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Efficient Numerical Technique for Solving Integral Equations

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Abstract In this paper, we apply an numerical technique to solve a solution of linear Volterra Integro-Differential Equations. The numerical technique originally developed by Huabsomboon et al. [P. Huabsomboon, B. Novaprateep, H. Kaneko, On Taylor-series expansion techniques for the second kind integral equations, J. Comput. Appl. Math. 234 (2010) 1446–1472] bases on Taylor-series expansion. Our results shown that the technique is simple and efficient.

MSC: 65D30

Keywords: Volterra Integral-Differential Equations; numerical technique; Taylor-series expansion technique

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1. Introduction

Many problems arising in applied mathematics or mathematical physics, can be formulated in two distinct ways; i.e, as differential equations or as integral equations. Integral equations are used as mathematical models for many and varied physical situations. Moreover, integral equations occur as reformulations of other mathematical problems. Many researchers use Galerkin, collocation or quadrature technique to numerically solve the integral equations. In general, these numerical techniques transform the integral equation to a linear system of algebraic equations that can be solved using direct techniques or iterative techniques. However, in many instances, matrices of the linear system is large and dense. As a result, the computational costs are expensive, and numerical solution is very difficult to obtain. On the other hand, an alternative method of approximating a solution was proposed [1]. The new procedure transforms the integral equation to a linear ordinary differential equation that can be solved easily. However,this alternative technique requires boundary conditions. In practical applications, boundary conditions may be difficult to obtain, as most often they must be found from experiments. One of

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alternative approaches is to use a Taylor-series expansion proposed by Y. Ren et al. [1]. In this new approach, boundary conditions are not required.

K. Maleknejad et al. [2], presented a Taylor-series expansion for a class of Volterra integral equations of second kind in form.

$$x(s) - \lambda \int_0^s k(s,t)x(t)dt = f(s), \quad 0 \leq s \leq 1. \tag{1.1}$$

and a system of the second kind Volterra integral equations [3].
Their technique used the Taylor series expansion to expand $x(t)$ defined as follows

$$x(t) \approx x(s) + x'(s)(t-s) + ... + \frac{1}{n!}x^n(t-s)^n. \tag{1.2}$$

They used equation (1.2) to approximate $x(s)$ by $x(t)$.

Huabsomboon et al. [4] applied their method in [6] to obtaining a solution of the second kind Volterra integral equation and solution of a system of the second kind Volterra integral equations. Their method performed on the examples in [2] and [3].They showed two examples that are taken from [2]. Numerical results came from their method are consistently better than those results in [2].

In this paper we use modified Taylor-series expansion technique based on Huabsomboon et al's technique [4] in the new approach for solving the first order linear volterra integro-differential equation of the form

$$x'(s) - \int_0^s k(s,t)x(t)dt = f(s), \quad x(0) = x_0,\ 0 \leq s \leq T \tag{1.3}$$

where the functions $f(s)$ and the kernel $k(s,t)$ are known. Also, Throughout the paper, we assume that $k \in C^n([0,1] \times [0,1]), f \in C^n[0,1]$ and the following condition to hold.

2. Leibnitz Rule for Differential of Integrals

Let $f(x,t)$ be a continuous function, and $\frac{\partial f}{\partial t}$ be continuous in the domain of the $x - t$ plane that includes the rectangle $a \leq x \leq b, t_0 \leq t \leq t_1$, and let

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)dt. \tag{2.1}$$

Then the differentiation of the integral in (2.1) exists and is given by

$$F'(x) = \frac{dF}{dx} = f(x,h(x))\frac{dh(x)}{dx} - f(x,g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}dt. \tag{2.2}$$

If $g(x) = a$ and $h(x) = b$ where a and b are constants, then the Leibnitz rule (2.2) reduces to

$$F'(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x,t)}{\partial x}dt, \tag{2.3}$$

which means that the differentiation and integration can be interchanged.

3. Reducing Double Integrals to Single Integrals

We will show that the double integral can be reduced to a single integral by using this formula

$$\int_0^x \int_0^{x_1} F(t)dtdx_1 = \int_0^x (x-t)F(t)dt. \tag{3.1}$$

This can be done by first setting

$$G(x) = \int_0^x (x-t)F(t)dt, \tag{3.2}$$

where $G(0) = 0$. Differentiating both sides of (2.5) using the reduced Leibnitz rule gives

$$G'(x) = \int_0^x F(t)dt. \tag{3.3}$$

Now by integrating both sides of (3.3) from 0 to x yields

$$G(x) = \int_0^x \int_0^{x_1} F(t)dtdx_1. \tag{3.4}$$

Since the right side of two equations (3.2) and (3.4) are equivalent, to obtain (3.1).

