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Approximate Quasi Solutions of Multiobjective Optimization Problems

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Abstract This paper deals with quasi approximate properly efficient solutions of multiobjective optimization problems involving nonsmooth functions. In this case, some necessary and sufficient conditions for these approximate solutions in the multiobjective are provided via max function. As a consequence, we obtain Fritz-John type necessary conditions for approximate properly efficient solutions of the considered problem by exploiting the corresponding results of the approximate properly efficient solutions.

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1. Introduction

Interest in getting approximate solutions of optimization problems has spread greatly during the past 30 years. This growing interest is essentially due to two reasons. First, mathematical models are an approximation of practical situations. Second, the use of an algorithm in a computer in order to solve an optimization problem often leads to an approximation of the solution.

Particularly, the interest has been emphasized in multiobjective optimization because this area studies decision models widely used in the practice. Moreover, the efficient set of multiobjective optimization might be empty in a non-compact instance, while approximate efficient set might be nonempty under very weaker requirements. The first concepts of approximate solution or ε-efficient solution in multiobjective optimization appear in the works of Kutateladze [1]. Then, White [2] studied six kinds of ε-efficiency for multiobjective optimization, and several researchers studied some properties of these concepts and the relationships between them.

One of the most important notions in multiobjective optimization theory is proper efficiency, to eliminate the situations in which the trade-off between criteria is unbounded.

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Geffrion [3] has studied this problem in finite dimensions with the coordinate ordering and has suggested a restriction to "proper" efficient points which allows for a reasonable characterization. Because of the importance of this concept, it is necessary to filter out the ε-strongly efficient points with unbounded trade-off between criteria, through defining ε-proper efficiency concept. Li and Wang [4] introduced the concept of ε-proper efficiency at first, and obtained necessary conditions for ε-proper efficiency through scalarization methods. Thereafter, Liu [5] derived a necessary and sufficient condition for ε-properly efficient solutions of convex multiobjective optimization. More definitions, generalizations, and characterizations of ε-proper efficiency can be found.

Recently, Beldiman et al. [6] considered approximate quasi (weak, proper) efficiency in multiobjective optimization and derived necessary conditions for these kinds of approximate solutions by using an alternative theorem. Their concepts, generalize definitions of (weak, proper) efficiency as well as ε-(weak, proper) efficiency. Also, Panaitescu and Dogaru [7] derived two necessary conditions for ε-quasi proper efficient points as well as two sufficient conditions for ε-quasi efficient solutions of a general multiobjective optimization.

The layout of the paper is as follows. Section 2 collects definitions, notations and preliminary results that will be used later in the paper. Section 3 establishes necessary conditions for approximate solution to two problems consist of unconstrained multiobjective optimization problem and constrain multiobjective optimizaton problem. Finally, apply multiobjective optimization problems to linear multiobjective optimization problems in Section 4.

2. Preliminaries

Throughout the paper, we use the standard notation of variational analysis. Unless otherwise specified, all spaces under consideration are Asplund spaces (i.e., Banach spaces whose separable subspaces have separable duals) whose norms are always denoted by $\|\cdot\|$. The canonical pairing between space X and its dual X^* is denoted by $\langle\cdot,\cdot\rangle$. The symbol B_X stands for the closed unit ball in X. As usual, the *polar cone* of $\Omega \subset X$ is the set

$$\Omega^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \le 0, \forall x \in \Omega\}.$$

Also, we denote by $\mathbb{R}^m_+$ the nonnegative orthant of $\mathbb{R}^m$ where $m \in \mathbb{N} := \{1, 2, \dots\}$.

Given a set-valued mapping $F : X \rightrightarrows X^*$ between X and its dual X^*, we denote by

$$\limsup_{x\to\bar{x}} F(x) := \left\{ x^* \in X^* \;\middle|\; \begin{array}{c} \exists \text{ sequences } x_n \to \bar{x} \text{ and } x_n^* \xrightarrow{w^*} x^* \\ \text{with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N} \end{array} \right\}, \tag{2.1}$$

the *sequential Painlevé-Kuratowski upper/outer limit* of F as $x \to \bar{x}$. Here the symbol $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

A set $\Omega \subset X$ is called *closed* around $\bar{x} \in \Omega$ if there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is closed. We say that Ω is *locally closed* if Ω is closed around x for every $x \in \Omega$. Let $\Omega \subset X$ be closed around $\bar{x} \in \Omega$.

The *Fréchet/regular normal cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$\hat{N}(\bar{x}, \Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0 \right\}, \tag{2.2}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $x \notin \Omega$, we put $\hat{N}(\bar{x}, \Omega) := \emptyset$.

The *Mordukhovich/limiting normal cone* $\hat{N}(\bar{x}, \Omega)$ to Ω at $\bar{x} \in \Omega$ is obtained from regular normal cones by taking the sequential Painlevé-Kuratowski upper limits as:

$$\hat{N}(\bar{x}, \Omega) := \limsup_{x \xrightarrow{\Omega} \bar{x}} \hat{N}(x; \Omega). \tag{2.3}$$

If $x \notin \Omega$, we put $N(x; \Omega) := \emptyset$.

When X is a finite-dimensional space, the Mordukhovich normal cone enjoys the so-called *robustness property* (see [8, Page 11]), that is,

$$N(\bar{x}; \Omega) = \limsup_{x \to \bar{x}} N(x; \Omega) \;\; \forall \bar{x} \in \Omega. \tag{2.4}$$

For an extended real-valued function $\varphi : X \to \bar{\mathbb{R}} := [-\infty, \infty]$, we set

$$\operatorname{dom} \varphi := \{x \in X \mid \varphi(x) < \infty\}, \quad \operatorname{epi} \varphi := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The *Mordukhovich/limiting subdifferential* of φ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < \infty$ is defined by

$$\partial \varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \operatorname{epi} \varphi)\}. \tag{2.5}$$

If $|\varphi(\bar{x})| = \infty$, then one puts $\partial \varphi(\bar{x}) := \emptyset$.

Considering the *indicator function* $\delta(\cdot; \Omega)$ defined by $\delta(x; \Omega) = 0$ for $x \in \Omega$ and by $\delta(x; \Omega) = \infty$ otherwise, we have (see [8, Proposition 1.79]):

$$N(\bar{x}; \Omega) = \partial \delta(\bar{x}; \Omega) \;\; \forall \bar{x} \in \Omega. \tag{2.6}$$

The *nonsmooth version of Fermat's rule* (see, [8, Proposition 1.114]) is formulated as follows: If $\bar{x}$ is a *local minimizer* for φ, then

$$0 \in \partial \varphi(\bar{x}). \tag{2.7}$$

For a function φ locally Lipschitz at $\bar{x}$ with modulus $\ell > 0$, it holds that (see [8, Corollary 1.81])

$$\|x^*\| \leq \ell \;\; \forall x^* \in \partial \varphi(\bar{x}). \tag{2.8}$$

In what follows, we also use the limiting/Mordukhovich subdifferential sum rule.

