

Existence of Solutions for Generalized Scalar Quasi-Equilibrium Problems Involving Two Bifunctions and Fixed Point Problems on Complete Metric Spaces

Narongrit Puturong¹ and Kasamsuk Ungchitrakool^{1,2,*}

¹Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
e-mail : naturong@gmail.com (N. Puturong)

²Research Center for Academic Excellence in Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
e-mail : kasamsuku@nu.ac.th (K. Ungchitrakool)

Abstract The purpose of this paper is to introduce and study the existence theorem of solutions for generalized scalar quasi-equilibrium problems involving two bifunctions on complete metric spaces. We show the uniqueness of its solution which is also a fixed point of some mappings. We also get new minimax theorem involving two bifunctions on complete metric spaces. Our results can be viewed as a general form and some extensions of some previously existing results.

MSC: 47H10; 49J27; 91B50

Keywords: generalized scalar quasi-equilibrium problem; fixed point; Banach limit

Submission date: 16.07.2018 / Acceptance date: 26.11.2018

1. INTRODUCTION

Let X be a nonempty closed convex subset of a topological space E , let $f : X \times X \rightarrow \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in X$. The scalar equilibrium problem is the problem of

(EP) finding $x \in X$ such that $f(x, y) \geq 0$ for all $y \in X$.

This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium, minimax inequalities, and saddle point problems as special cases (see [1]).

*Corresponding author.

Recently, extensions of the equilibrium problem for vector-valued or vector set-valued maps were introduced (see [2–10]) and references therein.

Let (X, d) be a complete metric space, $f : X \times X \rightarrow \mathbb{R}$ be a bifunction, and $T : X \rightarrow X$ be a mapping. Chuang and Lin [11, 12] provided some existence theorems for the scalar quasi-equilibrium problems on complete metric spaces.

(QEP₁) Find $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$ and $f(z, \bar{x}) \leq 0$ for all $z \in X$.

(QEP₂) Find $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$ and $f(\bar{x}, z) \geq 0$ for all $z \in X \setminus \{\bar{x}\}$.

They presented new existence theorem of solutions for scalar quasi-equilibrium problems, show that the uniqueness of its solution which is also fixed point of some mappings, and get new minimax theorem (MI) on complete metric spaces.

(MI) Find $\bar{x} \in X$ such that

$$\sup_{y \in X} \inf_{x \in X} f(x, y) = \inf_{x \in X} \sup_{y \in X} f(x, y) = f(\bar{x}, \bar{x}) = 0.$$

Furthermore, the solution of problem (QEP₁), [resp. (QEP₂)] can be obtained by the Picard iteration.

On the other hand, in this paper, we are interested in studying the new problem called generalized scalar quasi-equilibrium problems (GQEP) on complete metric spaces involving two bifunctions as follows:

(GQEP) Find $\bar{x} \in X$ such that

$$T\bar{x} = \bar{x}, g(\bar{x}, \bar{x}) = f(\bar{x}, \bar{x}) \leq 0 \text{ and } (f + g)(z, \bar{x}) \leq 0 \leq (f + g)(\bar{x}, z) \text{ for all } z \in X$$

where $f, g : X \times X \rightarrow \mathbb{R}$.

Furthermore, we can get a new minimax theorem on complete metric spaces by the existence theorem of problem (GQEP) as follows:

(GMI) Find $\bar{x} \in X$ such that

$$\sup_{x \in X} \inf_{y \in X} (f + g)(x, y) = \inf_{y \in X} \sup_{x \in X} (f + g)(x, y) = (f + g)(\bar{x}, \bar{x}) = 0.$$

The solution of (GQEP) and (GMI) can be obtained by Picard's iteration method. The obtained results can be reviewed as some generalizations of the previously existing results.