4. Numerical Technique

We first substitute (1.2) for $x(t)$ in the integral in (1.3) , gives

$$\int_0^s k(s,t)dtx(s) + \left[1 - \int_0^s k(s,t)dt\right] x'(s) \tag{4.1}$$
$$- \cdots - \left[\frac{1}{n!}\int_0^s k(s,t)(t-s)^n dt\right] x^{(n)}(s) \approx f(s).$$

Integrate (1.3) with respect to s, gives

$$x(s) - x_0 - \int_0^s \int_0^{s_1} k(s_1,t)x(t)dtds_1 = \int_0^s f(s_1)ds_1. \tag{4.2}$$

Substitute (1.2) for $x(t)$ in the integral in (4.2), gives

$$\left[1 - \int_0^s \int_0^{s_1} k(s_1,t)dtds_1\right] x(s) - \left[\int_0^s \int_0^{s_1} k(s_1,t)(t-s)dtds_1\right] x'(s)$$

$$- \cdots - \left[\frac{1}{n!}\int_0^s \int_0^{s_1} k(s_1,t)(t-s)^n dtds_1\right] x^{(n)}(s) \approx \int_0^s f(s_1)ds_1 + x_0. \tag{4.3}$$

Differentiate (1.3) $(n-1)$ times, we get

$$x''(s) - k(s,s)x(s) - \int_0^s k'_s(s,t)dt = f'(s).$$

$$x'''(s) - k(s,s)x'(s) - 2k'_s(s,s)x(s) - \int_0^s k''_s(s)(s,t)x(t)dt = f''(s).$$

$$\vdots$$

$$x^{(n)}(s) - k(s,s)x^{(n-2)}(s) - \binom{n-1}{1}k_s'(s,s)x^{(n-3)}(s) - \binom{n-1}{2}k_s''(s,s)x^{(n-4)}(s)$$

$$-... - \binom{n-1}{n-2}k_s^{(n-2)}(s,s)x(s) - \int_0^s k_s^{(n-1)}(s,t)x(t)dt = f^{(n-1)}(s). \tag{4.4}$$

where $k_s^{(n)} = \frac{\partial k(s,t)}{\partial s^n}$

Substitute (1.2) for each $x(t)$ in the integral (4.3), we get

$$-\left[k(s,s)+\int_0^s k_s'(s,t)dt\right]x(s)-\int_0^s k_s'(s,t)(t-s)dtx'(s)+\left[1-\frac{1}{2}\int_0^s k_s'(s,t)(t-s)^2\right]x''(s)\approx f'(s).$$

$$-\left[3k_s'(s,s)+\int_0^s k_s''(s,t)dt\right]x(s)-\left[k(s,s)+\int_0^s k_s''(s,t)(t-s)dt\right]x'(s)$$
$$-\frac{1}{2}\int_0^s k_s''(s,t)(t-s)^2dtx''(s)+\left[1-\frac{1}{3!}\int_0^s k_s''(s,t)(t-s)^3dt\right]x'''(s)\approx f''(s).$$

$$\vdots$$

$$-\left[\binom{n-1}{n-2}k_s^{(n-2)}(s,s)+\int_0^s k_s^{(n-1)}(s,t)dt\right]x(s)-\left[\binom{n-1}{n-3}k_s^{(n-3)}(s,s)+\right.$$
$$\left.\int_0^s k_s^{(n-1)}(s,t)(t-s)dt\right]x'(s)-\left[\binom{n-1}{n-4}k_s^{(n-4)}(s,s)+\frac{1}{2!}\int_0^s k_s^{(n-1)}(s,t)(t-s)^2dt\right]x''(s)$$
$$-\cdots-\left[\binom{n-1}{1}k_s'(s,s)+\frac{1}{(n-3)!}\int_0^s (s,t)(t-s)^{n-2}dt\right]x^{(n-2)}-$$
$$\left[k(s,s)+\frac{1}{(n-2)!}\int_0^s k_s^{(n-1)}(s,t)(t-s)^{n-2}dt\right]x^{(n-2)}(s)$$
$$-\frac{1}{(n-1)!}\int_0^s k_s^{(n-1)}(s,t)(t-s)^{n-1}dtx^{(n-1)}(s)+$$
$$\left[1-\frac{1}{n!}\int_0^s k_s^{(n-1)}(s,t)(t-s)^n dt\right]x^{(n)}(s)\approx f^{(n-1)}(s). \tag{4.5}$$

