_n\|u_n-x_n\|^2 &\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\
&\quad+2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|\\
&\quad+2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|\\
&\leq \alpha_n\|u-z\|^2+(\|x_n-z\|+\|x_{n+1}-z\|)(\|x_{n+1}-x_n\|)\\
&\quad+2\eta_n\delta_nr_n^1\|x_n-T_{r_n^1}(I-r_n^1G_n)x_n\|\|G_nx_n-G_nz\|\\
&\quad+2(1-\delta_n)\eta_nr_n^2\|x_n-T_{r_n^1}(I-r_n^2G_n)x_n\|\|G_nx_n-G_nz\|.
\end{aligned}$$

From the condition (i), (3.17) and (3.27), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.28}$$

By using the same method as (3.28), we have

$$\lim_{n\to\infty}\|v_n-x_n\|=0. \tag{3.29}$$

From the definition of y_n, we have

$$\begin{aligned}
\|y_n-x_n\| &= \|\delta_nu_n+(1-\delta_n)v_n-x_n\|\\
&\leq \delta_n\|u_n-x_n\|+(1-\delta_n)\|v_n-x_n\|.
\end{aligned}$$

From (3.28) and (3.29), we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.30}$$

From the nonexpansiveness of P_C and $z \in \mathcal{F}$, we have

$$\begin{aligned}\|P_C(I-\gamma_n J_n)y_n - z\|^2 &= \|P_C(I-\gamma_n J_n)y_n - P_C(I-\gamma_n J_n)z\|^2\\ &\le \|(I-\gamma_n J_n)y_n - (I-\gamma_n J_n)z\|^2\\ &= \|y_n - z - \gamma_n(J_n y_n - J_n z)\|^2\\ &= \|y_n-z\|^2 - 2\gamma_n\langle y_n-z, J_n y_n - J_n z\rangle + \gamma_n^2\|J_n y_n - J_n z\|^2\\ &\le \|x_n-z\|^2 - 2\gamma_n\langle y_n-z, J_n y_n - J_n z\rangle + \gamma_n^2\|J_n y_n - J_n z\|^2. \end{aligned} \tag{3.31}$$

For every $x, y \in C$, we have

$$\begin{aligned}\langle J_n x - J_n y, x-y\rangle &= a_n\langle Dx - Dy, x-y\rangle + (1-a_n)\langle Ex-Ey, x-y\rangle\\ &\ge a_n\frac{\mu}{L_D^2}\|Dx-Dy\|^2 + (1-a_n)\frac{\rho}{L_E^2}\|Ex-Ey\|^2. \end{aligned} \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\begin{aligned}\|P_C(I-\gamma_n J_n)y_n - z\|^2 &\le \|x_n-z\|^2 - 2\gamma_n\langle y_n-z, J_n y_n - J_n z\rangle + \gamma_n^2\|J_n y_n - J_n z\|^2\\ &\le \|x_n-z\|^2 - 2\gamma_n a_n\frac{\mu}{L_D^2}\|Dy_n - Dz\|^2\\ &\quad - 2\gamma_n(1-a_n)\frac{\rho}{L_E^2}\|Ey_n - Ez\|^2\\ &\quad + \gamma_n^2\|a_n(Dy_n - Dz) + (1-a_n)(Ey_n - Ez)\|^2\\ &\le \|x_n-z\|^2 - 2\gamma_n a_n\frac{\mu}{L_D^2}\|Dy_n - Dz\|^2\\ &\quad - 2\gamma_n(1-a_n)\frac{\rho}{L_E^2}\|Ey_n - Ez\|^2\\ &\quad + \gamma_n^2 a_n\|Dy_n - Dz\|^2 + \gamma_n^2(1-a_n)\|Ey_n - Ez\|^2\\ &= \|x_n-z\|^2 - \gamma_n a_n\left(\frac{2\mu}{L_D^2} - \gamma_n\right)\|Dy_n - Dz\|^2\\ &\quad - \gamma_n(1-a_n)\left(\frac{2\rho}{L_E^2} - \gamma_n\right)\|Ey_n - Ez\|^2. \end{aligned} \tag{3.33}$$