Lemma 2.1. [8, Theorem 3.36] *Let $\varphi_i : X \to \bar{\mathbb{R}}$, be lower semicontinuous around $\bar{x} \in X$ for $i = 1, 2, \cdots, n$, and $n \geq 2$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then one has*

$$\partial(\varphi_1 + \varphi_2 + \ldots + \varphi_n)(\bar{x}) \subset \partial \varphi_1(\bar{x}) + \partial \varphi_2(\bar{x}) + \ldots + \partial \varphi_n(\bar{x}). \tag{2.9}$$

Finally in this section, we recall the Ekeland variational principle (see [9]), which is needed for our investigation.

Lemma 2.2. (Ekeland Variational Principle) *Let (X, d) be a complete metric space and $\varphi : X \to \bar{\mathbb{R}}$ be a proper lower semicontinuous function bounded from below. Let $\varepsilon > 0$ and $x_0 \in X$ be given such that $\varphi(x_0) \leq \inf_{x \in X} \varphi(x) + \varepsilon$. Then for any $\lambda > 0$ there is $\bar{x} \in X$ satisfying*

(i) $\varphi(\bar{x}) \leq \varphi(x_0)$,

(ii) $d(\bar{x}, x_0) \leq \lambda$,

(iii) $\varphi(\bar{x}) < \varphi(x) + \dfrac{\varepsilon}{\lambda} d(x, \bar{x})$ *for all* $x \in X \backslash \{\bar{x}\}$.

3. Necessary Conditions for Approximate Solutions

Let Ω be a nonempty closed subset of X, and let $K := \{1, 2, \ldots, m\}$, and $I := \{1, 2, \ldots, p\}$ be index sets. Suppose that $f := (f_k), k \in K$ and $g := (g_i), i \in I$ are locally Lipschitz functions on X.

3.1. Unconstrained Multiobjective Optimization Problem

We consider the unconstrained multiobjective optimization problem:

$$\min\{f(x) \mid x \in X\}. \tag{3.1}$$

In what follows, we deal with approximate efficient solutions of problem (3.1)(see [10, 11]). The interested reader is referred to [6, 12] for various characterizations of weak/proper approximate efficient solutions via scalarization methods.

Definition 3.1. [5, 6] Let $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$.

1. We say that $\bar{x} \in X$ is an ε-weakly efficient solution of problem (3.1) if there is no $x \in X$ such that
$$f_k(x) + \varepsilon_k < f_k(\bar{x}), \quad k \in K. \tag{3.2}$$
2. We say that $\bar{x} \in X$ is an ε-efficient solution of problem (3.1) if there is no $x \in C$ such that
$$f_k(x) + \varepsilon_k \leq f_k(\bar{x}), \quad k \in K, \tag{3.3}$$
with at least one strict inequality.
3. A point $\bar{x} \in X$ is called an ε-quasi-weakly efficient solution of problem (3.1) if there is no $x \in X$ such that
$$f_k(x) + \varepsilon_k\|x - \bar{x}\| < f_k(\bar{x}), \quad k \in K. \tag{3.4}$$
4. A point $\bar{x} \in C$ is called an ε-quasi-efficient solution of problem (3.1) if there is no $x \in X$ such that
$$f_k(x) + \varepsilon_k\|x - \bar{x}\| \leq f_k(\bar{x}), \quad k \in K, \tag{3.5}$$
with at least one strict inequality.

Definition 3.2. [5] A point $\bar{x} \in X$ is called an ε-properly efficient solution of problem (3.1) if it is an ε-efficient solution and there is a real positive number $M > 0$ such that for each $k \in K$ and $x \in X$ satisfying $f_k(x) + \varepsilon_k < f_k(\bar{x})$, there exists at least one an index $j \in K$ such that $f_j(\bar{x}) < f_j(x) + \varepsilon_j$ and

$$\frac{f_k(\bar{x}) - f_k(x) - \varepsilon_k}{f_j(x) - f_j(\bar{x}) + \varepsilon_j} \leq M.$$

Definition 3.3. [6] A point $\bar{x} \in X$ is called an ε-quasi-properly efficient solution of problem (3.1) if it is an ε-quasi-efficient solution and there is a real positive number $M > 0$ such that for each $k \in K$ and $x \in X$ satisfying $f_k(x) + \varepsilon_k\|x - \bar{x}\| < f_k(\bar{x})$, there exists at least one an index $j \in K$ such that $f_j(\bar{x}) < f_j(x) + \varepsilon_j\|x - \bar{x}\|$ and

$$\frac{f_k(\bar{x}) - f_k(x) - \varepsilon_k\|x - \bar{x}\|}{f_j(x) - f_j(\bar{x}) + \varepsilon_j\|x - \bar{x}\|} \leq M.$$

Remark 3.4.

1. ε-proper efficiency $\Rightarrow$ ε-efficiency $\Rightarrow$ ε-weak efficiency;
2. ε-quasi-proper efficiency $\Rightarrow$ ε-quasi-efficiency $\Rightarrow$ ε-quasi-weak efficiency;
3. when $\varepsilon = 0$, an ε-(properly, weakly) efficient solution of problem (3.10) is a (properly, weakly) efficient solution see e.g., [13–15];
4. when $\varepsilon = 0$, an ε-quasi-(properly, weakly) efficient solution of problem (3.10) is aquasi-(properly, weakly) efficient solution.

Hence the case of $\varepsilon \neq 0$ is often of interest when dealing with approximate efficient solutions. For this reason, we always assume here after that $\varepsilon \in \mathbb{R}^m_+ \setminus \{0\}$.

Let $\phi : X \to \mathbb{R}$, θ be a given positive number and the scalar optimization problem:

$$\min\{\phi(x) \mid x \in X\} \tag{3.6}$$

Definition 3.5. [4]

1. A point $\bar{x} \in X$ is called an θ-optimal solution of (3.6) if $\phi(x) \geq \phi(\bar{x}) - \theta$ for any $x \in X$.

2. A point $\bar{x} \in X$ is called an θ-quasi optimal solution of (3.6) if $\phi(x) + \theta\|x - \bar{x}\| \geq \phi(\bar{x})$ for any $x \in X$.

To prove our main results, we quote an alternative theorem for nonconvex functions as follows:

Lemma 3.6. [16, Theorem 3.1] *One and only one of the following alternatives holds:*

1. there exists some $x \in X$ such that

$$f_k(x) < 0, \ \ \forall k \in K;$$

2. for any $k \in K$, and negative numbers δ_k, there exist positive numbers $\lambda_k = \delta_k^{-1}$, such that

$$\max_{k \in K} \lambda_k[f_k(x) - \delta_k] \geq 1, \ \ \forall x \in X.$$

Lemma 3.7. [4] *If $\bar{x} \in X$ is an ε-properly efficient solution of (3.1), then the system*

$$\alpha_k f_k(x) + \rho e f(x) < \alpha_k f_k(\bar{x}) + \rho e f(\bar{x}) - \alpha_k \varepsilon_k - \rho e \varepsilon, \forall k \in K$$

admits no solution $x \in X$, for some $\rho > 0$, where $\alpha_k > 0, k \in K$ and $e := (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

From Lemmas 3.6 and 3.7, we get the following necessary condition for a feasible solution to be an ε-properly efficient solution.