From the equation (4.1), (4.3) and (4.5), we solve the following system of linear equations for $x(s), x'(s), ..., x^{(n)}(s)$.

$$\begin{pmatrix} \int_0^s k(s,t)dt & 1-\int_0^s k(s,t)(t-s)dt & \cdots & -\frac{1}{n!}\int_0^s k(s,t)(t-s)^n dt \\ 1-\int_0^s\int_0^{s_1} k(s_1,t)dtds_1 & \int_0^s\int_0^{s_1} k(s_1,t)(t-s)dtds_1 & \cdots & -\frac{1}{n!}\int_0^s\int_0^{s_1} k(s_1,t)(t-s)^n dtds_1 \\ -k(s,s)-\int_0^s k_s'(s,t)dt & -\int_0^s k_s'(s,t)(t-s)dt & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\binom{n-1}{n-2}k_s^{(n-2)}(s,s) & -\binom{n-1}{n-3}k_s^{(n-3)}(s,s)+ & \cdots & 1-\frac{1}{n!}\int_0^s k_s^{(n-1)}(s,t)(t-s)^n dt \\ \int_0^s k_s^{(n-1)}(s,t)dt & \int_0^s k_s^{(n-1)}(s,t)(t-s)dt & & \end{pmatrix}\times$$

$$\begin{pmatrix} x(s) \\ x'(s) \\ \vdots \\ x^{(n)}(s) \end{pmatrix} = \begin{pmatrix} f(s) \\ \int_0^s f(s_1)ds_1 + x_0 \\ \vdots \\ f^{(n-1)}(s) \end{pmatrix}$$

By solving the above system, we show that this technique generates the excellent approximation for solving the linear integro-differential equation.

5. Numerical Results

In this section, we consider four examples. These examples are the linear volterra integro-differential equations given in [5, 7, 8]. We shown the approximate solutions and absolute errors with the exact solution.

Example 5.1. Consider the Volterra integro-differential equation in [5]

$$x'(s) + \int_0^s x(t)dt = 1, \quad s \in [0, 1], \quad x(0) = 0 \tag{5.1}$$

which has the exact solution $x(s) = \sin(s)$. The numerical approximations for $n = 3$, 4 and 5 are shown in table 1 and the numerical errors are shown in table 2. In Fig.1 also show the exact and approximate solution for $n = 4$.

Table 1. The numerical approximate solutions by using Taylor-series expansion with $n = 3$, 4 and 5.

s	$n = 3$	$n = 4$	$n = 5$	Exact solutions
0	0	0	0	0
0.1	9.9833×10^{-2}	9.9833×10^{-2}	9.9833×10^{-2}	9.9833×10^{-2}
0.2	1.9867×10^{-1}	1.9867×10^{-1}	1.9867×10^{-1}	1.9867×10^{-1}
0.3	2.9552×10^{-1}	2.9552×10^{-1}	2.9552×10^{-1}	2.9552×10^{-1}
0.4	3.8943×10^{-1}	3.8942×10^{-1}	3.8942×10^{-1}	3.8942×10^{-1}
0.5	4.7947×10^{-1}	4.7942×10^{-1}	4.7942×10^{-1}	4.7942×10^{-1}
0.6	5.6479×10^{-1}	5.6464×10^{-1}	5.6472×10^{-1}	5.6464×10^{-1}
0.7	6.4464×10^{-1}	6.4422×10^{-1}	6.4442×10^{-1}	6.4422×10^{-1}
0.8	7.1840×10^{-1}	7.1736×10^{-1}	7.1778×10^{-1}	7.1736×10^{-1}
0.9	7.8562×10^{-1}	7.8333×10^{-1}	7.8408×10^{-1}	7.8333×10^{-1}
1.0	8.4607×10^{-1}	8.4147×10^{-1}	8.4262×10^{-1}	8.4147×10^{-1}