From the definition of x_n, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n\|u-z\|^2 + \beta_n\|x_n-z\|^2 + \lambda_n\|x_n-z\|^2\\ &\quad + \eta_n\|P_C(I-\gamma_n J_{n+1})y_n - z\|^2\\ &\le \alpha_n\|u-z\|^2 + \beta_n\|x_n-z\|^2 + \lambda_n\|x_n-z\|^2\\ &\quad + \eta_n(\|x_n-z\|^2 - \gamma_n a_n\left(\frac{2\mu}{L_D^2} - \gamma_n\right)\|Dy_n - Dz\|^2\\ &\quad - \gamma_n(1-a_n)\left(\frac{2\rho}{L_E^2} - \gamma_n\right)\|Ey_n - Ez\|^2) \end{aligned}$$

$$\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \eta_n\gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n\right) \|Dy_n - Dz\|^2$$
$$- \eta_n\gamma_n (1-a_n) \left(\frac{2\rho}{L_E^2} - \gamma_n\right) \|Ey_n - Ez\|^2 .$$

It implies that

$$\begin{aligned}\eta_n\gamma_n a_n \left(\frac{2\mu}{L_D^2} - \gamma_n\right) \|Dy_n - Dz\|^2 &\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\quad - \eta_n\gamma_n (1-a_n) \left(\frac{2\rho}{L_E^2} - \gamma_n\right) \|Ey_n - Ez\|^2 \\ &\leq \alpha_n \|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &\leq \alpha_n \|u-z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|) \|x_{n+1} - x_n\| .\end{aligned}$$

From the condition (i) and (3.17), we have

$$\lim_{n\to\infty} \|Dy_n - Dz\| = 0. \tag{3.34}$$

By using the same method as (3.35), we have

$$\lim_{n\to\infty} \|Ey_n - Ez\| = 0. \tag{3.35}$$

From the definition of J_n, we have

$$\|J_n y_n - J_n z\| \leq a_n \|Dy_n - Dz\| + (1-a_n) \|Ey_n - Ez\| . \tag{3.36}$$

From (3.34), (3.35) and (3.36), we have

$$\lim_{n\to\infty} \|J_n y_n - J_n z\| = 0. \tag{3.37}$$

From the definition of $P_C(I - \gamma_n J)$ and Lemma 2.11, it implies that

$$\begin{aligned}\|P_C(I-\gamma_n J_n) y_n - z\|^2 &= \|P_C(I-\gamma_n J_n) y_n - P_C(I-\gamma_n J_n) z\|^2 \\ &\leq \langle (I-\gamma_n J_n) y_n - (I-\gamma_n J_n) z, P_C(I-\gamma_n J_n) y_n - z\rangle \\ &= \frac{1}{2}\Big[\|(I-\gamma_n J_n) y_n - (I-\gamma_n J_n) z\|^2 + \|P_C(I-\gamma_n J_n) y_n - z\|^2 \\ &\quad - \|(I-\gamma_n J_n) y_n - (I-\gamma_n J_n) z - (P_C(I-\gamma_n J_n) y_n - z)\|^2\Big] \\ &\leq \frac{1}{2}(\|y_n - z\|^2 + \|P_C(I-\gamma J_n) y_n - z\|^2 \\ &\quad - \|y_n - P_C(I-\gamma_n J_n) y_n - \gamma_n (J_n y_n - J_n z)\|^2) \\ &\leq \frac{1}{2}(\|x_n - z\|^2 + \|P_C(I-\gamma_n J) y_n - z\|^2 \\ &\quad - \|y_n - P_C(I-\gamma_n J_n) y_n\|^2 - \gamma_n^2 \|J_n y_n - J_n z\|^2 \\ &\quad + 2\gamma_n \langle y_n - P_C(I-\gamma_n J_n) y_n, J_n y_n - J_n z\rangle) \\ &\leq \frac{1}{2}(\|x_n - z\|^2 + \|P_C(I-\gamma_n J_n) y_n - z\|^2 \\ &\quad - \|y_n - P_C(I-\gamma_n J_n) y_n\|^2 - \gamma_n^2 \|J_n y_n - J_n z\|^2 \\ &\quad + 2\gamma_n \|y_n - P_C(I-\gamma_n J_n) y_n\| \|J_n y_n - J_n z\|).\end{aligned}$$

