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Generalized Growth of Special Monogenic Functions Having Finite Convergence Radius

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Abstract In the present paper, we study the growth of special monogenic functions having finite convergence radius. The characterizations of generalized order and generalized type of special monogenic functions having finite convergence radius have been obtained in terms of their Taylor's series coefficients.

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1. Introduction

Clifford analysis offers possibility of generalizing complex function theory to higher dimensions. Several complex variables theory is applicable only for even dimensional spaces but Clifford analysis is applicable for both even and odd dimensional spaces. It considers Clifford algebra valued functions that are defined in open subsets of $\mathbb{R}^n$ for arbitrary finite $n \in \mathbb{N}$ and that are solutions of higher dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions.

Constales et al. ([1] and [2]) have studied the growth properties of entire monogenic functions. They have established the explicit relationship between the order and type of the maximum modulus and the Taylor's series coefficients of entire monogenic functions. Almeida and Kraußhar [3] have obtained generalizations of Wiman-Valiron inequalities for entire monogenic functions. Using these inequalities they have studied the asymptotic growth of entire monogenic functions.

To refine the growth entities Srivastava and Kumar [4] have given the concept of generalized growth of entire monogenic functions. They have obtained the characterizations of generalized order, generalized lower order and generalized type of entire monogenic functions in terms of their Taylor's series coefficients.

Abul-Ez and Constales [5] have obtained several results concerning order and type of special monogenic functions. Recently, Abul-Ez and Almeida [6] have obtained the

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characterizations of growth of special monogenic functions in terms of their Taylor's series coefficients. Kumar [7] have refined the results of Abul-Ez and Almeida [6] by using the concept of generalized growth of special monogenic functions.

Constales et al. [8] have given the concept of growth of monogenic functions having finite convergence radius. They [8] have obtained the characterizations of growth of monogenic functions having finite convergence radius in terms of their Taylor's series coefficients. Recently, Kumar and Bala [9] have refined the results of Constales et al. [8] by using the concept of generalized growth and have obtained the characterizations of generalized order and generalized type of monogenic functions having finite convergence radius in terms of their Taylor's series coefficients.

To the best of our knowledge, no one has studied the growth properties of special monogenic functions having finite convergence radius. In the present paper we have given the concept of growth of special monogenic functions having finite convergence radius. We have obtained the characterizations of generalized order and generalized type of special monogenic functions having finite convergence radius in terms of their Taylor's series coefficients.

In the preliminaries section we have given the definitions of generalized order and generalized type of special monogenic functions having finite convergence radius. In the main section we have obtained the characterizations of generalized order and generalized type of special monogenic functions having finite convergence radius in terms of their Taylor's series coefficients.

2. Preliminaries

In order to make calculations more coincise we use following notations:

$$\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \dots x_n^{m_n} \ , \quad \mathbf{m!} = m_1! \dots m_n! \ , \quad |\mathbf{m}| = m_1 + \dots + m_n \ ,$$

where $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ be $n-$ dimensional multi-index and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Following Almeida and Kraußhar [3] and Constales et al. ([1] and [2]), we give some definitions and associated properties.

By $\{e_1, e_2, \dots, e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The associated real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo $\mathbf{x}^2 = -||\mathbf{x}||^2 e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \,, \qquad i, j = 1, 2, \dots, n \,,$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, 2, \dots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \dots e_{l_r}$, where $1 \leq l_1 < l_2 < \dots < l_r \leq n$, $e_\phi = e_0 = 1$. Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. The conjugation in Clifford algebra Cl_{0n} is defined by $\bar{a} = \sum_A a_A \, \bar{e}_A$, where $\overline{e_A} = \bar{e}_{l_r} \, \bar{e}_{l_{r-1}} \dots \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$ for $j = 1, 2, \dots n$, $\bar{e}_0 = e_0 = 1$. The linear subspace $\text{span}_R\{1, e_1, \dots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_{0n}$ is the so called space of para vectors $z = x_0 + x_1 e_1 + x_2 e_2 + \dots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. Here $x_0 = \text{Sc}(z)$ is scalar part and $\mathbf{x} = x_1 e_1 + x_2 e_2 + \dots + x_n e_n = \text{Vec}(z)$ is vector part of para vector z. The Clifford norm of an arbitrary $a = \sum_A a_A \, e_A$ is given by

