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On Generalized Sequence of Functions $B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x; a, k, s)$

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Abstract Authors defined generalized sequence of function $B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x; a, k, s)$ and investigated its various properties viz. generating relations, bilinear generating relation, bilateral generating relations, finite summation formulae and generating functions involving Stirling number.

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1. Introduction

Differential operators plays key role in study of various generalized Rodrigues formulae, Generating relations(including bilateral), Finite summation formulae etc.. In this paper, operator considered as follows

$$\theta \equiv x^a(s + xD) \tag{1.1}$$

where a and s are arbitrary and $D = \frac{d}{dx}$. In particular, consider

$$\theta_1 \equiv x^a(1 + xD) \tag{1.2}$$

Some noteworthy operational formulae obtained

$$\theta^n\{x^{\alpha+k}\} = a^n \left(\frac{s + \alpha + k}{a}\right)_n x^{\alpha+k+na} \tag{1.3}$$

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where k is an integer, n a non-negative integer and α is arbitrary.

$$\theta^n\{xuv\} = x\sum_{k=0}^{n}\binom{n}{k}\theta^k u\,\theta_1^{n-k}v \tag{1.4}$$

where n is non negative integer.

$$\theta^n\{x^\alpha f(x)\} = x^\alpha(\theta+\alpha x^a)^n\{f(x)\} \tag{1.5}$$

$$e^{t\theta}\{x^\alpha f(x)\} = x^\alpha[(1-ax^at)^{-\frac{1}{a}}]^{\alpha+s} f(x[1-ax^at]^{-\frac{1}{a}}) \tag{1.6}$$

Shukla and Prajapati [1–3] introduced a class of polynomials connected with Mittag-Leffler function and its generalization. Prajapati et. al. [4], Salehbhai et. al. [5] defined sequence of functions associated with Mittag-Leffler function. Recently, Ajudia and Prajapati [6, 7] investigated a sequence of functions associted with Wright function and studied its properties, in continuation of the study of sequence of functions. Here, we eshtablished a general sequence of functions defined as,

$$B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s) = \frac{x^{-\delta-an}}{n!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\theta^n\left[\frac{x^\delta}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right] \tag{1.7}$$

where $\alpha,\beta,\gamma\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0,q\in(0,1)\cup\mathbb{N}$ (furthermore, we can consider $q>0$) and $E_{\alpha,\beta}^{\gamma,q}(z)$ is generalized Mittag-Leffler function defined by Shukla and Prajapati [8] as

$$E_{\alpha,\,\beta}^{\gamma,\,q}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{1.8}$$

2. Generating Relations

A considerably large number of special functions (including all the classical orthogonal polynomials) are known to possess generating relations. We used operational technique by employing θ as a differential operator, for obtaining following generating relations of equation (1.7).

$$\sum_{n=0}^{\infty}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)t^n = (1-at)^{-\left(\frac{\delta+s}{a}\right)}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})} \tag{2.1}$$

$$\sum_{n=0}^{\infty}B_{qn}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)t^n = (1+at)^{\frac{\delta+s}{a}-1}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1+at)^{\frac{1}{a}}\})} \tag{2.2}$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n}B_{q(n+m)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)t^n$$
$$= (1-at)^{-m-\left(\frac{\delta+s}{a}\right)}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})}B_{qm}^{(\alpha,\beta,\gamma,\delta)}(x(1-at)^{-\frac{1}{a}};a,k,s) \tag{2.3}$$

$$\sum_{n=0}^{\infty}\binom{m+n}{n}B_{q(n+m)}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)t^n$$
$$=(1+at)^{\left(\frac{\delta+s}{a}\right)-1}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1+at)^{\frac{1}{a}}\})}B_{qm}^{(\alpha,\beta,\gamma,\delta)}(x(1+at)^{\frac{1}{a}};a,k,s) \tag{2.4}$$