It follows that

$$\|P_C(I-\gamma_n J_n)y_n - z\|^2 \le \|x_n - z\|^2 - \|y_n - P_C(I-\gamma_n J_n)y_n\|^2 + 2\gamma\|y_n - P_C(I-\gamma_n J_n)y_n\|\|J_n y_n - J_n z\|. \quad (3.38)$$

From the definition of x_n and (3.38), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2+\eta_n\|P_C(I-\gamma_nJ_n)y_n-z\|^2\\ &\le \alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\lambda_n\|x_n-z\|^2+\eta_n(\|x_n-z\|^2\\ &\quad -\|y_n-P_C(I-\gamma_nJ_n)y_n\|^2+2\gamma_n\|y_n-P_C(I-\gamma_nJ_n)y_n\|\|J_ny_n-J_nz\|)\\ &\le \alpha_n\|u-z\|^2+\|x_n-z\|^2-\eta_n\|y_n-P_C(I-\gamma_nJ_n)y_n\|^2\\ &\quad +2\eta_n\gamma_n\|y_n-P_C(I-\gamma_nJ_n)y_n\|\|J_ny_n-J_nz\|.\end{aligned}$$

It implies that

$$\begin{aligned}\eta_n\|y_n-P_C(I-\gamma_nJ_n)y_n\|^2 &\le \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\ &\quad +2\eta_n\gamma_n\|y_n-P_C(I-\gamma_nJ_n)y_n\|\|J_ny_n-J_nz\|\\ &\le \alpha_n\|u-z\|^2+(\|x_n-z\|-\|x_{n+1}-z\|)\|x_{n+1}-x_n\|\\ &\quad +2\eta_n\gamma_n\|y_n-P_C(I-\gamma_nJ_n)y_n\|\|J_ny_n-J_nz\|.\end{aligned} \quad (3.39)$$

From the condition (i), (3.17), (3.37) and (3.39), we obtain

$$\lim_{n\to\infty}\|y_n-P_C(I-\gamma_nJ_n)y_n\|=0. \quad (3.40)$$

Since

$$\|x_n-P_C(I-\gamma_nJ_n)y_n\|\le\|x_n-y_n\|+\|y_n-P_C(I-\gamma_nJ_n)y_n\|,$$

from (3.30) and (3.40), we have

$$\lim_{n\to\infty}\|x_n-P_C(I-\gamma_nJ_n)y_n\|=0. \quad (3.41)$$

Step 4. We show that $\lim_{n\to\infty}\sup\langle u-z_0, x_n-z_0\rangle\le 0$ where $z_0=P_{\mathcal{F}}u$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\sup\langle u-z_0,x_n-z_0\rangle=\lim_{k\to\infty}\langle u-z_0,x_{n_k}-z_0\rangle. \quad (3.42)$$

Without loss of generality, we may assume that $x_{n_k}\rightharpoonup\omega$ as $k\to\infty$ where $\omega\in C$. From (3.30), we obtain $y_{n_k}\rightharpoonup\omega$ as $k\to\infty$. From (3.28), we have $u_{n_k}\rightharpoonup\omega$ as $k\to\infty$. Assume $\omega\notin VI(C,D)\bigcap VI(C,E)$. From Lemma 2.11 and Lemma 2.4, we have

$$VI(C,D)\bigcap VI(C,E)=VI(C,J_{n_k})=Fix(P_C(I-\gamma_{n_k}J_{n_k})).$$

From the nonexpansiveness of $P_C(I-\gamma_{n_k}J_{n_k})$, (3.41) and Opial's condition , we obtain

$$\begin{aligned}\lim_{k\to\infty}\inf\|y_{n_k}-\omega\| &< \lim_{k\to\infty}\inf\|y_{n_k}-P_C(I-\gamma_{n_k}J_{n_k})\omega\|\\ &\le \lim_{k\to\infty}\inf\|y_{n_k}-P_C(I-\gamma_{n_k}J_{n_k})y_{n_k}\|\\ &\quad +\lim_{k\to\infty}\inf\|P_C(I-\gamma_nJ_{n_k})y_{n_k}-P_C(I-\gamma_{n_k}J_{n_k})\omega\|\\ &\le \lim_{k\to\infty}\inf\|y_{n_k}-\omega\|.\end{aligned}$$