TABLE 2. The absolute error solutions between Taylor-series expansion with n = 3,4 and 5

s	$n=3$	$n=4$	$n=5$
0	0	0	0
0.1	5.7×10^{-10}	1.2×10^{-10}	4.4×10^{-10}
0.2	7.30×10^{-8}	1.49×10^{-8}	5.35×10^{-8}
0.3	1.2334×10^{-6}	2.492×10^{-7}	8.683×10^{-7}
0.4	9.0974×10^{-6}	1.8028×10^{-6}	6.0291×10^{-6}
0.5	4.2515×10^{-5}	8.2145×10^{-6}	2.5985×10^{-5}
0.6	1.4863×10^{-4}	2.7807×10^{-5}	8.1836×10^{-5}
0.7	4.2467×10^{-4}	7.6341×10^{-5}	2.0484×10^{-4}
0.8	1.0456×10^{-3}	1.7899×10^{-4}	4.2676×10^{-4}
0.9	2.2959×10^{-3}	3.7020×10^{-4}	7.5760×10^{-4}
1.0	4.6023×10^{-3}	6.8986×10^{-4}	1.1521×10^{-3}

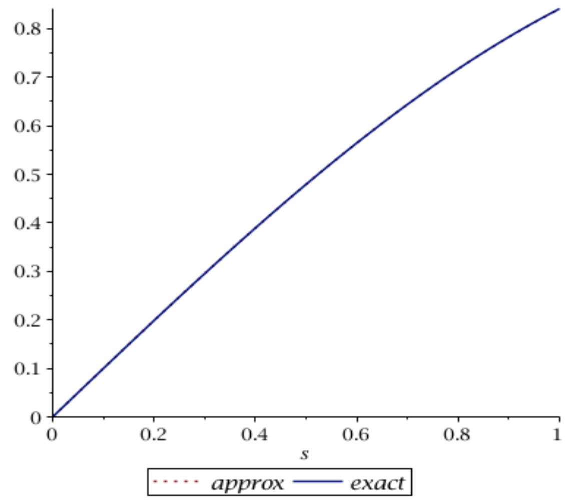

FIGURE 1. Result for Example 1 with n = 4.

Example 5.2. Consider the Volterra integro-differential equation in [5]

$$x'(s) - \int_0^s x(t)dt = -\sin(s), \quad s \in \left[0, \frac{\pi}{2}\right], \quad x(0) = 1. \tag{5.2}$$

which has the exact solution $x(s) = \cos(s)$. The numerical approximations for n = 3, 4 and 5 are shown in table 3 and the numerical error solutions in table 4. In Fig.2 also shows the exact and approximate solution for n = 6.

TABLE 3. The numerical approximate solutions by using Taylor-series expansion with n = 3, 4, 5 and 6 .

s	$n = 3$	$n = 4$	$n = 5$	n=6	Exact solutions
0	1	1	1	1	1
$\frac{\pi}{16}$	9.8078×10^{-1}	9.8078×10^{-1}	9.8078×10^{-1}	9.8078×10^{-1}	9.8078×10^{-1}
$\frac{\pi}{8}$	9.2385×10^{-1}	9.2388×10^{-1}	9.2388×10^{-1}	9.2388×10^{-1}	9.2388×10^{-1}
$\frac{3\pi}{16}$	8.3120×10^{-1}	8.3145×10^{-1}	8.3147×10^{-1}	8.3147×10^{-1}	8.3147×10^{-1}
$\frac{\pi}{4}$	7.0574×10^{-1}	7.0696×10^{-1}	7.0713×10^{-1}	7.0711×10^{-1}	7.0711×10^{-1}
$\frac{5\pi}{16}$	5.5094×10^{-1}	5.5471×10^{-1}	5.5568×10^{-1}	5.5559×10^{-1}	5.5557×10^{-1}
$\frac{3\pi}{8}$	3.7113×10^{-1}	3.7090×10^{-1}	3.8307×10^{-1}	3.8278×10^{-1}	3.8268×10^{-1}
$\frac{7\pi}{16}$	1.7829×10^{-1}	1.8270×10^{-1}	1.9607×10^{-1}	1.9544×10^{-1}	1.9509×10^{-1}
$\frac{\pi}{2}$	-3.1563×10^{-2}	-3.5850×10^{-2}	1.6536×10^{-3}	1.6677×10^{-3}	0