2. PRELIMINARIES

Let l^∞ be the Banach space of bounded sequences with the supremum norm. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. For $x = (x_1, x_2, \dots)$, the value $\mu(x)$ is also denoted by $\mu_n(x_n)$. A mean μ on l^∞ is called a Banach limit if it satisfies $\mu_n(x_n) = \mu_n(x_{n+1})$. If μ is a Banach limit on l^∞ , then for $x = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $x = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we have $\mu(x) = \mu_n(x_n) = a$. For details, we can refer [13].

Lemma 2.1 ([14, Lemma 3.1]). Let (X, d) be a metric space, let $\{x_n\}$ be a bounded sequence in X and let μ be a mean on l^∞ . If $g : X \rightarrow \mathbb{R}$ is defined by

$$g(y) = \mu_n d(x_n, y) \text{ for all } y \in X,$$

then g is a continuous function on X .

Lemma 2.2 ([11, Lemma 2.2]). Let (X, d) be a metric space, let $\{x_n\}$ be a bounded sequence in X and let μ be a mean on l^∞ . If $g : X \rightarrow \mathbb{R}$ is defined by

$$g(z) = \mu_n d(x_n, z) \text{ for each } z \in X,$$

then $g(x) = g(y) = 0$ implies $x = y$.

Definition 2.3 ([15, Definition 1.7.10]). Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for any $\epsilon > 0$, there is an $n_\epsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for any $m \geq n_\epsilon, n \geq n_\epsilon$.

Lemma 2.4 ([11, Lemma 2.3]). Let (X, d) be a metric space and let $\{x_n\}$ be a bounded sequence in X and let μ be a mean on l^∞ . Suppose that $\{y_n\}$ is a sequence in X with $\lim_{m \rightarrow \infty} \mu_n d(x_n, y_m) = 0$. Then $\{y_n\}$ is a Cauchy sequence.

Definition 2.5 ([13]). Let X be a topological space and let f be a function of X into \mathbb{R} .

- (i) f is called lower semicontinuous on X if for any real number a , the set $\{x \in X : f(x) \leq a\}$ is closed in X .
- (ii) f is called upper semicontinuous on X if for any real number a , the set $\{x \in X : f(x) \geq a\}$ is closed in X .
- (iii) f is continuous on X if and only if f is lower semicontinuous and upper semicontinuous on X .

Remark 2.6. f is lower semicontinuous on X if and only if $-f$ is upper semicontinuous on X .

Lemma 2.7 ([13, Theorem 1.3.2]). Let X be a topological space and let f be a function of X into $(-\infty, \infty]$. Then f is lower semicontinuous on X if and only if for any $x_0 \in X$, $x_\alpha \rightarrow x_0 \Rightarrow f(x_0) \leq \liminf_\alpha f(x_\alpha)$.

Lemma 2.8. Let X be a topological space and let f be a function of X into $[-\infty, \infty)$. Then f is upper semicontinuous on X if and only if for any $x_0 \in X$, $x_\alpha \rightarrow x_0 \Rightarrow \limsup_\alpha f(x_\alpha) \leq f(x_0)$.

Proof. By employing Remark 2.6 and Lemma 2.7, we have the desired result. ■

Definition 2.9 ([13]). Let X be a topological space and let f be a function of X into $[-\infty, \infty]$. f is called bounded above if there exists a real number M such that $f(x) \leq M$ for all $x \in X$.

3. MAIN RESULTS

In this section, we provide and study the existence theorem for generalized scalar quasi-equilibrium problems involving two bifunctions on complete metric spaces. The obtained results can be viewed as the tool for finding solutions (a unique solution) not only of (QEP_1) but also of (OEP_2) as follows:

Theorem 3.1. *Let $r \in [0, 1]$, (X, d) be a complete metric space, $T : X \rightarrow X$ be a mapping, and $f, g : X \times X \rightarrow \mathbb{R}$ be bifunctions. Let μ be a mean on l^∞ and let $\{x_n\}$ be a bounded sequence. Assume that :*