This is a contradiction. Hence

$$\omega \in VI(C,D)\bigcap VI(C,E). \tag{3.43}$$

From the definition of x_n, we have

$$x_{n+1} - x_n = \alpha_n(u - x_n) + \lambda_n(S_n x_n - x_n) + \eta_n(P_C(I - \gamma_n J_n)y_n - x_n).$$

From the condition (i), (3.15) and (3.41), we have

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.44}$$

Assume $\omega \notin \bigcap_{i=1}^{\infty} Fix(T_i)$. From Lemma 2.10 , we have $Fix(S) = \bigcap_{i=1}^{\infty} Fix(T_i)$. Then $\omega \notin Fix(S)$. From Remark 2.9, we have

$$\begin{aligned}\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - S\omega\| \\ &\leq \liminf_{k\to\infty} (\|x_{n_k} - S_{n_k}x_{n_k}\| + \|S_{n_k}x_{n_k} - S_{n_k}\omega\| + \|S_{n_k}\omega - S\omega\|) \\ &\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\| .\end{aligned}$$

This is a contradiction. Then

$$\omega \in Fix(S) = \bigcap_{i=1}^{\infty} Fix(T_i). \tag{3.45}$$

Since

$$F_1(u_n, y) + \varphi_1(y) - \varphi_1(u_n) + \langle G_n x_n, y - u_n\rangle + \frac{1}{r_n^1}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C,$$

from (A2), we have

$$\varphi_1(y) - \varphi_1(u_n) + \langle G_n x_n, y - u_n\rangle + \frac{1}{r_n^1}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n), \forall y \in C.$$

In particular

$$\varphi_1(y) - \varphi_1(u_{n_i}) + \langle G_{n_i} x_{n_i}, y - u_{n_i}\rangle + \frac{1}{r_{n_i}^1}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq F_1(y, u_{n_i}), \forall y \in C.$$

It follows that

$$\varphi_1(y) - \varphi_1(u_{n_i}) + \langle G_{n_i} x_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}^1}\rangle \geq F_1(y, u_{n_i}). \tag{3.46}$$

For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\omega$. From (3.46), we have

$$\begin{aligned}&\varphi_1(y_t) - \varphi_1(u_{n_i}) + \langle y_t - u_{n_i}, G_{n_i} y_t\rangle \\ &\qquad \geq \langle y_t - u_{n_i}, G_{n_i} y_t\rangle - \langle y_t - u_{n_i}, G_{n_i} x_{n_i}\rangle \\ &\qquad\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}^1}\rangle + F_1(y_t, u_{n_i}) \\ &\qquad = \langle y_t - u_{n_i}, G_{n_i} y_t - G_{n_i} u_{n_i} + G_{n_i} u_{n_i}\rangle - \langle y_t - u_{n_i}, G_{n_i} x_{n_i}\rangle \\ &\qquad\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}^1}\rangle + F_1(y_t, u_{n_i}) \\ &\qquad = \langle y_t - u_{n_i}, G_{n_i} y_t - G_{n_i} u_{n_i}\rangle + \langle y_t - u_{n_i}, G_{n_i} u_{n_i} - G_{n_i} x_{n_i}\rangle \\ &\qquad\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}^1}\rangle + F_1(y_t, u_{n_i}).\end{aligned}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|G_{n_i}u_{n_i} - G_{n_i}x_{n_i}\| \to 0$. Since $\frac{u_{n_i}-x_{n_i}}{r^1_{n_i}} \to 0$ and $\langle y_t - u_{n_i}, G_{n_i}y_t - G_{n_i}u_{n_i}\rangle \geq 0$ and (A4), we have

$$\varphi_1(y_t) - \varphi_1(\omega) + \langle y_t - \omega, G_{n_i}y_t\rangle \geq F_1(y_t, \omega). \tag{3.47}$$