TABLE 4. The absolute error solutions between Taylor-series expansion with n = 3,4,5 and 6.

s	$n = 3$	$n = 4$	$n = 5$	n=6
0	0	0	0	0
$\frac{\pi}{16}$	3.941×10^{-7}	1.4×10^{-9}	4×10^{-10}	4×10^{-10}
$\frac{\pi}{8}$	2.4466×10^{-5}	5.737×10^{-7}	9.40×10^{-8}	1.5×10^{-9}
$\frac{3\pi}{16}$	2.6419×10^{-4}	1.4632×10^{-5}	2.2776×10^{-6}	9.82×10^{-8}
$\frac{\pi}{4}$	1.3677×10^{-3}	1.4520×10^{-4}	2.0833×10^{-5}	1.7299×10^{-6}
$\frac{5\pi}{16}$	4.6248×10^{-3}	8.5780×10^{-4}	1.0904×10^{-4}	1.5988×10^{-5}
$\frac{3\pi}{8}$	1.1558×10^{-2}	3.6487×10^{-3}	3.8655×10^{-4}	9.7827×10^{-5}
$\frac{7\pi}{16}$	2.2198×10^{-2}	1.2389×10^{-2}	9.8078×10^{-4}	4.4924×10^{-4}
$\frac{\pi}{2}$	3.1563×10^{-2}	3.5850×10^{-2}	1.6536×10^{-3}	1.6677×10^{-3}

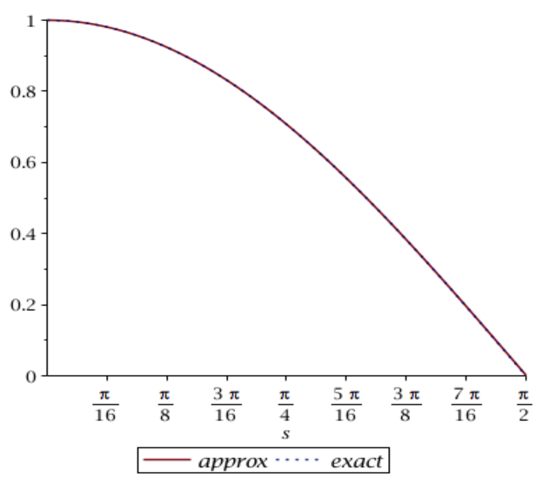

FIGURE 2. Result for Example 2 with n = 6.

Example 5.3. Consider the Volterra integro-differential equation in [7].

$$x'(s) - \int_0^s (s-t)x(t)dt = 1 + s, \quad s \in [0,1], \quad x(0) = 1. \tag{5.3}$$

which has the exact solution $x(s) = e^s$. The numerical approximations for $n = 4$,5,6 and 7 are shown in table 5 and the numerical error solutions are shown in table 6.In Fig.3 also shows the exact and approximate solution for $n = 7$.

Table 5. The numerical approximate solutions by using Taylor-series expansion with $n = 4$, 5, 6 and 7.

s	$n=4$	$n=5$	$n=6$	$n=7$	Exact solutions
0	1	1	1	1	1
0.1	1.1052	1.1052	1.1052	1.1052	1.1052
0.2	1.2214	1.2214	1.2214	1.2214	1.2214
0.3	1.3498	1.3498	1.3498	1.3498	1.3498
0.4	1.4916	1.4916	1.4916	1.4918	1.4918
0.5	1.6479	1.6479	1.6479	1.16487	1.6487
0.6	1.8198	1.8198	1.8198	1.8221	1.8221
0.7	2.0079	2.0079	2.0079	2.0138	2.0137
0.8	2.2125	2.2123	2.2124	2.2259	2.2255
0.9	2.4331	2.4328	2.4328	2.4607	2.4596
1.0	2.6684	2.6676	2.6678	2.7211	2.7183

Table 6. The absolute error solutions between Taylor-series expansion with $n = 4$, 5, 6 and 7.