Theorem 3.8. [4] *Any ε-properly efficient solution $\bar{x}$ of (3.1), is an ε_0-optimal solution to the scalar optimization problem*

$$\min\left\{\max_{k \in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] \mid x \in X\right\} \tag{3.7}$$

for some $\rho > 0$, where $\varepsilon_0 = \max_{k \in K} \lambda_k(\varepsilon_k + \rho e \varepsilon)$, $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$, y_k^* is any number such that $\lambda_k > 0, k \in K$ and $y^* := (y_1^*, y_2^*, \ldots, y_m^*)$.

Since [4] has lemma for $\bar{x} \in X$ is ε-properly efficient solution of (3.1), we will prove lemma for $\bar{x} \in X$ is ε-quasi-properly efficient solution of (3.1) as follows.

Lemma 3.9. *If $\bar{x} \in X$ is ε-quasi-properly efficient solution of (3.1), then the system*

$$\alpha_k f_k(x) + \rho e f(x) < \alpha_k f_k(\bar{x}) + \rho e f(\bar{x}) - \alpha_k \varepsilon_k \|x - \bar{x}\| - \rho e \varepsilon \|x - \bar{x}\|, \forall k \in K,$$

admits no solution $x \in X$ for some $\rho > 0$, where $\alpha_k > 0, k \in K$ and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

Proof. Suppose that $\bar{x} \in X$, is an ε-quasi-properly efficient solution of (3.1). Thus $\bar{x}$ is also an ε-quasi efficient solution of (3.1). By the ε-quasi efficiency of $\bar{x}$, the system

$$f_k(x) < f_k(\bar{x}) - \varepsilon_k \|x - \bar{x}\|, \forall k \in K,$$

has no solution x in X. Hence, for any $\alpha_k > 0, k \in K$, the system

$$\alpha_k f_k(x) < \alpha_k f_k(\bar{x}) - \alpha_k \varepsilon_k \|x - \bar{x}\|, \tag{3.8}$$

has no solution x in X. Let $\hat{x} \in X$ be fixed. We discuss in the following two cases.

Case I: We prove that if $ef(\bar{x}) - e\varepsilon\|\hat{x} - \bar{x}\| \leq ef(\hat{x})$, then the system of k inequalities

$$\alpha_k f_k(\hat{x}) + \rho e f(\hat{x}) < \alpha_k f_k(\bar{x}) + \rho e f(\bar{x}) - \alpha_k \varepsilon_k \|\hat{x} - \bar{x}\| - \rho e \varepsilon \|\hat{x} - \bar{x}\|, \forall k \in K,$$

is inconsistent for any $\rho > 0$. This is because if it was not the case, we would have

$$\alpha_k f_k(\bar{x}) - \alpha_k f_k(\hat{x}) - \alpha_k \varepsilon_k \|\hat{x} - \bar{x}\| > \rho e f(\hat{x}) - \rho e f(\bar{x}) + \rho e \varepsilon \|\hat{x} - \bar{x}\| \geq 0, \forall k \in K,$$

which is a contradiction to (3.8).

Case II: If $ef(\bar{x}) - e\varepsilon\|\hat{x} - \bar{x}\| > ef(\hat{x})$, then

$$\bar{K} := \{k \in K \mid f_k(\bar{x}) - \varepsilon_k \|\hat{x} - \bar{x}\| > f_k(\hat{x})\} \neq \emptyset.$$

Since $\bar{x} \in X$ is an ε-quasi-properly efficient solution of (3.1) and by Remark 3.4, $\bar{x}$ is an ε-quasi efficient solution of (3.1). There exists some $\bar{k} \in K$ such that $f_{\bar{k}}(\hat{x}) > f_{\bar{k}}(\bar{x}) - \varepsilon_{\bar{k}} \|\hat{x} - \bar{x}\|$. Let $f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l \|\hat{x} - \bar{x}\| = \max_{k \in K}\{f_k(\hat{x}) - f_k(\bar{x}) + \varepsilon_k \|\hat{x} - \bar{x}\|\}$. Thus $f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l \|\hat{x} - \bar{x}\| > 0$. Since $\bar{x} \in X$ is an ε-quasi-properly efficient solution of (3.1), there exists an $M > 0$ such that for any $k \in K$, there exists an $j \in K$ satisfying $f_j(\hat{x}) > f_j(\bar{x}) - \varepsilon_j \|\hat{x} - \bar{x}\|$ and

$$\frac{f_k(\bar{x}) - f_k(\hat{x}) - \varepsilon_k \|\hat{x} - \bar{x}\|}{f_j(\hat{x}) - f_j(\bar{x}) + \varepsilon_j \|\hat{x} - \bar{x}\|} \leq M.$$

Thus for any $k \in K$,

$$\begin{aligned} f_k(\bar{x}) - f_k(\hat{x}) - \varepsilon_k \|\hat{x} - \bar{x}\| &\leq M[f_j(\hat{x}) - f_j(\bar{x}) + \varepsilon_j \|\hat{x} - \bar{x}\|] \\ &\leq M[f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l \|\hat{x} - \bar{x}\|], \end{aligned}$$

and hence

$$[f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l \|\hat{x} - \bar{x}\|]^{-1} \sum_{k \in K} [f_k(\bar{x}) - f_k(\hat{x}) - \varepsilon_k \|\hat{x} - \bar{x}\|] \leq Mm.$$

Since

$$0 < ef(\bar{x}) - ef(\hat{x}) - e\varepsilon \|\hat{x} - \bar{x}\| \leq \sum_{k \in K} [f_k(\bar{x}) - f_k(\hat{x}) - \varepsilon_k \|\hat{x} - \bar{x}\|],$$

we have

$$[f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l \|\hat{x} - \bar{x}\|]^{-1} [ef(\bar{x}) - ef(\hat{x}) - e\varepsilon \|\hat{x} - \bar{x}\|] \leq Mm.$$

Let $\rho \leq \min_{k\in K} \alpha_k (Mm)^{-1}$. Then

$$\begin{aligned}\rho &\leq \alpha_l (Mm)^{-1}\\ &\leq \frac{\alpha_l[f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l\|\hat{x}-\bar{x}\|]}{ef(\bar{x}) - ef(\hat{x}) - e\varepsilon\|\hat{x}-\bar{x}\|},\end{aligned}$$

i.e.,

$$\rho ef(\bar{x}) - \rho ef(\hat{x}) - \rho e\varepsilon\|\hat{x}-\bar{x}\| \leq \alpha_l[f_l(\hat{x}) - f_l(\bar{x}) + \varepsilon_l\|\hat{x}-\bar{x}\|].$$

Hence

$$\alpha_l f_l(\bar{x}) + \rho ef(\bar{x}) - \alpha_l\varepsilon_l\|\hat{x}-\bar{x}\| - \rho e\varepsilon\|\hat{x}-\bar{x}\| \leq \alpha_l f_l(\hat{x}) + \rho ef(\hat{x}).$$

Therefore, the system of k inequalities

$$\alpha_k f_k(\hat{x}) + \rho ef(\hat{x}) < \alpha_k f_k(\bar{x}) + \rho ef(\bar{x}) - \alpha_k\varepsilon_k\|\hat{x}-\bar{x}\| - \rho e\varepsilon\|\hat{x}-\bar{x}\|, \forall k \in K,$$

is inconsistent.