$$||a|| = \left(\sum_A |a_A|^2\right)^{1/2}.$$

Also for $b \in Cl_{0n}$, we have $\|ab\| \leq 2^{n/2}\|a\|\,\|b\|$. Each para vector $z \in \mathbb{R}^{n+1}\backslash\{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ which can be represented in the form $z^{-1} = \overline{z}/||z||^2$. In order to make calculations more coincise we use following notations:

$$k_{\mathbf{m}} = \begin{pmatrix} n + |\mathbf{m}| - 1 \\ |\mathbf{m}| \end{pmatrix} = \frac{(n)_{\mathbf{m}}}{\mathbf{m}!},$$
$$(n)_{\mathbf{m}} = n(n+1)\ ...\ (n + |\mathbf{m}| - 1).$$

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, then a function $g : U \to Cl_{0n}$ is called monogenic at a point $z \in U$ if $D\,g(z) = 0$. The functions which are monogenic in the whole space are called entire monogenic functions.
Following Abul-Ez and Constales [10], we consider the class of monogenic polynomials $p_{\mathbf{m}}$ of degree $|\mathbf{m}|$, defined as

$$p_{\mathbf{m}}(z) = \sum_{i+j=|\mathbf{m}|}^{\infty} \frac{\left(\frac{n-1}{2}\right)_i}{i!} \frac{\left(\frac{n+1}{2}\right)_j}{j!}\ (\overline{z})^i\ z^j. \tag{2.1}$$

Let w_n be n-dimensional surface area of $(n+1)$-dimensional unit ball and S^n be $n-$dimensional sphere. Then the class of monogenic polynomials described in (2.1) satisfy ([6], pp. 1259)

$$\frac{1}{w_{\mathbf{m}}} \int_{S^n} \overline{p_{\mathbf{m}}(z)}\ p_{\mathbf{l}}(z)\ dS_z\ = k_{\mathbf{m}} \delta_{|\mathbf{m}||\mathbf{l}|}.$$

Also following Abul-Ez and Almeida [6], we have

$$\max_{\|z\|=r} \|p_{\mathbf{m}}(z)\| = k_{\mathbf{m}} r^{\mathbf{m}}.$$

Now following Abul-Ez and Almeida [6], we give some definitions which shall be used in next section.

Definition 2.1. Let g be a monogenic function in a ball $\|z\| < R$, R is finite real number. Then g is called special monogenic function in $\|z\| < R$, if and only if its Taylor's series has the form

$$g(z) = \sum_{|\mathbf{m}|=0}^{\infty} p_{\mathbf{m}}(z)\ c_{\mathbf{m}}, \quad c_{\mathbf{m}} \in Cl_{0n}. \tag{2.2}$$

Proposition 2.2. (Cauchy's Inequality). *Let* $g : \mathbb{R}^{n+1} \to Cl_{0n}$ *be a special monogenic function in* $\|z\| < R$ *whose Taylor's series representation is given by* (2.2). *Then for* $0 < r < R$

$$\|c_m\| \leq \frac{1}{\sqrt{k_m}} M(r)\, r^{-m},$$

where $M(r) = M(r, g) = \max_{\|z\|=r} \|g(z)\|$ *is the maximum modulus of* g.

Definition 2.3. Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a special monogenic function in $\|z\| < R$ whose Taylor's series representation is given by (2.2). Then we define the order ρ of g as

$$\rho = \limsup_{r \to R} \frac{\log^+ \log^+ M(r)}{\log\left[R/(R-r)\right]}, \tag{2.3}$$

where

$$\log^+ x = \begin{cases} \log x \ , & x > 1 \\ 0 \ , & x \leq 1 \end{cases} .$$

Definition 2.4. Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a special monogenic function in $\|z\| < R$ whose Taylor's series representation is given by (2.2). Then we define the type σ of g having non-zero finite order as

$$\sigma = \limsup_{r \to \infty} \frac{\log^+ M(r)}{\left[R/(R-r)\right]^{\rho}}. \tag{2.4}$$

For generalization of the classical characterizations of growth of analytic functions, Seremeta [11] introduced the concept of generalized order and generalized type with the help of general growth functions as follows:
Let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim_{x \to \infty} \frac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$, for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.
The functions of the form $f(x) = ax + b, 0 < a < \infty, 0 < b < \infty$, are in class L^0 ([12] pp.420).