Proof. (Proof of (2.1)) From (1.7), consider

$$\sum_{n=0}^{\infty}x^{an}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)t^n=x^{-\delta}E_{\alpha,\beta}^{\gamma,q}(p_k(x))e^{t\theta}\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}$$

above equation reduces to

$$\sum_{n=0}^{\infty}x^{an}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)t^n=x^{-\delta}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\frac{x^{\delta}(1-ax^at)^{-\left(\frac{\delta+s}{a}\right)}}{E_{\alpha,\beta}^{\gamma,q}[p_k\{x(1-ax^at)^{-\frac{1}{a}}\}]}$$

and replacing t by tx^a, which gives (2.1). ■

Proof. (Proof of (2.2)) From (1.7), one can get

$$\sum_{n=0}^{\infty}B_{qn}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)t^n=x^{-\delta}E_{\alpha,\beta}^{\gamma,q}(p_k(x))e^{t\theta}\left\{\frac{x^{\delta-an}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}$$

The simplification of above equation gives,

$$\sum_{n=0}^{\infty}B_{qn}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)t^n=x^{-\delta}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\frac{x^{\delta}(1+ax^at)^{\frac{\delta+s}{a}-1}}{E_{\alpha,\beta}^{\gamma,q}[p_k\{x(1+ax^at)^{\frac{1}{a}}\}]}$$

which proves (2.2). ■

Proof. (Proof of (2.3)) Writing (1.7) as

$$\theta^n\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}=n!\frac{x^{\delta+an}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)$$

therefore,

$$e^{t\theta}\theta^n\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}=n!e^{t\theta}\frac{x^{\delta+an}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)$$

above equation can be written as

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}\theta^{m+n}\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}$$
$$=\frac{n!x^{\delta+an}(1-ax^at)^{-\left(\frac{\delta+s}{a}\right)-n}}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-ax^at)^{-\frac{1}{a}}\})}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x(1-ax^at)^{-\frac{1}{a}};a,k,s)$$

this reduces to

$$\sum_{m=0}^{\infty}\binom{m+n}{m}x^{am}B_{q(n+m)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)t^{m}$$
$$=(1-ax^{a}t)^{-(\frac{\delta+s}{a})-n}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-ax^{a}t)^{-\frac{1}{a}}\})}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x(1-ax^{a}t)^{-\frac{1}{a}};a,k,s)$$

replacing t by tx^a, this leads to (2.3). ∎

Proof. (Proof of (2.4)) Multiply $\dfrac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}$ and then employing θ^m on both the sides of equation (2.2), we have

$$\sum_{n=0}^{\infty}\theta^{m}\left\{\frac{x^{\delta}B_{qn}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}t^{n}=(1+at)^{\frac{\delta+s}{a}-1}\theta^{m}\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1+at)^{\frac{1}{a}}\})}\right\} \tag{2.5}$$

Equation (1.7) can be written as

$$(m+n)!x^{\delta+a(m+n)}\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{q(m+n)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)=\theta^{m+n}\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}$$

i.e.

$$(m+n)!\frac{x^{\delta+a(m+n)}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{q(m+n)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)$$
$$=\theta^{m}\left\{n!\frac{x^{\delta+an}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)\right\} \tag{2.6}$$

replacing δ by $\delta-an$, above equation reduces to

$$\theta^{m}\left\{\frac{x^{\delta}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{qn}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)\right\}$$
$$=\frac{(m+n)!}{n!}\frac{x^{\delta+am}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{q(m+n)}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s) \tag{2.7}$$