Noting that $\hat{x}$ can be any element of X, we conclude that the system

$$\alpha_k f_k(x) + \rho ef(x) < \alpha_k f_k(\bar{x}) + \rho ef(\bar{x}) - \alpha_k\varepsilon_k\|x-\bar{x}\| - \rho e\varepsilon\|x-\bar{x}\|, \forall k \in K, x \in X,$$

has no solution for some $\rho > 0$. This completes the proof. ∎

Theorem 3.10. *Any ε-quasi-properly efficient solution $\bar{x}$ of (3.1) is an ε_0-quasi optimal solution to the scalar optimization problem*

$$\min\left\{\max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] \mid x \in X\right\} \tag{3.9}$$

for some $\rho > 0$, where $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$, y_k^* is any number such that $\lambda_k > 0, k \in K$ and $\varepsilon_0 = \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon)$.

Proof. By Lemma 3.9, let $\alpha_k = 1, k \in K$, there exists a $\rho > 0$ such that

$$f_k(x) + \rho ef(x) < f_k(\bar{x}) + \rho ef(\bar{x}) - \varepsilon_k\|x-\bar{x}\| - \rho e\varepsilon\|x-\bar{x}\|, \forall k \in K,$$

has no solution. By Lemma 3.6, for any $k \in K$, and negative numbers δ_k, there exist positive numbers $\lambda_k = -\delta_k^{-1} > 0$, such that

$$1 \leq \max_{k\in K} \lambda_k[(f_k(x) - f_k(\bar{x}) + \varepsilon_k\|x-\bar{x}\|) + \rho e(f(x) - f(\bar{x}) + \varepsilon\|x-\bar{x}\|) - \delta_k], \forall x \in X.$$

Let y_k^* be a number such that $\delta_k = (y_k^* - f_k(\bar{x})) + \rho e(y^* - f(\bar{x})) < 0$, for all $k \in K$. Denote $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$. For any $x \in X$,

$$\begin{aligned}1 &\leq \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*) + (\varepsilon_k + \rho e\varepsilon)\|x-\bar{x}\|]\\ &\leq \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] + \max_{k\in K}(\varepsilon_k + \rho e\varepsilon)\|x-\bar{x}\|\\ &= \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] + \varepsilon_0\|x-\bar{x}\|.\end{aligned}$$

Since

$$\begin{aligned}1 &= \max_{k\in K} \frac{(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)}{(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)}\\ &= \max_{k\in K} \lambda_k[(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)],\end{aligned}$$

we have for any $x \in X$,

$$\max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] + \varepsilon_0\|x - \bar{x}\|$$
$$\geq \max_{k\in K} \lambda_k[(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)].$$

The proof is completed. ∎

3.2. Constrain Multiobjective Optimizaton Problem

The main results in this section are two necessary conditions for a feasible solution to be ε-properly efficient solutions and ε-quasi-properly efficient solutions of this following problem.

We focus on the constrained multiobjective optimization problem :

$$\min\{f(x) \mid x \in C\}, \tag{3.10}$$

where C is the feasible set given by

$$C := \{x \in \Omega \mid g_i(x) \leq 0, i \in I\}. \tag{3.11}$$

To simplify the statements concerning problem (3.10), for fixed $\bar{x} \in X$ and ε-properly efficient solution, we define a real-valued function ψ on X as follows:

$$\psi(x) := \max_{k\in K, i\in I} \{\lambda_k[(f_k(x) - y_k^* + \varepsilon_k) + \rho e(f(x) - y^* + \varepsilon)], g_i(x)\}, \quad x \in X, \tag{3.12}$$

for some $\rho > 0$, where $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$ and y_k^* is any number such that $\lambda_k > 0, k \in K$, which motivated by Theorem 3.8.

Note that since $e := (1, 1, \ldots, 1)^T \in \mathbb{R}^m, \varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \mathbb{R}^m_+$ and $f := (f_1, f_2, \ldots, f_m)$, we can see that $e\varepsilon = (1, 1, \ldots, 1)^T(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) = \sum_{l\in K} \varepsilon_l$ and $ef = (1, 1, \ldots, 1)^T(f_1, f_2, \ldots, f_m) = \sum_{l\in K} f_l$ where $l \in K := \{1, 2, \ldots, m\}$.

The following theorem provides a Fritz-John type necessary condition in a fuzzy form for ε-properly efficient solutions of problem (3.10).

Theorem 3.11. *Let $\bar{x}$ be ε-properly efficient solution of (3.10). For any $\nu > 0$, there exist $x_\nu \in \Omega, \rho > 0, \eta_k \geq 0, k \in K, \mu_i \geq 0, i \in I$ with $\Sigma_{k\in K}\eta_k + \Sigma_{i\in I}\mu_i = 1$, such that $\|x_\nu - \bar{x}\| \leq \nu$ and*

$$0 \in \sum_{k\in K} \eta_k\lambda_k \left(\partial f_k(x_\nu) + \rho \sum_{l\in K} \partial f_l(x_\upsilon)\right) + \sum_{i\in I} \mu_i \partial g_i(x_\nu) + \frac{\varepsilon_0 + 1}{\nu} B_{X^*} + N(x_\nu; \Omega),$$
$$\eta_k\lambda_k \left[(f_k(x_\upsilon) - y_k^* + \varepsilon_k) + \rho\left(\sum_{l\in K} f_l(x_\upsilon) - \sum_{l\in K} y_l^* + \sum_{l\in K} \varepsilon_l\right) - \psi(x_\upsilon)\right] = 0, k \in K,$$
$$\mu_i\left[g_i(x_\nu) - \psi(x_\nu)\right] = 0, \quad i \in I,$$

where $\varepsilon_0 = \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon), \lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}, y_k^*$ is any number such that $\lambda_k > 0$ and the function ψ was defined in (3.12).

Proof. Let $\bar{x}$ be an ε-properly efficient solution of (3.10). Then $\bar{x}$ is an ε-efficient solution and there is a real positive number $M > 0$ such that for each $k \in K$ and $x \in X$ satisfying

$f_k(x)+\varepsilon_k < f_k(\bar{x})$, there exists at least one an index $j \in K$ such that $f_j(\bar{x}) < f_j(x)+\varepsilon_j$ and

$$\frac{f_k(\bar{x}) - f_k(x) - \varepsilon_k}{f_j(x) - f_j(\bar{x}) + \varepsilon_j} \le M.$$