- (i) *there exists $\hat{x} \in X$ such that*
 - (a) *$y \rightarrow f(\hat{x}, y)$ is lower semicontinuous and bounded above function;*
 - (b) *$f(\hat{x}, y) - f(z, y) + g(y, z) \leq f(\hat{x}, z)$ for each $y, z \in X$;*
 - (ii) *$f(x, y) + f(y, x) \leq 0$ for each $x, y \in X$;*
 - (iii) *if $d(x, Tx) \leq (1 + r)d(x, y)$, then $\mu_n d(x_n, Ty) \leq f(x, y) + g(x, y)$. Then there exists $\bar{x} \in X$ such that*
 - (a) *$\mu_n d(x_n, \bar{x}) = 0$;*
 - (b) *$\lim_{n \rightarrow \infty} T^n x = \bar{x}$ for each $x \in X$;*
 - (c) *$g(\bar{x}, \bar{x}) \leq f(\bar{x}, \bar{x}) \leq 0$ and $(f + g)(z, \bar{x}) \leq 0$ for each $z \in X$.*
- Furthermore, if $d(Tx, T^2x) \leq (1 + r)d(x, Tx)$ for each $x \in X$, then*
- (d) *\bar{x} is the unique fixed point of T ;*
 - (e) *\bar{x} is the unique solution of problem (GQEP).*

Proof. Start with any $x \in X$. It is clear that $d(x, Tx) \leq (1 + r)d(x, Tx)$. By using (iii), (i)(b) and $f(x, y) + f(y, x) \leq 0$ for each $x, y \in X$, we have that

$$\mu_n d(x_n, T^2x) \leq f(x, Tx) + g(x, Tx) \leq g(x, Tx) - f(Tx, x) \leq f(\hat{x}, Tx) - f(\hat{x}, x).$$

Repeating this process, we obtain

$$\begin{aligned} 0 &\leq \mu_n d(x_n, T^{k+1}x) \leq g(T^{k-1}x, T^kx) - f(T^kx, T^{k-1}x) \\ &\leq f(\hat{x}, T^kx) - f(\hat{x}, T^{k-1}x) \end{aligned} \tag{3.1}$$

for each $k \in \mathbb{N}$. So, $\{f(\hat{x}, T^kx)\}$ is a nondecreasing sequence. It follows from (i)(a) that $\lim_{k \rightarrow \infty} f(\hat{x}, T^kx)$ exists. By virtue of (3.1), it implies that

$$\lim_{k \rightarrow \infty} \mu_n d(x_n, T^{k+1}x) = 0. \tag{3.2}$$

So, by Lemma 2.4, $\{T^kx\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $\bar{x} \in X$ such that $T^kx \rightarrow \bar{x}$ as $k \rightarrow \infty$. By (3.2) and Lemma 2.1, we obtain that

$$\mu_n d(x_n, \bar{x}) = 0. \tag{3.3}$$

Thus for any $u \in X$, there exists $\bar{u} \in X$ such that $\lim_{k \rightarrow \infty} T^k u = \bar{u}$ and $\mu_n d(x_n, \bar{u}) = 0$. By Lemma 2.2, $\bar{u} = \bar{x}$. Therefore, we have that $\bar{x} = \lim_{k \rightarrow \infty} T^k z$ for all $z \in X$.

For each $k \in \mathbb{N} \cup \{0\}$, let $u_k = T^kx$. Take any $z \in X \setminus \{\bar{x}\}$ and let z be fixed. Since $\bar{x} \neq z$ and $u_k \rightarrow \bar{x}$ as $k \rightarrow \infty$, there exists $N \in \mathbb{N}$ such that for each $k \geq N$, we have

$$\frac{1}{1 + r} d(u_k, u_{k+1}) \leq d(u_k, u_{k+1}) \leq d(u_k, z). \tag{3.4}$$

By (3.4), (iii), $f(x, y) + f(y, x) \leq 0$ for each $x, y \in X$, (i)(b), and for $k \geq N$, we have that

$$\begin{aligned} \mu_n d(x_n, Tz) &\leq f(u_k, z) + g(u_k, z) \leq g(u_k, z) - f(z, u_k) \\ &\leq f(\hat{x}, z) - f(\hat{x}, u_k). \end{aligned} \tag{3.5}$$