From equation (2.5) and (2.7), we get

$$\sum_{n=0}^{\infty}\frac{(m+n)!}{n!}\frac{x^{\delta+am}}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{q(m+n)}^{(\alpha,\beta,\gamma,\delta-an)}(x;a,k,s)t^{n}$$
$$=(1+at)^{\frac{\delta+s}{a}-1}m!\left\{\frac{x^{\delta+am}}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1+at)^{\frac{1}{a}}\})}\right\}B_{qm}^{(\alpha,\beta,\gamma,\delta)}(x(1+at)^{\frac{1}{a}};a,k,s)$$

this follows (2.4). ∎

3. Generating Functions Involving Stirling Number

Riordan [9], defined Stirling number of second kind as

$$S(n,m)=\frac{1}{m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}j^n \tag{3.1}$$

so that

$$S(n,1)=S(n,n)=1, S(n,n-1)=\binom{n}{2} \text{ and } S(n,0)=\begin{cases}1, & n=0\\ 0 & n\in\mathbb{N}\end{cases}$$

The following theorem is useful to get some generating relation for $B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)$;

Theorem 3.1 (Srivastava (Theorem 1) [10]). Let the sequence $\{\zeta_n(x)\}_{n=0}^{\infty}$ be generated by

$$\sum_{m=0}^{\infty}\binom{n+m}{m}\zeta_{m+n}(x)t^m=f(x,t)\{g(x,t)\}^{-n}\zeta_n(h(x,t)) \tag{3.2}$$

where f, g and h are suitable function of x and t.
Then the following family of generating function

$$\sum_{m=0}^{\infty}m^n\zeta_m(h(x,-z))z^m\{g(x,-z)\}^{-m}=\{f(x,-z)\}^{-1}\sum_{m=0}^{n}m!S(n,m)\zeta_m(x)z^m \tag{3.3}$$

holds true provided that each member of (3.3) exists.

The generating functions (2.3) and (2.4) related to the family given by (3.2). By comparing (2.3) and (3.2); $g(x,t)=(1-at), \zeta_m(x)=B_{qm}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)$, $h(x,t)=x(1-at)^{-\frac{1}{a}}$ and $f(x,t)=(1-at)^{-\left(\frac{\delta+s}{a}\right)}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})}$, we get

$$\sum_{m=0}^{\infty}m^nB_{qm}^{(\alpha,\beta,\gamma,\delta)}(x(1+az)^{-\frac{1}{a}};a,k,s)z^m(1+az)^{-m}$$
$$=(1+az)^{\left(\frac{\delta+s}{a}\right)}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1+az)^{-\frac{1}{a}}\})}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\sum_{m=0}^{n}m!S(n,m)B_{qm}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)z^m \tag{3.4}$$

replacing z by $\frac{z}{1+az}$ and x by $\frac{x}{(1+az)^{\frac{1}{a}}}$ above equation gives

$$\sum_{m=0}^{\infty}m^nB_{qm}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)z^m=(1-az)^{-\left(\frac{\delta+s}{a}\right)}\frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-az)^{-\frac{1}{a}}\})}\times$$
$$\sum_{m=0}^{n}m!S(n,m)B_{qm}^{(\alpha,\beta,\gamma,\delta)}\left(x(1-az)^{-\frac{1}{a}};a,k,s\right)\left(\frac{z}{1-az}\right)^m \tag{3.5}$$

with $|z| < |a|^{-1}; a \neq 0$.
Similarly, use of Theorem 3.1, to generating relation (2.4) gives generating relation as

$$\sum_{m=0}^{\infty} m^n B_{qm}^{(\alpha,\beta,\gamma,\delta-am)}(x(1-az)^{-\frac{1}{a}}; a, k, s) z^m (1-az)^{-m}$$
$$= (1+az)^{1-(\frac{\delta+s}{a})} \frac{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-az)^{-\frac{1}{a}}\})}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))} \sum_{m=0}^{n} m! S(n,m) B_{qm}^{(\alpha,\beta,\gamma,\delta)}(x; a, k, s) z^m \tag{3.6}$$

replacing z by $\frac{z}{1-az}$ and x by $x(1-az)^{-\frac{1}{a}}$ above equation gives

$$\sum_{m=0}^{\infty} m^n B_{qm}^{(\alpha,\beta,\gamma,\delta-am)}(x; a, k, s) z^m = (1-az)^{(\frac{\delta+s}{a})-1} \frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1+az)^{\frac{1}{a}}\})} \times$$
$$\sum_{m=0}^{n} m! S(n,m) B_{qm}^{(\alpha,\beta,\gamma,\delta)}\left(x(1+az)^{\frac{1}{a}}; a, k, s\right) \left(\frac{z}{1+az}\right)^m \tag{3.7}$$

with $|z| < |a|^{-1}; a \neq 0$.