Let us consider the function ψ defined in (3.12),

$$\psi(x) := \max_{k\in K, i\in I} \{\lambda_k[(f_k(x) - y_k^* + \varepsilon_k) + \rho e(f(x) - y^* + \varepsilon)], g_i(x)\}, \ \ x \in X,$$

for some $\rho > 0$, where $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$ and y_k^* is any number such that $\lambda_k > 0, k \in K$. If $x \in C$, then $g_i(x) \le 0, \forall i \in I$. If $x \notin C$ but $x \in \Omega$, then there exists $\bar{i} \in I$ such that $g_{\bar{i}}(x) > 0$. Since $\bar{x} \in \Omega$, $\bar{x}$ is an ε-properly efficient solution and by Lemma 3.7, we obtain that there are $\rho > 0, \alpha_k > 0$ and there is no $x \in \Omega$ such that

$$\alpha_k f_k(x) + \rho e f(x) < \alpha_k f_k(\bar{x}) + \rho e f(\bar{x}) - \alpha_k \varepsilon_k - \rho e \varepsilon, \ \ \forall k \in K.$$

This implies that there is no $x \in \Omega$ such that

$$(f_k(x) - f_k(\bar{x}) + \varepsilon_k) + \rho e(f(x) - f(\bar{x}) + \varepsilon) < 0, \ \ \forall k \in K.$$

Thus there exists $\hat{k} \in K$ such that for each $x \in \Omega$,

$$\begin{aligned} 0 &\le (f_{\hat{k}}(x) - f_{\hat{k}}(\bar{x}) + \varepsilon_{\hat{k}}) + \rho e(f(x) - f(\bar{x}) + \varepsilon) \\ &\le [(f_{\hat{k}}(x) - y_{\hat{k}}^*) - (f_{\hat{k}}(\bar{x}) - y_{\hat{k}}^*) + \varepsilon_{\hat{k}}] + \rho e[(f(x) - y^*) - (f(\bar{x}) - y^*) + \varepsilon], \end{aligned}$$

it follows that

$$(f_{\hat{k}}(\bar{x}) - y_{\hat{k}}^*) + \rho e(f(\bar{x}) - y^*) \le [(f_{\hat{k}}(x) - y_{\hat{k}}^*) + \varepsilon_{\hat{k}}] + \rho e[(f(x) - y^*) + \varepsilon],$$

so

$$\begin{aligned} 0 &\le \frac{(f_{\hat{k}}(x) - f_{\hat{k}}(\bar{x}) + \varepsilon_{\hat{k}}) + \rho e(f(x) - f(\bar{x}) + \varepsilon)}{(f_{\hat{k}}(\bar{x}) - y_{\hat{k}}^*) + \rho e(f(\bar{x}) - y^*)} \\ &= \lambda_{\hat{k}} \left[(f_{\hat{k}}(x) - f_{\hat{k}}(\bar{x}) + \varepsilon_{\hat{k}}) + \rho e(f(x) - f(\bar{x}) + \varepsilon)\right], \end{aligned}$$

where $\lambda_{\hat{k}} = [(f_{\hat{k}}(\bar{x}) - y_{\hat{k}}^*) + \rho e(f(\bar{x}) - y^*)]^{-1}, y_{\hat{k}}^*$ is any number such that $\lambda_{\hat{k}} > 0$. Thus

$$0 \le \psi(x) \ \ \forall x \in \Omega, \tag{3.13}$$

which infers that ψ is bounded from below on Ω. This implies that $\inf_{x\in\Omega} \psi(x)$ exists, which $\inf_{x\in\Omega} \psi(x) \ge 0$.

In addition, due to $\bar{x} \in C$, we obtain that $g_i(\bar{x}) \le 0, \forall i \in I$. We consider

$$\begin{aligned} \psi(\bar{x}) &= \max_{k\in K, i\in I} \{\lambda_k[(f_k(\bar{x}) - y_k^* + \varepsilon_k) + \rho e(f(\bar{x}) - y^* + \varepsilon)], g_i(\bar{x})\} \\ &= \max_{k\in K, i\in I} \{\lambda_k(\varepsilon_k + \rho e\varepsilon) + 1, g_i(\bar{x})\} \\ &= \max_{k\in K} \{\lambda_k(\varepsilon_k + \rho e\varepsilon) + 1\} \\ &= \varepsilon_0 + 1, \end{aligned}$$

where $\varepsilon_0 := \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon)$. This implies that

$$\psi(\bar{x}) \le \inf_{x\in\Omega} \psi(x) + \varepsilon_0 + 1.$$

For any $\upsilon > 0$, applying the Ekeland variational principle Lemma 2.2, there exists $x_\upsilon \in \Omega$ such that $\|x_\upsilon - \bar{x}\| \leq \upsilon$ and

$$\psi(x_\upsilon) < \psi(x) + \frac{\varepsilon_0 + 1}{\upsilon}\|x - x_\upsilon\|, \quad \forall x \in \Omega.$$

This means that x_υ is a minimizer to the scalar optimization problem

$$\min_{x\in\Omega} \varphi(x),$$

where

$$\varphi(x) := \psi(x) + \frac{\varepsilon_0 + 1}{\upsilon}\|x - x_\upsilon\|, \quad x \in \Omega. \tag{3.14}$$

Thus x_υ is a minimizer to the unconstrained scalar optimization problem

$$\min_{x\in X} \varphi(x) + \delta(x; \Omega). \tag{3.15}$$

Applying the nonsmooth version of Fermat's rule (2.7) to the unconstrained scalar optimization problem (3.15), we obtain

$$0 \in \partial\left(\varphi + \delta(\cdot; \Omega)\right)(x_\upsilon). \tag{3.16}$$

Since the function φ is Lipschitz continuous around x_υ and the function $\delta(\cdot; \Omega)$ is lower semicontinuous around x_υ, it follows from the sum rule (2.9) applied to (3.16) and from the relation in (2.6) that

$$0 \in \partial\varphi(x_\upsilon) + N(x_\upsilon; \Omega). \tag{3.17}$$

Note that from evaluated the subdifferential of norm in Banach space [18, Example 4, Page 198],

$$\partial(\|\cdot - x_\upsilon\|)(x_\upsilon) = B_{X^*}.$$

Applying again the sum rule (2.9) to φ defined in (3.14), we get (3.17) that

$$0 \in \partial\psi(x_\upsilon) + \frac{\varepsilon_0 + 1}{\upsilon}B_{X^*} + N(x_\upsilon; \Omega). \tag{3.18}$$

Now, invoking the formula for the Mordukhovich/limiting subdifferential of maximum functions (see [8, Theorem 3.46(ii)]) and the sum rule (2.9) applied to ψ in (3.4) we have

$$\partial\psi(x_\upsilon) \subset \sum_{k\in K} \eta_k\lambda_k\left(\partial f_k(x_\upsilon) + \rho\partial(ef)(x_\upsilon)\right) + \sum_{i\in I} \mu_i\partial g_i(x_\upsilon), \tag{3.19}$$

where

$$\eta_k \geq 0, k \in K, \mu_i \geq 0, i \in I,$$
$$\sum_{k\in K} \eta_k + \sum_{i\in I} \mu_i = 1,$$
$$\eta_k\lambda_k\left[(f_k(x_\upsilon) - y_k^* + \varepsilon_k) + \rho e\left(f(x_\upsilon) - y^* + \varepsilon\right) - \psi(x_\upsilon)\right] = 0, k \in K,$$
$$\mu_i[g_i(x_\upsilon) - \psi(x_\upsilon)] = 0, i \in I.$$