By employing (i)(a) and (3.5), we obtain that

$$\begin{aligned} \mu_n d(x_n, Tz) + f(\hat{x}, \bar{x}) &\leq \mu_n d(x_n, Tz) + \liminf_{k \rightarrow \infty} f(\hat{x}, u_k) \\ &\leq \liminf_{k \rightarrow \infty} [\mu_n d(x_n, Tz) + f(\hat{x}, u_k)] \\ &\leq \limsup_{k \rightarrow \infty} [\mu_n d(x_n, Tz) + f(\hat{x}, u_k)] \\ &\leq f(\hat{x}, z). \end{aligned} \tag{3.6}$$

It implies by (3.6) and (i)(b) that

$$\mu_n d(x_n, Tz) \leq f(\hat{x}, z) - f(\hat{x}, \bar{x}) \leq f(\bar{x}, z) - g(z, \bar{x}). \tag{3.7}$$

Thus (3.7) gives us for the following result

$$g(z, \bar{x}) \leq g(z, \bar{x}) + \mu_n d(x_n, Tz) \leq f(\bar{x}, z) \text{ for each } z \in X \setminus \{\bar{x}\}.$$

On the other hand, it is easy for checking from (i)(b) that $g(\bar{x}, \bar{x}) \leq f(\bar{x}, \bar{x})$. Therefore, we have the desired result as follows

$$g(z, \bar{x}) \leq f(\bar{x}, z) \text{ for all } z \in X. \tag{3.8}$$

By virtue of $f(x, y) + f(y, x) \leq 0$ for each $x, y \in X$ and it follows from (3.8), we get that $g(z, \bar{x}) \leq -f(z, \bar{x})$ for all $z \in X$. It implies that

$$(f + g)(z, \bar{x}) \leq 0 \text{ for all } z \in X. \tag{3.9}$$

Also, by $f(x, y) + f(y, x) \leq 0$ for each $x, y \in X$, we get that $f(\bar{x}, \bar{x}) \leq 0$.

Furthermore, if $d(Tx, T^2x) \leq (1 + r)d(x, Tx)$ for each $x \in X$, then we have

$$d(T\bar{x}, T(T\bar{x})) \leq (1 + r)d(T\bar{x}, \bar{x}).$$

By (iii) and $f(x, y) + f(y, x) \leq 0$ for each $x, y \in X$, we have the consequence that

$$\mu_n d(x_n, T\bar{x}) + f(\bar{x}, T\bar{x}) \leq g(T\bar{x}, \bar{x}). \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$f(\bar{x}, T\bar{x}) \leq g(T\bar{x}, \bar{x}) \leq f(\bar{x}, T\bar{x}).$$

Thus $f(\bar{x}, T\bar{x}) = g(T\bar{x}, \bar{x})$. By using (3.10) again, we obtain that

$$0 \leq \mu_n d(x_n, T\bar{x}) \leq g(T\bar{x}, \bar{x}) - f(\bar{x}, T\bar{x}) = 0.$$

That is

$$\mu_n d(x_n, T\bar{x}) = 0. \tag{3.11}$$

By using (3.3), (3.11) and Lemma 2.2, we obtain that $T\bar{x} = \bar{x}$. Consequently, $g(\bar{x}, \bar{x}) = f(\bar{x}, \bar{x}) \leq 0$.

Let \bar{y} be an arbitrary fixed point of T . Then $\lim_{k \rightarrow \infty} T^k \bar{y} = \lim_{k \rightarrow \infty} \bar{y} = \bar{y}$ and by using the conclusion below (3.3), it is not hard to verify that $\bar{y} = \bar{x}$. So, \bar{x} is the unique fixed point of T .