Form (A1), (A4) and (3.47), we have

$$\begin{aligned}
0 &= F_1(y_t, y_t) + \varphi_1(y_t) - \varphi_1(y_t) \\
&\leq tF_1(y_t, y) + (1-t)F_1(y_t, \omega) + t\varphi_1(y) + (1-t)\varphi_1(\omega) - \varphi_1(y_t) \\
&\leq tF_1(y_t, y) + (1-t)\varphi_1(y_t) - (1-t)\varphi_1(\omega) + (1-t)\langle y_t - \omega, G_{n_i}y_t\rangle \\
&\quad + t\varphi_1(y) + (1-t)\varphi_1(\omega) - \varphi_1(y_t) \\
&= tF_1(y_t, y) + t\varphi_1(y) - t\varphi_1(y_t) + (1-t)\langle ty + (1-t)\omega - \omega, G_{n_i}y_t\rangle \\
&= tF_1(y_t, y) + t\varphi_1(y) - t\varphi_1(y_t) + (1-t)t\langle y - \omega, G_{n_i}y_t\rangle.
\end{aligned}$$

Dividing by t, we have

$$0 \leq F_1(y_t, y) + \varphi_1(y) - \varphi_1(y_t) + (1-t)\langle y - \omega, G_{n_i}y_t\rangle.$$

Letting $t \to 0$, it follows from (A3), we have

$$0 \leq F_1(\omega, y) + \varphi_1(y) - \varphi_1(\omega) + \langle y - \omega, G_{n_i}\omega\rangle, \forall y \in C. \tag{3.48}$$

From Lemma 2.13, we have

$$\omega \in GMEP(F_1, \varphi_1, a_{n_i}A + (1-a_{n_i})B) = GMEP(F_1, \varphi_1, A)\bigcap GMEP(F_1, \varphi_1, B).$$

By using the same method as (3.48), we have

$$\omega \in GMEP(F_2, \varphi_2, A)\bigcap GMEP(F_2, \varphi_2, B).$$

Hence $\omega \in \mathcal{F}$. Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathcal{F}$, we have

$$\limsup_{n\to\infty} \langle u - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle u - z_0, x_{n_k} - z_0\rangle = \langle u - z_0, \omega - z_0\rangle \leq 0. \tag{3.49}$$

Step 5. Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{\mathcal{F}}u$. From the nonexpansiveness of $P_C(I - \gamma J_n)$, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(u - z_0) + \beta_n(x_n - z_0) + \lambda_n(S_nx_n - z_0) + \eta_n(P_C(I - \gamma_nJ_n)y_n - z_0)\|^2 \\
&\leq \|\beta_n(x_n - z_0) + \lambda_n(S_nx_n - z_0) + \eta_n(P_C(I - \gamma_nJ_n)y_n - z_0)\|^2 \\
&\quad + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&= (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

Applying Lemma 2.2 and (3.49), we have the sequence $\{x_n\}$ converse strongly to $z_0 = P_{\mathcal{F}}u$. This complete the proof. ■

Using our main theorem (Theorem 3.1), we obtain the following results.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfy $A_1) - A_4)$ and $F_i(x,z) \leq F_i(x,y) + F_i(y,z)$ for all $x,y,z \in C$ and $i = 1,2$. Let A, B be α,β-inverse strongly monotone, respectively and D, E be L_D, L_E-Lipschitz continuous and μ,ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap EP(F_1,A) \bigcap EP(F_1,B) \bigcap EP(F_2,A) \bigcap EP(F_2,B) \bigcap VI(C,D) \bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Assume the either $B_1)$ or $B_2)$ holds and let the sequence $\{x_n\}$ generated by $x_1, u \in C$ and*