This implies that

$$0 \in \sum_{k\in K} \eta_k\lambda_k\left(\partial f_k(x_\upsilon) + \rho\sum_{l\in K}\partial f_l(x_\upsilon)\right) + \sum_{i\in I}\mu_i\partial g_i(x_\upsilon) + \frac{\varepsilon_0 + 1}{\upsilon}B_{X^*} + N(x_\upsilon; \Omega), \tag{3.20}$$

where

$$\eta_k \geq 0, k \in K, \mu_i \geq 0, i \in I,$$
$$\sum_{k\in K} \eta_k + \sum_{i\in I} \mu_i = 1,$$
$$\eta_k \lambda_k \left[(f_k(x_\upsilon) - y_k^* + \varepsilon_k) + \rho \left(\sum_{l\in K} f_l(x_\upsilon) - \sum_{l\in K} y_l^* + \sum_{l\in K} \varepsilon_l \right) - \psi(x_\upsilon) \right] = 0, k \in K,$$
$$\mu_i [g_i(x_\upsilon) - \psi(x_\upsilon)] = 0, i \in I.$$

■

To simplify the statements concerning problem (3.10), for fixed $\bar{x}$ is ε-quasi-properly efficient solution, we define a real-valued function Φ on X as follows:

$$\Phi(x) := \max_{k\in K, i\in I} \{\lambda_k[(f_k(x) - y_k^* + \varepsilon_k\|x - \bar{x}\|) + \rho e(f(x) - y^* + \varepsilon\|x - \bar{x}\|)], g_i(x)\}. \tag{3.21}$$

for some $\rho > 0$, where $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$ and y_k^* is any number such that $\lambda_k > 0, k \in K$ which motivated by Theorem 3.10 with $\bar{x}$ is an ε_0-quasi optimal solution of (3.9).

The following theorem provides a Fritz-John type necessary condition in a fuzzy form for ε-quasi-properly efficient solutions of problem (3.10).

Theorem 3.12. *Let $\bar{x}$ be an ε-quasi-properly efficient solution of (3.10). Then there exist $\rho > 0, \eta_k \geq 0, k \in K, \mu_i \geq 0, i \in I$ with $\Sigma_{k\in K}\eta_k + \Sigma_{i\in I}\mu_i = 1$ such that*

$$0 \in \sum_{k\in K} \eta_k \lambda_k \left(\partial f_k(\bar{x}) + \rho \sum_{l\in K} \partial f_l(\bar{x}) + \left(\varepsilon_k + \rho \sum_{l\in K} \varepsilon_l \right) B_{X^*} \right) + \sum_{i\in I} \mu_i g_i(\bar{x}) + N(\bar{x}; \Omega),$$
$$\mu_i g_i(\bar{x}) = 0, i \in I. \tag{3.22}$$

Proof. Let $\bar{x}$ be an ε-quasi-properly efficient solution of (3.10).

By Theorem 3.10, there exists $\rho > 0$ such that $\bar{x}$ is an ε_0-quasi optimal solution to the scalar optimization problem (3.9), that is

$$\min \left\{ \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] \mid x \in X \right\},$$

where $\lambda_k = [(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)]^{-1}$ which y_k^* is any number such that $\lambda_k > 0, k \in K$ and $\varepsilon_0 = \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon)$. Let us consider for any $x \in \Omega$ and the function Φ defined in (3.21), that is

$$\Phi(x) := \max_{k\in K, i\in I} \{\lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*) + (\varepsilon_k + \rho e\varepsilon)\|x - \bar{x}\|], g_i(x)\}.$$

If $x \in C$, then $g_i(x) \leq 0, \forall i \in I$. If $x \notin C$ but $x \in \Omega$, then there exists $\hat{i} \in I$ such that $g_{\hat{i}}(x) > 0$. Since $\bar{x}$ is an ε_0-quasi optimal solution to the scalar optimization problem

(3.9), we obtain that for each $x \in \Omega$,

$$\begin{aligned}1 &= \max_{k\in K} \lambda_k[(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)] \\ &\leq \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] + \varepsilon_0\|x - \bar{x}\| \\ &= \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*)] + \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon)\|x - \bar{x}\| \\ &= \max_{k\in K} \lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*) + (\varepsilon_k + \rho e\varepsilon)\|x - \bar{x}\|].\end{aligned}$$

Since $\bar{x} \in C$, we obtain that $g_i(\bar{x}) \leq 0$, for all $i \in I$, so we can consider

$$\begin{aligned}&\Phi(\bar{x}) \\ &= \max_{k\in K, i\in I} \{\lambda_k[(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*) + (\varepsilon_k + \rho e\varepsilon)\|\bar{x} - \bar{x}\|], g_i(\bar{x})\} \\ &= \max_{k\in K, i\in I} \{\lambda_k[(f_k(\bar{x}) - y_k^*) + \rho e(f(\bar{x}) - y^*)], g_i(\bar{x})\} \\ &\leq \max_{k\in K, i\in I} \{\lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*) + (\varepsilon_k + \rho e\varepsilon)\|x - \bar{x}\|], g_i(x)\} \\ &= \Phi(x), \forall x \in \Omega.\end{aligned}$$

This means that $\bar{x}$ is a minimizer to the scalar optimization problem

$$\min_{x\in\Omega} \Phi(x).$$

Thus $\bar{x}$ is a minimizer to the unconstrained scalar optimization problem

$$\min_{x\in X} \Phi(x) + \delta(x; \Omega). \tag{3.23}$$

Applying the nonsmooth version of Fermat's rule (2.7) to the unconstrained scalar optimization problem (3.23), we obtain

$$0 \in \partial\left(\Phi + \delta(\cdot; \Omega)\right)(\bar{x}). \tag{3.24}$$

Since the function Φ is Lipschitz continuous around $\bar{x}$ and the function $\delta(\cdot; \Omega)$ is lower semicontinuous around $\bar{x}$, it follows from the sum rule (2.9) applied to (3.24) and from the relation in (2.6) that

$$0 \in \partial\Phi(\bar{x}) + N(\bar{x}; \Omega). \tag{3.25}$$

Next consider for each $x \in X$,

$$\Phi(x) = \max_{k\in K, i\in I} \{\lambda_k[(f_k(x) - y_k^*) + \rho e(f(x) - y^*) + (\varepsilon_k + \rho e\varepsilon)\|x - \bar{x}\|], g_i(x)\},$$

by evaluated the subdifferential of norm in Banach space [18, Example 4, Page 198],

$$\partial(\|\cdot - \bar{x}\|)(\bar{x}) = B_{X^*}.$$

Now, invoking the formula for the Mordukhovich/limiting subdifferential of maximum functions (see [8, Theorem 3.46(ii)]) and the sum rule (2.9) applied to Φ in (3.21) we have