We next show that $0 \leq (f + g)(\bar{x}, z)$ for all $z \in X$. For \bar{x} is a fixed point of T , we have $d(\bar{x}, T\bar{x}) \leq (1 + r)d(\bar{x}, \bar{x}) \leq (1 + r)d(\bar{x}, z)$. By (iii), we have that

$$0 \leq \mu_n d(x_n, Tz) \leq f(\bar{x}, z) + g(\bar{x}, z).$$

That is

$$0 \leq (f + g)(\bar{x}, z) \quad \text{for all } z \in X. \quad (3.12)$$

It follows from (3.9) and (3.12), we have that

$$(f + g)(z, \bar{x}) \leq 0 \leq (f + g)(\bar{x}, z) \quad \text{for all } z \in X.$$

For each $y \in X$, following the above proof, we know that there exists $\bar{z} \in X$ such that $\lim_{k \rightarrow \infty} T^k y = \bar{z}$, $T\bar{z} = \bar{z}$, and $(f + g)(z, \bar{z}) \leq 0 \leq (f + g)(\bar{z}, z)$ for all $z \in X$. Since \bar{x} is the unique fixed point of T , $\bar{x} = \bar{z}$. The proof is completed. ■

The following corollary is some applications and consequences of Theorem 3.1. It can be seen that Theorem 3.1 is the tool for finding solutions (a unique solution) not only of (QEP_1) but also of (OEP_2) as follows:

Corollary 3.2 ([11, Theorem 3.1]). *Let $r \in [0, 1]$, (X, d) be a complete metric space, $T : X \rightarrow X$ be a mapping, and $f : X \times X \rightarrow \mathbb{R}$ be a function. Let μ be a mean on l^∞ and let $\{x_n\}$ be a bounded sequence. Assume that :*

- (i) *there exists $\hat{x} \in X$ such that*
 - (a) *$y \rightarrow f(\hat{x}, y)$ is a lower semicontinuous and bounded above function;*
 - (b) *$f(\hat{x}, y) + f(y, z) \leq f(\hat{x}, z)$ for each $y, z \in X$;*
- (ii) *if $d(x, Tx) \leq (1 + r)d(x, y)$, then $\mu_n d(x_n, Ty) \leq f(x, y)$.*

Then there exists $\bar{x} \in X$ such that

- (a) *$\mu_n d(x_n, \bar{x}) = 0$;*
- (b) *$\lim_{n \rightarrow \infty} T^n x = \bar{x}$ for each $x \in X$;*
- (c) *$f(z, \bar{x}) \leq 0$ for each $z \in X$.*

Furthermore, if $d(Tx, T^2x) \leq (1 + r)d(x, Tx)$ for each $x \in X$, then

- (d) *\bar{x} is the unique fixed point of T ;*
- (e) *\bar{x} is the unique solution of problem (QEP_1) and (QEP_2) .*

Proof. By (i)(b), we have that

$$f(y, z) \leq f(\hat{x}, z) - f(\hat{x}, y) \quad \text{and} \quad f(z, y) \leq f(\hat{x}, y) - f(\hat{x}, z)$$

for all $y, z \in X$. Combining the above two inequalities, we get that

$$f(y, z) + f(z, y) \leq 0 \quad \text{for all } y, z \in X. \quad (3.13)$$

Therefore, the condition (ii) of the mapping f in Theorem 3.1 is satisfied. In the particular case where g is the zero mapping and by using (3.13), the condition (i)(b) of Theorem 3.1 holds, that is,

$$f(\hat{x}, y) - f(z, y) + g(y, z) \leq f(\hat{x}, z) \quad \Leftrightarrow \quad f(\hat{x}, y) + f(y, z) \leq f(\hat{x}, z)$$

for all $y, z \in X$ where g is the zero mapping. Under the assumption $d(Tx, T^2x) \leq (1 + r)d(x, Tx)$ for each $x \in X$ and by applying Theorem 3.1, we have that there exists \bar{x} is the unique fixed point of T such that

$$f(z, \bar{x}) \leq 0 \leq f(\bar{x}, z) \quad \text{for all } z \in X.$$

Therefore, \bar{x} is the unique solution of (QEP_1) and (QEP_2) . The proof is completed. ■

Motivated by [11, Example 3.1], the following example is established for supporting the main result.