4. Bilateral Generating Functions

A class of function $\{S_n(x), n = 0, 1, 2, ...\}$ is generated by

$$\sum_{n=0}^{\infty} A_{m,n} S_{m+n} t^n = f(x,t)\{g(x,t)\}^{-m} S_m(h(x,t)) \tag{4.1}$$

where m is non negative integer, the coefficient $A_{m,n}$ are arbitrary constant and f, g, h are suitable functions of x and t. Equation (4.1) is well-known in the literature as the Singhal-Srivastava generating function [11].

Theorem 4.1 (Srivastava and Manocha [12])**.** For the sequence $\{S_n(x)\}$ generated by (4.1), let

$$F(x,t) = \sum_{n=0}^{\infty} a_n S_n(x) t^n \tag{4.2}$$

where $a_n \neq 0$ are arbitrary constants, then

$$f(x,t) F[h(x,t), yt\{g(x,t)\}^{-1}] = \sum_{n=0}^{\infty} S_n(x) \sigma_n(y) t^n \tag{4.3}$$

where $\sigma_n(y)$ is a polynomial (of degree n in y) defined by

$$\sigma_n(y) = \sum_{k=0}^{n} a_k A_{n-k} y^k \tag{4.4}$$

Theorem 4.1 play an important role to obtain a bilateral generating relation from (2.3). Considering $S_n(x) = B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x; a, k, s)$, $A_{m,n} = \binom{m+n}{n}$, $g(x,t) = (1-at)$, $h(x,t) = x(1-at)^{-\frac{1}{a}}$ and $f(x,t) = (1-at)^{-(\frac{\delta+s}{a})} \frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})}$ in above theorem, produces

$$\sum_{n=0}^{\infty} B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)\sigma_n(y)t^n$$
$$= (1-at)^{-\left(\frac{\delta+s}{a}\right)} \frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})} F\left[\frac{x}{(1-at)^{\frac{1}{a}}}, \frac{yt}{(1-at)}\right] \tag{4.5}$$

where $F(x,t) = \sum_{n=0}^{\infty} a_n S_n(x)t^n$

Srivastava and Lavoie have generalized McBride's theorem [13] in 1975 as follows.

Theorem 4.2 (Srivastava and Lavoie [14]). If sequence $\{\zeta_\mu(x) : \mu \text{is a complex number}\}$ is generated by

$$\sum_{n=0}^{\infty} A_{\mu,n}\zeta_{\mu+n}(x)t^n = f(x,t)\{g(x,t)\}^{-\mu}\zeta_\mu(h(x,t)) \tag{4.6}$$

where $A_{\mu,n}$ arbitrary constants, and f, g, h are arbitrary functions of x and t. Let

$$\Phi_{\lambda,\nu}(x,t) = \sum_{n=0}^{\infty} a_{\nu,n}\zeta_{\mu+\lambda n}(x)t^n, (a_{\nu,n} \neq 0), \tag{4.7}$$

where λ is positive integer and ν is an arbitrary complex parameter.
Then

$$\sum_{n=0}^{\infty} \zeta_{\mu+n}(x)P_{n,\nu}^{\lambda}(y)t^n = f(x,t)\{g(x,t)\}^{-\mu}\Phi_{\lambda,\nu}\left[h(x,t), yt^\lambda\{g(x,t)\}^{-\lambda}\right] \tag{4.8}$$

where $P_{n,\nu}^{\lambda}(y)$ is a polynomial of degree $\left[\frac{n}{\lambda}\right]$ in y, which is defined as,

$$P_{n,\nu}^{\lambda}(y) = \sum_{r=0}^{\left[\frac{n}{\lambda}\right]} A_{\nu+\lambda r, n-\lambda r}\ a_{\nu,n}\ y^r$$