$$F_1(u_n, y) + \langle a_n A x_n + (1-a_n) B x_n, y - u_n \rangle + \frac{1}{r_n^1} \langle y - u_n, u_n - x_n \rangle \geq 0,$$

$$F_2(v_n, y) + \langle a_n A x_n + (1-a_n) B x_n, y - v_n \rangle + \frac{1}{r_n^2} \langle y - v_n, v_n - x_n \rangle \geq 0,$$

$$y_n = \delta_n u_n + (1-\delta_n) v_n,$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + \lambda_n S_n x_n + \eta_n P_C(I - \gamma_n(a_n D + (1-a_n)E)) y_n, \forall n \geq 1.$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0,1]$ with $\alpha_n + \beta_n + \lambda_n + \eta_n = 1$ for all $n \in \mathbb{N}$, $\{a_n\} \subset (0,1)$ and $\{r_n^j\} \subseteq [b,c] \subset (0, 2min\{\alpha,\beta\})$for all $j = 1,2$. Suppose the following conditions hold:

(i): $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0,\ \{\beta_n\} \subseteq [d,e] \subset (0,1),$

(ii): $0 < \gamma_n \leq min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\},$

(iii): $\lim_{n\to\infty} \delta_n = \delta \in (0,1), \sum_{n=1}^{\infty} \alpha_1^n < \infty,$

(iv): $\sum_{n=1}^{\infty} \left| r_{n+1}^j - r_n^j \right| < \infty \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$
$\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty, \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$
$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$ *for all* $j = 1,2.$

Then, the sequence $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}} u$.

Proof. Put $\varphi_1 \equiv \varphi_2 \equiv 0$ in Theorem 3.1. So, from Theorem 3.1, we obtain the desired result. ∎

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A, B be α,β-inverse strongly monotone, respectively and D, E be L_D, L_E-Lipschitz continuous and μ,ρ-strongly monotone mapping, respectively. Let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strictly pseudo-contractive mapping of C into itself with $\mathcal{F} := \bigcap_{i=1}^{\infty} Fix(T_i) \bigcap VI(C,A) \bigcap VI(C,B) \bigcap VI(C,D) \bigcap VI(C,E) \neq \emptyset$ and $\kappa = sup_{i\in\mathbb{N}}\kappa_i$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\ldots$. For every $n \in \mathbb{N}$, let S_n be S-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$. Let the* *sequence* $\{x_n\}$ *generated by* $x_1, u \in C$ *and*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_n x_n + \eta_n P_C(I - \gamma_n(a_n D + (1-a_n)E))P_C(I - r_n^1(a_n A + (1-a_n)B))x_n, \forall n \geq 1.$$

where the sequence $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\eta_n\}, \{\delta_n\} \subseteq [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n + \eta_n = 1$ *for all* $n \in \mathbb{N}$, $\{a_n\} \subset (0,1)$ *and* $\{r_n^1\} \subseteq [b,c] \subset (0, 2min\{\alpha, \beta\})$ *for all* $j = 1,2$. *Suppose the following conditions hold:*

(i): $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0, \{\beta_n\} \subseteq [d,e] \subset (0,1),$

(ii): $0 < \gamma_n \leq min\{\frac{2\mu}{L_D^2}, \frac{2\rho}{L_E^2}\},$

(iii): $\sum_{n=1}^{\infty} \alpha_1^n < \infty,$

(iv): $\sum_{n=1}^{\infty} |r_{n+1}^1 - r_n^1| < \infty \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$

$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty.$

Then, the sequence $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathcal{F}} u$.

Proof. Putting $F_1 \equiv F_2 \equiv \varphi_1 \equiv \varphi_2 \equiv 0, r_n^1 = r_n^2$ and $v_n = u_n$ in Theorem 3.1, we have.

$$\langle y - u_n, x_n - r_n^1(a_n A x_n + (1 - a_n)B x_u) - u_n \rangle, \forall y \in C.$$

It implies that

$$u_n = P_C(I - r_n^1(a_n A + (1 - a_n)B))x_n.$$

So, from Theorem 3.1 and Remark 2.12, we obtain the desired result. ∎

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