Example 3.3. Let $X = \{z_n : z_n = \frac{1}{2^n}, n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ with the usual metric $d(x, y) = |x - y|$. Let $r = 0, T : X \rightarrow X$, and $f, g : X \times X \rightarrow \mathbb{R}$ be defined by $Tx = 1$,

$$f(x, y) = \begin{cases} 0 & \text{if } x = 1 \text{ or } y = 1, \\ -1 & \text{others,} \end{cases}$$

and

$$g(x, y) = \begin{cases} 0 & \text{if } x = 1 \text{ or } y = 1, \\ -2 & \text{others.} \end{cases}$$

Let $\hat{x} = 1$. For each $n \in \mathbb{N}$, let $x_n := 1$. Then $f(\hat{x}, y) = 0$ and $g(\hat{x}, y) = 0$ for each $y \in X$. Then by definition of f and g , we get that the condition (i), (ii) of Theorem 3.1 hold. Hence, we only need to consider the condition (iii) of Theorem 3.1.

(1) If $x = 0$ and $d(0, T0) \leq d(0, y)$, then $y = 1$. And this implies that

$$\mu_n d(x_n, T1) = 0 \leq f(0, 1) + g(0, 1) = 0 + 0 = 0.$$

(2) If $x = 1$ and $d(1, T1) \leq d(1, y)$, then y is any point of X . And this implies that

$$\mu_n d(x_n, Ty) = 0 \leq f(1, y) + g(1, y) = 0 + 0 = 0.$$

(3) if $x = z_n, n \in \mathbb{N}$, and $d(z_n, Tz_n) \leq d(z_n, y)$, then $y = 1$. And this implies that

$$\mu_n d(x_n, Ty) = \mu_n d(x_n, T1) = 0 \leq f(z_n, 1) + g(z_n, 1) = 0 + 0 = 0.$$

By Theorem 3.1, there exists $\bar{x} \in X$ such that $f(\bar{x}, \bar{x}) \leq 0$ and $g(z, \bar{x}) \leq f(\bar{x}, z)$ for each $z \in X$. Indeed, $\bar{x} = 1$. Furthermore, $d(Tx, T^2x) \leq d(x, Tx)$ for each $x \in X$ and \bar{x} is a fixed point of T .

By application of Theorem 3.1, we can provide the minimax theorem involving two bifunctions on complete metric spaces as follows:

Theorem 3.4. Assume that all assumptions are the same as in Theorem 3.1. Let \bar{x} be solution of (GQEP) and $f(\bar{x}, \bar{x}) = 0$. Then

$$\sup_{x \in X} \inf_{y \in X} (f + g)(x, y) = \inf_{y \in X} \sup_{x \in X} (f + g)(x, y) = (f + g)(\bar{x}, \bar{x}) = 0.$$

Proof. Let \bar{x} be a solution of (GQEP). By Theorem 3.1, we get that \bar{x} is the unique fixed point of T , $g(\bar{x}, \bar{x}) = f(\bar{x}, \bar{x}) \leq 0$, and

$$(f + g)(z, \bar{x}) \leq 0 \leq (f + g)(\bar{x}, z)$$

for each $z \in X$. Then, if $f(\bar{x}, \bar{x}) = 0$, we have $(f + g)(\bar{x}, \bar{x}) = 0$. So, we get that

$$\max_{x \in X} (f + g)(x, \bar{x}) \leq (f + g)(\bar{x}, \bar{x}) \leq \min_{y \in X} (f + g)(\bar{x}, y). \quad (3.14)$$

By using (3.14), we have that

$$\inf_{y \in X} \sup_{x \in X} (f + g)(x, y) \leq (f + g)(\bar{x}, \bar{x}) \leq \sup_{x \in X} \inf_{y \in X} (f + g)(x, y). \quad (3.15)$$