In the generating relation (2.3), consider $\zeta_n(x) = B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s), \mu = m$, $A_{\mu,n} = \binom{m+n}{n}$, $f(x,t) = (1-at)^{-\left(\frac{\delta+s}{a}\right)} \frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})}$, $g(x,t) = (1-at)$ and $h(x,t) = x(1-at)^{-\frac{1}{a}}$. Then

$$\sum_{n=0}^{\infty} B_{q(m+n)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)P_{n,\nu}^{\lambda}(y)t^n$$
$$= (1-at)^{-\left(\frac{\delta+s}{a}\right)} \frac{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}{E_{\alpha,\beta}^{\gamma,q}(p_k\{x(1-at)^{-\frac{1}{a}}\})} \Phi_{\lambda,\nu}\left[\frac{x}{(1-at)^{\frac{1}{a}}}, \frac{yt^\lambda}{(1-at)^\lambda}\right] \tag{4.9}$$

where,

$$\Phi_{\lambda,\nu}(x,t) = \sum_{n=0}^{\infty} a_{\nu,n}B_{q(m+\lambda n)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)t^n, \text{ with } a_{\nu,n} \neq 0, \tag{4.10}$$

and

$$P_{n,\nu}^{\lambda}(y) = \sum_{r=0}^{\left[\frac{n}{\lambda}\right]} \binom{\nu+n}{n-\lambda r}\ a_{\nu,n}\ y^r$$

is a polynomial of degree $\left[\frac{n}{\lambda}\right]$ in y, λ is positive integer and ν is an arbitrary complex number.

Theorem 4.3 (Srivastava [15]). For the function $\zeta_\mu(x)$ defined by (4.6), let

$$\Theta^{p,\nu}_{\lambda,\mu}[x;y_1,...,y_l;t]=\sum_{n=0}^{\infty}C^{\mu,\nu}_n\zeta_{\mu+\lambda n}(x)\Omega_{\nu+pn}(y_1,...,y_l)t^n, C^{\mu,\nu}_n\neq 0, \tag{4.11}$$

where μ and ν are arbitrary complex numbers, p and λ are positive integers, and $\Omega_\nu(y_1,...,y_l)$ is non-vanishing function of l variables $y_1,...,y_l$, $l\geq 1$. Then

$$\sum_{n=0}^{\infty}\zeta_{\mu+n}(x)Q^{p,\nu}_{n,\lambda,\mu}(y_1,...,y_l;z)t^n$$
$$=f(x,t)\{g(x,t)\}^{-\mu}\Theta^{p,\nu}_{\lambda,\mu}\left[h(x,t);y_1,...,y_l;z\left\{\frac{t}{g(x,t)}\right\}^{\lambda}\right] \tag{4.12}$$

where $Q^{p,\nu}_{n,\lambda,\mu}(y_1,...,y_l;z)$ is a polynomial of degree $\left[\frac{n}{\lambda}\right]$ in z (with coefficient's dependent on $y_1,...,y_l$) defined by

$$Q^{p,\nu}_{n,\lambda,\mu}(y_1,...,y_l;z)=\sum_{r=0}^{\left[\frac{n}{\lambda}\right]}A_{\mu+\lambda r,n-\lambda r}\ C^{\mu,\nu}_r\Omega_{\nu+pr}(y_1,...,y_l)\ z^r \tag{4.13}$$