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii) $\lim_{x \to \infty} \frac{h(cx)}{h(x)} = 1$, for every $c{>}0$, that is $h(x)$ is slowly increasing. The functions of the form $f(x) = \log(ax), 0 < a < \infty$, are in class Λ ([12] pp.420).

Now following Kumar and Bala ([13], [12] and [9]) and Srivastava and Kumar ([14], [15] and [4]), we give definitions of generalized order and generalized type of monogenic functions.

Definition 2.5. Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a special monogenic function in $\|z\| < R$, whose Taylor's series representation is given by (2.2). Then for $\alpha, \beta \in \Lambda$, we define the generalized order $\rho(\alpha, \beta, g)$ of g as

$$\rho = \rho(\alpha, \beta, g) = \limsup_{r\to R} \frac{\alpha\left[\log^+ M(r)\right]}{\beta\left[R/(R-r)\right]}. \tag{2.5}$$

If in the above equation we put $\alpha(r) = \log^+ r$ and $\beta(r) = \log r$, then we get definition of order as defined by (2.3).

Definition 2.6. Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a special monogenic function in $\|z\| < R$, whose Taylor's series representation is given by (2.2). Then for $\alpha,\ \beta,\ \gamma \in L^0$, we define the generalized type $\sigma(\alpha, \beta, \rho, g)$ of g having non-zero finite generalized order as

$$\sigma(\alpha, \beta, \rho, g) = \limsup_{r\to R} \frac{\alpha\left[\log^+ M(r)\right]}{\beta\left(\left[\gamma\left\{R/(R-r)\right\}\right]^{\rho}\right)}. \tag{2.6}$$

If in above equation we put $\alpha(r) = \beta(r) = \gamma(r) = r$, then we get definition of type as defined by (2.4).

3. Main Results

We now prove

Theorem 3.1. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a special monogenic function in $\|z\| < R$, whose Taylor's series representation is given by (2.2). For positive real numbers x and μ_1 set $U(x, \mu_1) = \beta^{-1}\{\mu_1\, \alpha(x)\}$. Assume that for sufficiently large value of x*

$$U\left(x/U\left(x, \mu_1^{-1}\right), \mu_1^{-1}\right) = c_1\, U\left(x, \mu_1^{-1}\right) \ , 0 < c_1 < \infty.$$

Then for $\alpha,\ \beta \in \Lambda$, the generalized order $\rho(\alpha, \beta, g)$ of g is given by

$$\rho = \rho(\alpha, \beta, g) = \limsup_{|\boldsymbol{m}|\to\infty} \frac{\alpha(|\boldsymbol{m}|)}{\beta\left\{|\boldsymbol{m}| / \log^+\left(\|c_{\boldsymbol{m}}\|\, R^{|\boldsymbol{m}|}\right)\right\}}. \tag{3.1}$$

Proof. Write

$$\eta_1 = \limsup_{|\mathbf{m}|\to\infty} \frac{\alpha(|\mathbf{m}|)}{\beta\left\{|\mathbf{m}| / \log^+\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right)\right\}}.$$

Now first we prove that $\eta_1 \le \rho$. The coefficients of a monogenic Taylor's series satisfy Cauchy's inequality, that is

$$\|c_{\mathbf{m}}\| \le \frac{1}{\sqrt{k_{\mathbf{m}}}} M(r)\ r^{-|\mathbf{m}|}. \tag{3.2}$$

Also from (2.5), for $\mu_1 > \rho$ and r sufficiently close to R, we have

$$M(r) \le \exp\left[\alpha^{-1}\left(\mu_1\, \beta\left[R/(R-r)\right]\right)\right].$$

Now from inequality (3.2), we get

$$\|c_{\mathbf{m}}\| \ \leq \frac{1}{\sqrt{k_{\mathbf{m}}}}\, r^{-|\mathbf{m}|} \exp\left[\alpha^{-1}\left(\mu_1\, \beta\left[R/(R-r)\right]\right)\right].$$