$$\begin{aligned}
\partial\Phi(\bar{x}) &= \partial\left[\max_{k\in K, i\in I}\{\lambda_k[(f_k(\cdot)-y_k^*)+\rho e(f(\cdot)-y^*)+(\varepsilon_k+\rho e\varepsilon)\|\cdot-\bar{x}\|], g_i(\cdot)\}\right](\bar{x})\\
&\subseteq \partial\left[\sum_{k\in K}\eta_k\lambda_k[(f_k(\cdot)-y_k^*)+\rho e(f(\cdot)-y^*)+(\varepsilon_k+\rho e\varepsilon)\|\cdot-\bar{x}\|]+\sum_{i\in I}\mu_i g_i(\cdot)\right](\bar{x})\\
&\subseteq \sum_{k\in K}\eta_k\lambda_k\left[\partial f_k(\bar{x})+\rho\partial(ef)(\bar{x})+(\varepsilon_k+\rho e\varepsilon)\partial\|\cdot-\bar{x}\|(\bar{x})\right]+\sum_{i\in I}\mu_i g_i(\bar{x})\\
&= \sum_{k\in K}\eta_k\lambda_k\left[\partial f_k(\bar{x})+\rho\partial(ef)(\bar{x})+(\varepsilon_k+\rho e\varepsilon)B_{X^*}\right]+\sum_{i\in I}\mu_i g_i(\bar{x})
\end{aligned}$$

where

$$\begin{aligned}
&\eta_k\geq 0, k\in K, \mu_i\geq 0, i\in I,\\
&\sum_{k\in K}\eta_k+\sum_{i\in I}\mu_i=1,\\
&\mu_i g_i(\bar{x})=0, i\in I.
\end{aligned}$$

This implies that

$$0\in\sum_{k\in K}\eta_k\lambda_k\left(\partial f_k(\bar{x})+\rho\sum_{l\in K}\partial f_l(\bar{x})+\left(\varepsilon_k+\rho\sum_{l\in K}\varepsilon_l\right)B_{X^*}\right)+\sum_{i\in I}\mu_i g_i(\bar{x})+N(\bar{x};\Omega),$$

where

$$\begin{aligned}
&\eta_k\geq 0, k\in K, \mu_i\geq 0, i\in I,\\
&\sum_{k\in K}\eta_k+\sum_{i\in I}\mu_i=1,\\
&\mu_i g_i(\bar{x})=0, i\in I.
\end{aligned}$$

■

4. Applications to Necessary Conditions for Approximate Solutions of Linear Optimization Problem

Now we apply multiobjective optimization problems to linear multiobjective optimization problems. Let Ω' be a nonempty closed subset of $\mathbb{R}^n$, and let $K := \{1, 2, \ldots, m\}$, and $I := \{1, 2, \ldots, p\}$ be index sets.

4.1. Unconstrained Linear Multiobjective Optimization Problem

First of all, we considered the unconstrained linear multiobjective optimization problem:

$$\min\{(c_1^T x, \ldots, c_m^T x) \mid x\in\mathbb{R}^n\} \tag{4.1}$$

where $c_k, x\in\mathbb{R}^n$ and the superscript T stands for transposition.

We start with definitions of efficiency and proper efficiency with refer to [19–22]. Let $\bar{x}^k$ denote a feasible solution where the k-th objective function is minimized. Mostly, the single-objective maximum solutions $\bar{x}^1, \ldots, \bar{x}^m$ do not coincide so that we are forced to find a compromise in an efficient solution. By definition, a feasible solution $\bar{x}$ is referred to as an efficient solution if there is no feasible solution x such that

$$c_k^T x \leq c_k^T \bar{x}, \;\; k \in K,$$

with at least one an index $j \in K$ such that

$$c_j^T x < c_j^T \bar{x}. \tag{4.2}$$

Moreover, the feasible solution $\bar{x}$ is weakly efficient if there is no feasible solution x such that

$$c_k^T x < c_k^T \bar{x}, \;\; k \in K. \tag{4.3}$$

The specification of approximate efficient solutions in Definition 3.1 to problem (4.1) is as follows:

Definition 4.1. Let $\varepsilon := (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{R}^m_+$.

1. We say that $\bar{x} \in \mathbb{R}^n$ is an ε-weakly efficient solution of problem (4.1) if there is no $x \in \mathbb{R}^n$ such that

$$c_k^T x + \varepsilon_k < c_k^T \bar{x}, \;\; k \in K. \tag{4.4}$$

2. We say that $\bar{x} \in \mathbb{R}^n$ is an ε-efficient solution of problem (4.1) if there is no $x \in \mathbb{R}^n$ such that

$$c_k^T x + \varepsilon_k \leq c_k^T \bar{x}, \;\; k \in K, \tag{4.5}$$

with at least one strict inequality.

3. A point $\bar{x} \in \mathbb{R}^n$ is called an ε-quasi-weakly efficient solution of problem (4.1) if there is no $x \in \mathbb{R}^n$ such that

$$c_k^T x + \varepsilon_k \|x - \bar{x}\| < c_k^T \bar{x}, \;\; k \in K. \tag{4.6}$$

4. A point $\bar{x} \in \mathbb{R}^n$ is called an ε-quasi-efficient solution of problem (4.1) if there is no $x \in \mathbb{R}^n$ such that

$$c_k^T x + \varepsilon_k \|x - \bar{x}\| \leq c_k^T \bar{x}, \;\; k \in K, \tag{4.7}$$

with at least one strict inequality.

Definition 4.2. A point $\bar{x} \in \mathbb{R}^n$ is called an ε-properly efficient solution of problem (4.1) if it is an ε-efficient solution and there is a real positive number $M > 0$ such that for each $k \in K$ and $x \in \mathbb{R}^n$ satisfying $c_k^T x + \varepsilon_k < c_k^T \bar{x}$, there exists at least one an index $j \in K$ such that $c_j^T \bar{x} < c_j^T x + \varepsilon_j$ and

$$\frac{c_k^T \bar{x} - c_k^T x - \varepsilon_k}{c_j^T x - c_j^T \bar{x} + \varepsilon_j} \leq M.$$

Definition 4.3. A point $\bar{x} \in \mathbb{R}^n$ is called an ε-quasi-properly efficient solution of problem (4.1) if it is an ε-quasi-efficient solution and there is a real positive number $M > 0$ such that for each $k \in K$ and $x \in \mathbb{R}^n$ satisfying $c_k^T x + \varepsilon_k \|x - \bar{x}\| < c_k^T \bar{x}$, there exists at least one an index $j \in K$ such that $c_j^T \bar{x} < c_j^T x + \varepsilon_j \|x - \bar{x}\|$ and

$$\frac{c_k^T \bar{x} - c_k^T x - \varepsilon_k \|x - \bar{x}\|}{c_j^T x - c_j^T \bar{x} + \varepsilon_j \|x - \bar{x}\|} \leq M.$$

The following Lemma and Theorem show that Lemma and Theorem in section 3.1 can apply to linear multiobjective optimization problems by taking $f_k(x) = c_k^T x$. So $ef(x) = e\bar{C}x$.