considering,

$$\Theta^{p,\nu}_{\lambda,m}[x;y_1,...,y_l;t]=\sum_{n=0}^{\infty}C^{m,\nu}_n V^{(\alpha,\beta,\delta)}_{m+\lambda n}(x;a,k,s)\Omega_{\nu+pn}(y_1,...,y_l)t^n, C^{m,\nu}_n\neq 0, \tag{4.14}$$

where ν is arbitrary complex number, p and λ are positive integers and $\Omega_\nu(y_1,...,y_l)$ is non-vanishing function of l variables $y_1,...,y_l$, $l\geq 1$.
Taking $\zeta_n(x)=B^{(\alpha,\beta,\gamma,\delta)}_{qn}(x;a,k,s)$, $f(x,t)=(1-at)^{-\left(\frac{\delta+s}{a}\right)}\frac{E^{\gamma,q}_{\alpha,\beta}(p_k(x))}{E^{\gamma,q}_{\alpha,\beta}(p_k\{x(1-at)^{-\frac{1}{a}}\})}$, $g(x,t)=(1-at), h(x,t)=x(1-at)^{-\frac{1}{a}}, \mu=m, A_{\mu,n}=\binom{m+n}{n}$ in above theorem, and generating relation (2.3) yields bilateral generating relation as

$$\sum_{n=0}^{\infty}B^{(\alpha,\beta,\gamma,\delta)}_{q(m+n)}(x;a,k,s)Q^{p,\nu}_{n,q,\mu}(y_1,...,y_l;z)t^n=(1-at)^{-\left(\frac{\delta+s}{a}\right)}\frac{E^{\gamma,q}_{\alpha,\beta}(p_k(x))}{E^{\gamma,q}_{\alpha,\beta}(p_k\{x(1-at)^{-\frac{1}{a}}\})}$$
$$\times\Theta^{p,\nu}_{q,\mu}\left[x(1-at)^{-\frac{1}{a}};y_1,...,y_l;z\left\{\frac{t}{1-at}\right\}^{\lambda}\right] \tag{4.15}$$

where $Q^{p,\nu}_{n,\lambda,m}(y_1,...,y_l;z)$ is a polynomial of degree $\left[\frac{n}{\lambda}\right]$ in z (with coefficient's dependent on $y_1,...,y_l$) defined by

$$Q^{p,\nu}_{n,\lambda,m}(y_1,...,y_l;z)=\sum_{r=0}^{\left[\frac{n}{\lambda}\right]}\binom{m+n}{n-\lambda r}\ C^{m,\nu}_r\Omega_{\nu+pr}(y_1,...,y_l)\ z^r \tag{4.16}$$

5. Finite Summation Formulae

In this section, We obtained finite summation formulae for equation (1.7) as

$$B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)=\sum_{m=0}^{n}\frac{a^m}{m!}\left(\frac{\delta}{a}\right)_m B_{q(n-m)}^{(\alpha,\beta,\gamma,0)}(x;a,k,s) \tag{5.1}$$

$$B_{qn}^{(\alpha,\beta,\gamma,\sigma)}(x;a,k,s)=\sum_{m=0}^{n}\frac{a^m}{m!}\left(\frac{\sigma}{a}\right)_m B_{q(n-m)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s-\delta) \tag{5.2}$$

$$B_{qn}^{(\alpha,\beta,\gamma,\delta+\mu+1)}(x;a,k,s)=\sum_{m=0}^{n}\frac{a^m}{m!}\left(\frac{\mu+1}{a}\right)_m B_{q(n-m)}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s) \tag{5.3}$$

Proof. (Proof of (5.1)) We can write (1.7) as,

$$B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)=\frac{x^{-\delta-an}}{n!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\theta^n\left\{xx^{\delta-1}\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}$$

using (1.4) we get,

$$B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)=\frac{x^{-\delta-an}}{n!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))x\sum_{m=0}^{n}\binom{n}{m}\theta^{n-m}\left\{\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}\theta_1^m\left\{x^{\delta-1}\right\}$$