s	$n=4$	$n=5$	$n=6$	$n=7$
0	0	0	0	0
0.1	5×10^{-8}	5×10^{-8}	5×10^{-8}	5×10^{-8}
0.2	3.1980×10^{-6}	3.2×10^{-6}	3.2×10^{-6}	3×10^{-9}
0.3	3.6426×10^{-5}	3.6463×10^{-5}	3.6461×10^{-5}	9.2×10^{-8}
0.4	2.0459×10^{-4}	2.0497×10^{-4}	2.0494×10^{-4}	1.0180×10^{-6}
0.5	7.8018×10^{-4}	7.8250×10^{-4}	7.8225×10^{-4}	6.7200×10^{-6}
0.6	2.3288×10^{-3}	2.3394×10^{-3}	2.3379×10^{-3}	3.1976×10^{-5}
0.7	5.8708×10^{-3}	5.9092×10^{-3}	5.9029×10^{-3}	1.2152×10^{-4}
0.8	1.3079×10^{-2}	1.3198×10^{-2}	1.3175×10^{-2}	3.9189×10^{-4}
0.9	2.6512×10^{-2}	2.6840×10^{-2}	2.6766×10^{-2}	1.1154×10^{-3}
1.0	4.9893×10^{-2}	5.0706×10^{-2}	5.0497×10^{-2}	2.8782×10^{-3}

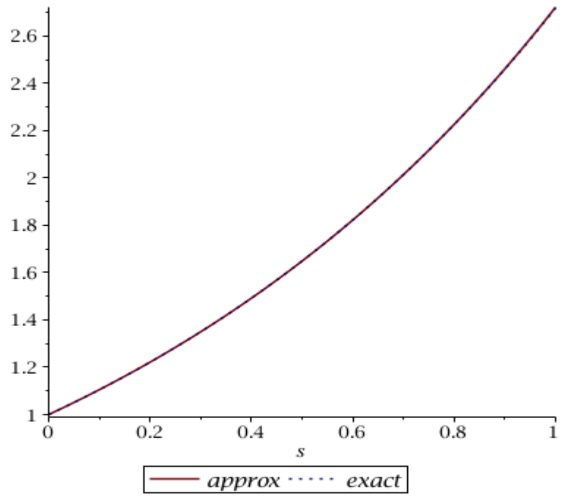

FIGURE 3. Result for Example 3 with $n = 7$.

Example 5.4. Consider the Volterra integro-differential equation in [8].

$$x'(s) - \int_0^s e^{(t+s)} x(t)dt = -se^s - e^{-s}, \quad s \in [0,1], \quad x(0) = 1. \tag{5.4}$$

which has the exact solution $x(s) = e^{-s}$. The numerical approximations and the numerical error solutions for $n = 3$ are shown in table 7.In Fig.4 also shows the exact and approximate solution for $n = 3$.

TABLE 7. The numerical approximate solutions by using Taylor-series expansion with n =3.

s	$n = 3$	Exact solutions	Absolute Error Solutions
0	1	1	0
0.1	0.9048	0.9048	9.7×10^{-9}
0.2	0.8187	0.8187	4.352×10^{-7}
0.3	0.7408	0.7408	4.977×10^{-6}
0.4	0.6702	0.6703	2.805×10^{-6}
0.5	0.6064	0.6065	1.076×10^{-5}
0.6	0.5484	0.5488	3.206×10^{-5}
0.7	0.4958	0.4965	7.785×10^{-5}
0.8	0.4478	0.4493	1.521×10^{-4}
0.9	0.4042	0.4065	2.228×10^{-4}
1	0.3652	0.3678	2.619×10^{-4}

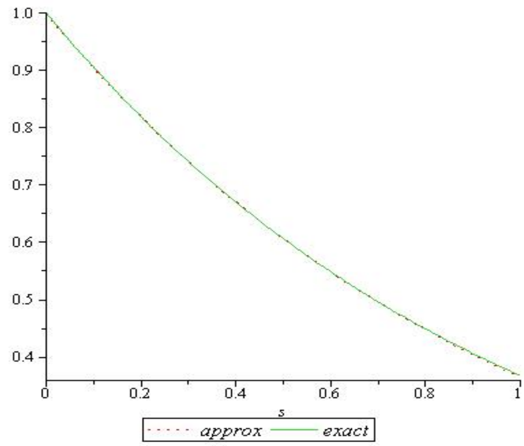

FIGURE 4. Result for Example 4 with $n = 3$.

6. Conclusion

In this study, we apply a new Taylor series technique based on Huabsomboon and his coworker to solve the linear Volterra Integro-Differential Equations. We have found that, this technique is easy to use and yield the accurate solution in a few terms between another method.

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