Following Abul-Ez and Constales ([5], pp. 148), we have

$$\frac{1}{\sqrt{k_{\mathbf{m}}}} \leq 1,$$

So the above inequality reduces to

$$\|c_{\mathbf{m}}\| \ \leq r^{-|\mathbf{m}|} \exp\left[\alpha^{-1}\left(\mu_1\, \beta\left[R/(R-r)\right]\right)\right].$$

Hence for every r sufficiently close to R, we get

$$\log^{+}\left(\|c_{\mathbf{m}}\|\ R^{|\mathbf{m}|}\right) \ \leq \ -|\mathbf{m}|\ \log^{+}(r/R) + \alpha^{-1}\left(\mu_1\, \beta\left[R/(R-r)\right]\right).$$

Putting

$$r = R\left[1 - 1/U\left(|\mathbf{m}|/U\left(|\mathbf{m}|, \mu_1^{-1}\right), \mu_1^{-1}\right)\right] \ ,$$

we get

$$\log^{+}\left(\|c_{\mathbf{m}}\|\ R^{|\mathbf{m}|}\right) \ \leq -|\mathbf{m}|\log^{+}\left[1 - 1/U\left(|\mathbf{m}|/U\left(|\mathbf{m}|, \mu_1^{-1}\right), \mu_1^{-1}\right)\right] + \\ + |\mathbf{m}|/U\left(|\mathbf{m}|, \mu_1^{-1}\right).$$

Now using the properties of logarithm and assumptions of the theorem, we get for sufficiently large value of $|\mathbf{m}|$

$$\log^{+}\left(\|c_{\mathbf{m}}\|\ R^{|\mathbf{m}|}\right) \ \leq \ C_1\ \left(|\mathbf{m}|/\beta^{-1}\left\{\mu_1^{-1}\,\alpha\left(|\mathbf{m}|\right)\right\}\right),$$

where C_1 is a positive constant. Hence by using the properties of β, we get for sufficiently large value of $|\mathbf{m}|$

$$\frac{\alpha\left(|\mathbf{m}|\right)}{\beta\left\{|\mathbf{m}|/\log^{+}\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right)\right\}} \leq \mu_1.$$

Proceeding to limits as $|\mathbf{m}| \to \infty$, we get $\eta_1 \leq \mu_1$. Since $\mu_1 > \rho$ is arbitrary, we finally get

$$\eta_1 \leq \rho. \tag{3.3}$$

Now we will prove that $\rho \leq \eta_1$. If $\eta_1 = \infty$, then there is nothing to prove. So let us assume that $0 \leq \eta_1 < \infty$. Therefore for a given $\varepsilon > 0$ there exist $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \leq \frac{\alpha\left(|\mathbf{m}|\right)}{\beta\left\{|\mathbf{m}|/\log^{+}\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right)\right\}} \leq \eta_1 + \varepsilon = \overline{\eta_1}$$

or

$$\|c_{\mathbf{m}}\| R^{|\mathbf{m}|} \leq \exp\left[|\mathbf{m}| / \beta^{-1}\left\{\overline{\eta_1}^{-1} \alpha(|\mathbf{m}|)\right\}\right]$$

or

$$\|c_{\mathbf{m}}\| r^{|\mathbf{m}|} \leq r^{|\mathbf{m}|} R^{-|\mathbf{m}|} \exp\left[|\mathbf{m}| / \beta^{-1}\left\{\overline{\eta_1}^{-1} \alpha(|\mathbf{m}|)\right\}\right].$$

Following Abul-Ez and Constales ([5], pp. 148), we have

$$M(r) \leq \sum_{|\mathbf{m}|=0}^{\infty} \|c_{\mathbf{m}}\| k_{\mathbf{m}} r^{|\mathbf{m}|}$$

or

$$M(r) \leq \sum_{|\mathbf{m}|=0}^{\infty} k_{\mathbf{m}} r^{|\mathbf{m}|} R^{-|\mathbf{m}|} \exp\left[|\mathbf{m}| / \beta^{-1}\left\{\overline{\eta_1}^{-1} \alpha(|\mathbf{m}|)\right\}\right].$$