Besides, we have that

$$\sup_{x \in X} \inf_{y \in X} (f + g)(x, y) \leq \inf_{y \in X} \sup_{x \in X} (f + g)(x, y). \quad (3.16)$$

By using (3.15) and (3.16), we have that

$$\sup_{x \in X} \inf_{y \in X} (f + g)(x, y) = \inf_{y \in X} \sup_{x \in X} (f + g)(x, y) = (f + g)(\bar{x}, \bar{x}) = 0.$$

Therefore, the proof is completed. ■

Remark 3.5. It follows from Example 3.3, solution of (GMI) is $\bar{x} = 1$.

4. CONCLUSION

In the present paper, we study the existence theorem of solutions for generalized scalar quasi-equilibrium problems involving two bifunctions on complete metric spaces. The obtained results can be applied to minimax theorem involving two bifunctions on complete metric spaces. Our results can be viewed as a general form and some extensions of some previously existing results.

ACKNOWLEDGEMENTS

The second author would like to thank Naresuan University for financial support.

REFERENCES

- [1] E. Blum, W. Oettli, From optimization and variational inequalities, *Math. Stud.* 63 (1994) 123–146.
- [2] Q.H. Ansari, I.V. Konnov, J.C. Yao, Existence of a solution and variational principles for vector equilibrium problems, *J. Optim. Theory Appl.* 110 (2001) 481–492.
- [3] A. Capata, G. Kassay, On vector equilibrium problems and applications, *Taiwan J. Math.* 15 (2011) 365–380.
- [4] G.Y. Chen, X.Q. Yang, H. Yu, A nonlinear scalarization function and generalized quasi-vector equilibrium problems, *J. Glob. Optim.* 32 (2005) 451–466.
- [5] G.Y. Chen, X.X. Huang, X.Q. Yang, *Vector Optimization*, Springer, Berlin, 2005.
- [6] L.J. Lin, Existence results for primal and dual generalized vector equilibrium problems with applications to generalized semi-infinite programming, *J. Glob. Optim.* 32 (2005) 579–597.
- [7] L.J. Lin, Existence theorem of simultaneous equilibrium problems and generalized vector quasi-saddle points, *J. Glob. Optim.* 32 (2005) 613–632.
- [8] L.J. Lin, Y.J. Huang, Generalized vector quasi-equilibrium problem with applications to common fixed point theorems and optimization problems, *Nonlinear Anal.* 66 (2007) 1275–1289.
- [9] L.J. Lin, System of generalized vector quasi-equilibrium problems with applications to fixed point theorem for a family of nonexpansive multivalued mappings, *J. Glob. Optim.* 34 (2006) 15–32.
- [10] N. Puturong, K. Ungchittrakool, Existence of solutions for set-valued quasi-equilibrium problems and fixed point problems on Euclidean spaces, *JP J. Fixed Point Theory Appl.* 13(3) (2018) 137–165.
- [11] C.S. Chuang, L.J. Lin, Existence and uniqueness of solution for quasi-equilibrium problems and fixed point problems on complete metric spaces with applications, *J. Glob. Optim.* 57 (2013) 829–841.

-
- [12] C.S. Chuang, L.J. Lin, New existence theorem for quasi-equilibrium problems and a minimax theorem on complete metric spaces, *J. Glob. Optim.* 57 (2013) 533–547.
 - [13] W. Takahashi, *Nonlinear Functional Analysis - Fixed Point Theory and Its Applications*, Yokohama Publishers, Yokohama, 2000.
 - [14] K. Hasegawa, T. Komiya, W. Takahashi, Fixed point theorems of general contractive mappings in metric spaces and estimating expressions, *Sci. Math. Japonicae.* 74 (1) (2011) 15–27.
 - [15] P.K. Jain, O.P. Ahuja, K. Ahmad, *Functional Analysis*, Wiley Eastern Limited, New Delhi, 1995.