Equation (1.3) yields

$$\begin{aligned}&B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)\\&=\frac{x^{-an}}{n!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\sum_{m=0}^{n}\binom{n}{m}\theta^{n-m}\left\{\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}a^m\left(\frac{\delta}{a}\right)_m x^{am}\end{aligned} \tag{5.4}$$

Putting $\delta=0$ and replacing n by $n-m$ in (1.7) gives

$$B_{q(n-m)}^{(\alpha,\beta,\gamma,0)}(x;a,k,s)=\frac{x^{-a(n-m)}}{(n-m)!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\theta^{n-m}\left\{\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\} \tag{5.5}$$

use of (5.4) and (5.5), follows (5.1) ∎

Proof. (Proof of (5.2)) From (1.7) and (1.5) it follows that,

$$B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)=\frac{x^{-an}}{n!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))(\theta+\delta x^a)^n\left\{\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}$$

this gives

$$\theta^n\left\{\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}=\frac{x^{an}n!}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s-\delta) \tag{5.6}$$

with the help of (1.4), equation (1.7) gives

$$\begin{aligned}&B_{qn}^{(\alpha,\beta,\gamma,\sigma)}(x;a,k,s)\\&=\frac{x^{-(\sigma+an)+1}}{n!}E_{\alpha,\beta}^{\gamma,q}(p_k(x))\sum_{m=0}^{n}\binom{n}{m}\theta_1^m\{x^{\sigma-1}\}\theta^{n-m}\left\{\frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))}\right\}\end{aligned} \tag{5.7}$$

Using (5.6), simplification of (5.7) reduces to (5.2). ■

Proof. (Proof of (5.3)) Using (1.4), equation (1.7) can be written as

$$B_{qn}^{(\alpha,\beta,\gamma,\delta+\mu+1)}(x;a,k,s) = \frac{x^{-(\delta+\mu+an)}}{n!} E_{\alpha,\beta}^{\gamma,q}(p_k(x)) \sum_{m=0}^{n} \binom{n}{m} \theta_1^m \{x^\mu\} \theta^{n-m} \left\{ x^\delta \frac{1}{E_{\alpha,\beta}^{\gamma,q}(p_k(x))} \right\}$$

this leads to (5.3). ■

6. Special Cases

Special cases of $B_{qn}^{(\alpha,\beta,\gamma,\delta)}(x;a,k,s)$ obtained by considering suitable values of parameters. Putting $\alpha=\beta=\gamma=q=1$ and $p_k(x)=\beta x^k$ in (1.7) reduces to

$$B_n^{(1,1,1,\delta)}(x;a,k,s) = x^{a(s-n)} \tau_n^\delta(x;k,\beta,a,s) \tag{6.1}$$

where

$$\tau_n^\alpha(x;r,\beta,\kappa,\eta) = \frac{x^{-\alpha-\kappa\eta}}{n!} \exp\left(\beta x^r\right) \left[x^\kappa(\eta+xD)\right]^n \left\{x^\alpha \exp(-\beta x^r)\right\} \tag{6.2}$$

is defined by Chen et. al. [16].
If $\alpha=\beta=\gamma=q=1$ then (1.7) reduces to

$$B_n^{(1,1,1,\delta)}(x;a,k,s) = M_{kn}^\delta(x;s,a) \tag{6.3}$$

where

$$M_{kn}^\delta(x;s,a) = \frac{x^{-\delta-an}}{n!} \exp\{p_k(x)\} \left[x^a(s+xD)\right]^n \left\{x^\delta \exp -p_k(x)\right\} \tag{6.4}$$

given by Joshi and Prajapat [17].
If $\alpha=\beta=\gamma=q=1$ and $p_k(x)=x$ then (1.7) gives

$$B_n^{(1,1,1,\delta)}(x;a,1,s) = a^n Y_n^{\delta-1}(x;a) \tag{6.5}$$

where $Y_n^\alpha(x;a)$ is Konhauser polynomial of second kind defined by Konhauser [18].

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