On the lines of proof of the theorem given by Kumar and Bala ([9], Theorem 2.1, pp. 130), we get

$$\rho \leq \eta_1. \tag{3.4}$$

Combining (3.3) and (3.4), we get (3.1). Hence Theorem 3.1 is proved. ■

Next we prove

Theorem 3.2. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a special monogenic function in $\|z\| < R$, whose Taylor's series representation is given by (2.2). For positive real numbers x, μ_2 and ρ set $V(x, \mu_2, \rho) = \gamma^{-1}\left(\left[\beta^{-1}\{\mu_2 \alpha(x)\}\right]^{1/\rho}\right)$.*
Assume that for sufficiently large value of x

$$V\left(\frac{x(\rho+1)}{\rho\, V(x/\rho,\, 1/\mu_2,\, \rho+1)}, \frac{1}{\mu_2}, \rho\right)$$
$$= c_2\, V(x/\rho,\, 1/\mu_2,\, \rho+1), \quad 0 < c_2 < \infty.$$

Then for $\alpha, \beta, \gamma \in L^0$ the generalized type $\sigma(\alpha, \beta, \rho, g)$ of g having generalized order $\rho = \rho(\alpha, \beta, g)$ $(0 < \rho(\alpha, \beta, g) < \infty)$ is given by

$$\sigma(\alpha, \beta, \rho, g)$$
$$= \limsup_{|\mathbf{m}| \to \infty} \frac{\alpha(|\mathbf{m}|/\rho)}{\beta\left\{\left(\gamma\left[(\rho+1)\left\{\rho \log^+\left(\|c_{\mathbf{m}}\| R^{|\mathbf{m}|}\right)^{1/|\mathbf{m}|}\right\}^{-1}\right]\right)^{(\rho+1)}\right\}}. \tag{3.5}$$

Proof. Write

$$\eta_2$$
$$= \limsup_{|\mathbf{m}| \to \infty} \frac{\alpha(|\mathbf{m}|/\rho)}{\beta\left\{\left(\gamma\left[(\rho+1)\left\{\rho \log^+\left(\|c_{\mathbf{m}}\| R^{|\mathbf{m}|}\right)^{1/|\mathbf{m}|}\right\}^{-1}\right]\right)^{(\rho+1)}\right\}}.$$

Now first we prove that $\eta_2 \le \sigma$. From (2.6), for $\mu_2 > \sigma$ and r sufficiently close to R, we have

$$M(r) \ \le \ \exp\left[\alpha^{-1}\left\{\mu_2\, \beta\left(\left[\gamma\left\{R/(R-r)\right\}\right]^{\rho}\right)\right\}\right].$$

Now using (3.2), we get

$$\|c_{\mathbf{m}}\| \ \le \ \frac{1}{\sqrt{k_{\mathbf{m}}}}\, r^{-|\mathbf{m}|} \exp\left[\alpha^{-1}\left\{\mu_2\, \beta\left(\left[\gamma\left\{R/(R-r)\right\}\right]^{\rho}\right)\right\}\right].$$

Now as in the proof of Theorem 3.1, here we have

$$\begin{aligned}\log^{+}\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right) \ &\le \ -\left|\mathbf{m}\right| \log^{+}(r/R)+ \\ &+\alpha^{-1}\left\{\mu_2\, \beta\left(\left[\gamma\left\{R/(R-r)\right\}\right]^{\rho}\right)\right\}.\end{aligned}$$

Putting

$$r = R\left[1-\left\{V\left(\frac{|\mathbf{m}|\ (\rho+1)}{\rho\ V\left(|\mathbf{m}|/\rho,\ 1/\mu_2,\ \rho+1\right)}, \frac{1}{\mu_2}, \rho\right)\right\}^{-1}\right],$$

we get

$$\begin{aligned}&\log^{+}\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right) \\ &\le \ -\left|\mathbf{m}\right| \log^{+}\left[1-\left\{V\left(\tfrac{|\mathbf{m}|\ (\rho+1)}{\rho\ V(|\mathbf{m}|/\rho,\ 1/\mu_2,\ \rho+1)}, \tfrac{1}{\mu_2}, \rho\right)\right\}^{-1}\right]+ \\ &+\left|\mathbf{m}\right| \tfrac{\rho+1}{\rho}\ \left[\gamma^{-1}\left(\left[\beta^{-1}\left\{\mu_2^{-1}\, \alpha\left(\left|\mathbf{m}\right|/\rho\right)\right\}\right]^{1/(\rho+1)}\right)\right]^{-1}.\end{aligned}$$