Lemma 4.4. *If $\bar{x} \in \mathbb{R}^n$ is an ε-properly efficient solution of (4.1), then the system*

$$\alpha_k c_k^T x + \rho e\bar{C}x < \alpha_k c_k^T \bar{x} + \rho e\bar{C}\bar{x} - \alpha_k \varepsilon_k - \rho e\varepsilon, \forall k \in K, \tag{4.8}$$

admits no solution $x \in \mathbb{R}^n$ for some $\rho > 0$, where $\alpha_k > 0, k \in K$ and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

Theorem 4.5. *Any ε-properly efficient solution $\bar{x}$ of (4.1), is an ε_0-optimal solution to the scalar linear optimization problem*

$$\min\left\{\max_{k\in K} \lambda_k \left[(c_k^T x - y_k^*) + \rho e(\bar{C}x - y^*)\right] \mid x \in \mathbb{R}^n\right\},$$

for some $\rho > 0$, where $\varepsilon_0 = \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon), \lambda_k = [(c_k^T \bar{x} - y_k^*) + \rho e(\bar{C}\bar{x} - y^*)]^{-1}$ and y_k^* is any number such that $\lambda_k > 0, k \in K$.

Lemma 4.6. *If $\bar{x} \in \mathbb{R}^n$ is an ε-quasi-properly efficient solution of (4.1), then the system*

$$\alpha_k c_k^T x + \rho e\bar{C}x < \alpha_k c_k^T \bar{x} + \rho e\bar{C}\bar{x} - \alpha_k \varepsilon_k \|x - \bar{x}\| - \rho e\varepsilon\|x - \bar{x}\|, \forall k \in K, \tag{4.9}$$

admits no solution $x \in \mathbb{R}^n$ for some $\rho > 0$, where $\alpha_k > 0, k \in K$ and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

Theorem 4.7. *Any ε-quasi-properly efficient solution $\bar{x}$ of (4.1) is an ε_0-optimal solution to the scalar optimization problem*

$$\min\left\{\max_{k\in K} \lambda_k[(c_k^T x - y_k^*) + \rho e(\bar{C}x - y^*)] + \varepsilon_0\|x - \bar{x}\| \mid x \in X\right\}, \tag{4.10}$$

for some $\rho > 0$, where $\lambda_k = [(c_k^T \bar{x} - y_k^*) + \rho e(\bar{C}\bar{x} - y^*)]^{-1}$, y_k^* is any number such that $\lambda_k > 0, k \in K$ $\varepsilon_0 = \max_{k\in K} \lambda_k(\varepsilon_k + \rho e\varepsilon)$.

4.2. Constrained Linear Multiobjective Optimization Problem

We consider the following constrained linear multiobjective optimization problem

$$\min\{(c_1^T x, \ldots, c_m^T x) \mid x \in C'\}, \tag{4.11}$$

which C' is the feasible set given by

$$C' := \{x \in \Omega' \mid a_i^T x \geq b_i, i \in I\}, \tag{4.12}$$

where $c_k, a_i, x \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, where $k \in K = \{1, \ldots, m\}, i \in I = \{1, \ldots, p\}$ and $\bar{C} = (c_1, \ldots, c_m)^T$ is $m \times n$ matrix. The superscript T stands for transposition.

The following Theorem show that Theorem in section 3.2 can apply to linear multiobjective optimization problems by taking $f_k(x) = c_k^T x$ and $g_i(x) = b_i - a_i^T x$. So $ef(x) = e\bar{C}x$.

Let $\bar{x} \in \mathbb{R}^n$ and be an ε-properly efficient solution of (4.11), we define a real-valued function $\bar{\psi}$ on $\mathbb{R}^n$ as follows:

$$\bar{\psi}(x) := \max_{k\in K, i\in I} \left\{\lambda_k[(c_k^T x - y_k^* + \varepsilon_k) + \rho e(\bar{C}x - y^* + \varepsilon)], b_i - a_i^T x\right\}, \ \ x \in \mathbb{R}^n, \tag{4.13}$$

for some $\rho > 0$, where $\lambda_k = [(c_k^T\bar{x} - y_k^*) + \rho e(\bar{C}\bar{x} - y^*)]^{-1}$ and y_k^* is any number such that $\lambda_k > 0, k \in K$.

The following theorem provides a Fritz-John type necessary condition in a fuzzy form for ε-properly efficient solutions of problem (4.11).

Theorem 4.8. *Let $\bar{x}$ be an ε-properly efficient solution of (4.11). For any $\nu > 0$, there exist $x_\nu \in \Omega, \rho > 0, \eta_k \geq 0, k \in K, \mu_i \geq 0, i \in I$ with $\Sigma_{k\in K}\eta_k + \Sigma_{i\in I}\mu_i = 1$, such that $\|x_\nu - \bar{x}\| \leq \nu$ and*

$$
\begin{aligned}
&0 \in \sum_{k\in K}\eta_k\lambda_k\left(c_k + \rho\sum_{l\in K}c_l\right) + \sum_{i\in I}\mu_i a_i + \frac{\varepsilon_0 + 1}{\nu}B + N(x_\nu;\Omega),\\
&\eta_k\lambda_k[(c_k^T x_\upsilon - y_k^* + \varepsilon_k) + \rho e(\bar{C}x_\upsilon - y^* + \varepsilon) - \bar{\psi}(x_\upsilon)] = 0, k \in K,\\
&\mu_i\left[a_i^T x_\nu - \bar{\psi}(x_\nu)\right] = 0, \quad i \in I,
\end{aligned}
$$

where $\lambda_k = [(c_k^T\bar{x} - y_k^*) + \rho e(\bar{C}\bar{x} - y^*)]^{-1}$, and y_k^* is any number such that $\lambda_k > 0$.

Let $\bar{x}$ be an ε-quasi-properly efficient solution of (4.11), we define a real-valued function $\bar{\Phi}$ on $\mathbb{R}^n$ as follows:

$$
\bar{\Phi}(x) := \max_{k\in K, i\in I}\left\{\lambda_k[(c_k^T x - y_k^* + \varepsilon_k\|x - \bar{x}\|) + \rho e(\bar{C}x - y^* + \varepsilon\|x - \bar{x}\|)], b_i - a_i^T x\right\}, \tag{4.14}
$$

for some $\rho > 0$, where $\lambda_k = [(c_k^T\bar{x} - y_k^*) + \rho e(\bar{C}\bar{x} - y^*)]^{-1}$ and y_k^* is any number such that $\lambda_k > 0, k \in K$.

The following theorem provides a Fritz-John type necessary condition in a fuzzy form for an ε-quasi-properly efficient solutions of problem (4.11).

Theorem 4.9. *Let $\bar{x}$ be an ε-quasi-properly efficient solution of (4.11). Then there exist $\rho > 0, \eta_k \geq 0, k \in K, \mu_i \geq 0, i \in I$ with $\Sigma_{k\in K}\eta_k + \Sigma_{i\in I}\mu_i = 1$, such that*

$$
\begin{aligned}
&0 \in \sum_{k\in K}\eta_k\lambda_k\left[c_k + \rho\sum_{l\in K}c_l + \left(\varepsilon_k + \rho\sum_{l\in K}\varepsilon_l\right)B\right] - \sum_{i\in I}\mu_i a_i + N(\bar{x};\Omega),\\
&\mu_i b_i - \mu_i a_i^T\bar{x} = 0, i \in I.
\end{aligned} \tag{4.15}
$$

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