Now using the properties of logarithm and assumptions of theorem, we get for sufficiently large value of $|\mathbf{m}|$

$$\begin{aligned}\log^{+}\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right) \ &\le \quad C_2\ \left|\mathbf{m}\right| \tfrac{\rho+1}{\rho}\times \\ &\times\ \left[\gamma^{-1}\left(\left[\beta^{-1}\left\{\mu_2^{-1}\, \alpha\left(\left|\mathbf{m}\right|/\rho\right)\right\}\right]^{1/(\rho+1)}\right)\right]^{-1},\end{aligned}$$

where C_2 is a positive constant. Hence by using the properties of α, β and γ, we get for sufficiently large value of $|\mathbf{m}|$

$$\frac{\alpha\left(\left|\mathbf{m}\right|/\rho\right)}{\beta\left\{\left(\gamma\left[(\rho+1)\ \left\{\rho \log^{+}\left(\|c_{\mathbf{m}}\|\, R^{|\mathbf{m}|}\right)^{1/|\mathbf{m}|}\right\}^{-1}\right]\right)^{(\rho+1)}\right\}} \ \le \ \mu_2\,.$$

Proceeding to limits as $|\mathbf{m}| \to \infty$, we get $\eta_2 \le \mu_2$. Since $\mu_2 > \sigma$ is arbitrary, we finally get

$$\eta_2 \le \sigma. \tag{3.6}$$

Now we will prove that $\sigma \le \eta_2$. If $\eta_2 = \infty$, then there is nothing to prove. So let us assume that $0 \le \eta_2 < \infty$. Therefore for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \leq \frac{\alpha(|\mathbf{m}|/\rho)}{\beta\left\{\left(\gamma\left[(\rho+1)\left\{\rho\log^{+}\left(\|c_{\mathbf{m}}\|R^{|\mathbf{m}|}\right)^{1/|\mathbf{m}|}\right\}^{-1}\right]\right)^{(\rho+1)}\right\}}$$
$$\leq \eta_2 + \varepsilon = \overline{\eta_2}$$

or

$$\|c_{\mathbf{m}}\| \, R^{|\mathbf{m}|}$$
$$\leq \ \exp\left\{ \ |\mathbf{m}| \ \tfrac{\rho+1}{\rho} \ \left[\gamma^{-1}\left(\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1} \ \alpha\left(|\mathbf{m}|/\rho\right)\right\}\right]^{1/(\rho+1)}\right)\right]^{-1}\right\}$$

or

$$\|c_{\mathbf{m}}\| \, r^{|\mathbf{m}|} \ \leq \ r^{|\mathbf{m}|} \ R^{-|\mathbf{m}|} \times$$
$$\times \exp\left\{ \ |\mathbf{m}| \ \tfrac{\rho+1}{\rho} \ \left[\gamma^{-1}\left(\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1} \ \alpha\left(|\mathbf{m}|/\rho\right)\right\}\right]^{1/(\rho+1)}\right)\right]^{-1}\right\}.$$

Following Abul-Ez and Constales ([5], pp. 148), we have

$$M(r) \leq \sum_{|\mathbf{m}|=0}^{\infty} \|c_{\mathbf{m}}\| \ k_{\mathbf{m}} \, r^{|\mathbf{m}|}$$

or

$$M(r) \leq \textstyle\sum_{|\mathbf{m}|=0}^{\infty} k_{\mathbf{m}} \ r^{|\mathbf{m}|} \ R^{-|\mathbf{m}|} \exp\left\{ \ |\mathbf{m}| \ \tfrac{\rho+1}{\rho} \times\right.$$
$$\left. \times \left[\gamma^{-1}\left(\left[\beta^{-1}\left\{(\overline{\eta_2})^{-1} \ \alpha\left(|\mathbf{m}|/\rho\right)\right\}\right]^{1/(\rho+1)}\right)\right]^{-1}\right\}$$

On the lines of proof of the theorem given by Kumar and Bala ([9], Theorem 2.2, pp. 135), we get

$$\sigma \leq \eta_2. \tag{3.7}$$

Combining (3.6) and (3.7), we get (3.5). Hence Theorem 3.2 is proved. ∎

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