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Szász-Type Operators which Preserves e_0 and e_2

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Abstract In the present article, we introduce a generalization of Szász-type operators which preserves test functions e_0 and e_2 $(e_i = t^i, i = 0, 2)$. By these sequence of positive linear operators, we give better error estimation analytically using modulus of continuity. Moreover, we have shown the better approximation graphically. Further, weighted Korovkin and Voronovskaya type theorems are established.

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1. Introduction

For $f \in C[0,1]$ and $x \in [0,1]$, Bernstein [1] defined a sequence of positive linear operators

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \ n \in \mathbb{N}, \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. The objective of these sequences was to give an elegant proof of Weierstrass approximation theorem which plays a central role in the development of approximation theory. A generalization of Bernstein operators was given by Szász [2] to an infinite interval as

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \ n \in \mathbb{N}, \tag{1.2}$$

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where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, $x \in (0, \infty)$. In this paper, he showed the manner in which the operators $S_n(f; x)$ tends to $f(x)$. In recent past, there are many important generalizations of Szász-Mirakjan operators (see [3], [4] and references therein).

In 1968, for $f \in C[0, 1]$ and $n \in \mathbb{N}$, Stancu [5] presented another generalization of operators (1.1) with two real parameters α, β satisfying $0 \le \alpha \le \beta$

$$P_n^{\alpha,\beta}(f; x) = \sum_{k=0}^{n} \tbinom{n}{i} x^k (1 - x)^{n-k} f\left(\frac{k + \alpha}{n + \beta}\right),\ n \in \mathbb{N}.$$

For $\alpha = \beta = 0$, these linear positive operators reduce to Bernstein operators defined by (1.1). Many researchers studied this approach to different type of operators and studied approximation properties (see [6]–[11]).

Various well known positive linear operators L_n preserve the constant as well as the linear functions i.e. $L_n(e_0; x) = e_0(x)$ and $L_n(e_1; x) = e_1(x)$ for $e_i(x) = x^i (i = 0, 1)$. But, these operators do not preserve $e_2(x)$ and it is rather difficult to nearing $e_2(x)$ for large value of n. In [12], King introduced a method by which every linear positive operators preserve $e_2(x)$ i.e. $L_n(e_2; x) = e_2(x)$ and provide the better error estimation. Several authors use this powerful tool to different type of positive linear operators and discuss the better error estimation for instance see Duman [13]. Recently, S. Varma and F. Tasdelen [14], gave a generalization of well known Szász-Mirakjan operators using Charlier polynomials [15] having the generating function of the form

$$e^t\left(1 - \frac{t}{a}\right)^u = \sum_{k=0}^{\infty} C_k^{(a)}(u)\frac{t^k}{k!}, \qquad |t| < a \tag{1.3}$$

and the explicit representation

$$C_k^{(a)}(u) = \sum_{r=0}^{k} \tbinom{k}{r} (-u)_r \left(\frac{1}{a}\right)^r,$$

where $(\alpha)_k$ is the Pochhammer's symbol given by

$$(\alpha)_0 = 1, (\alpha)_k = \alpha(\alpha + 1)...(\alpha + k - 1),\ k \in \mathbb{N}.$$

We note that for $a > 0$ and $u \le 0$, Charlier polynomials are positive. Then, they [14] defined the Szász-type operators by the following equality

$$L_n(f; x, a) = e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} C_k^{(u)}(-(a - 1)nx) f\left(\frac{k}{n}\right),\ n \in \mathbb{N}, \tag{1.4}$$

where $a > 1$ and $x \ge 0$. In the light of above, we define a generalization of operators L_n as follows

$$\begin{aligned} T_{n,a}^{\alpha,\beta}(f; r_{n,a}(x; \alpha, \beta), a) &= e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)} \sum_{k=0}^{\infty} C_k^{(u)}(-(a - 1)nr_{n,a}(x; \alpha, \beta)) \\ &\times f\left(\frac{k + \alpha}{n + \beta}\right). \end{aligned} \tag{1.5}$$

where $f \in C[0, \infty)$, $x \in [0, \infty)$ and $0 \le \alpha \le \beta$ with

$$r_{n,a}(x; \alpha, \beta) = \frac{-(3 + 2\alpha + \frac{1}{a-1}) + \sqrt{(3 + 2\alpha + \frac{1}{a-1})^2 + 4((n + \beta)^2x^2 - 2 - 2\alpha - \alpha^2)}}{2n}. \tag{1.6}$$

For $r_{n,a}(x;\alpha,\beta)=x$ and $\alpha=\beta=0$, these sequence of operators reduce to operators (1.4). In this article we have obtained better error estimation using the operators (1.5) than the operators (1.4) in terms of modulus of continuity. It is shown graphically as well as numerically with the help of examples. Moreover, we prove weighted Korovkin and Voronovskaya type theorems.

2. Basic Estimates

Lemma 2.1. *For the operators $T_{n,a}^{\alpha,\beta}$ defined by (1.5), we have*

$$\begin{aligned} T_{n,a}^{\alpha,\beta}(1;x) &= 1,\\ T_{n,a}^{\alpha,\beta}(t;x) &= \frac{-(1+\frac{1}{a-1})+\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}}{2(n+\beta)},\\ T_{n,a}^{\alpha,\beta}(t^2;x) &= x^2. \end{aligned}$$

Proof. Using $t=1, u=-(a-1)nr_{n,a}(x;\alpha,\beta)$ in (1.3) and on differentiation (1.3), we get

$$\begin{aligned} \sum_{k=0}^{\infty}\frac{C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))}{k!} &= e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x;\alpha,\beta)},\\ \sum_{k=0}^{\infty}k\frac{C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))}{k!} &= e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x;\alpha,\beta)}(1+nr_{n,a}(x;\alpha,\beta)), \end{aligned}$$

$$\sum_{k=0}^{\infty}k^2\frac{C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))}{k!}=e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x;\alpha,\beta)}\left(2+\left(3+\frac{1}{a-1}\right)nr_{n,a}(x;\alpha,\beta)+n^2r_n^2(x;\alpha,\beta)(x)\right).$$

Now, using these equalities, we have

$$\begin{aligned} (i)\quad T_{n,a}^{\alpha,\beta}(1;x,a) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x;\alpha,\beta)}\\ &= 1,\\ (ii)\quad T_{n,a}^{\alpha,\beta}(t;x,a) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\frac{k+\alpha}{n+\beta}\\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\frac{k}{n+\beta}\\ &\quad+e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\frac{\alpha}{n+\beta}\\ &= \frac{nr_{n,a}(x;\alpha,\beta)+1}{n+\beta}+\frac{\alpha}{n+\beta}. \end{aligned}$$

$$T_{n,a}^{\alpha,\beta}(t;x,a) = \frac{-\left(1+\frac{1}{a-1}\right)+\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}}{2(n+\beta)}$$

(*iii*) $T_{n,a}^{\alpha,\beta}(t^2;x,a)$

$$\begin{aligned}
&= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\left(\frac{k+\alpha}{n+\beta}\right)^2\\
&= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\frac{k^2}{(n+\beta)^2}\\
&\quad +2\alpha e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\frac{k}{(n+\beta)^2}\\
&\quad +\alpha^2 e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\frac{1}{(n+\beta)^2}
\end{aligned}$$

$$T_{n,a}^{\alpha,\beta}(t^2;x,a) = x^2.$$

∎

Lemma 2.2. *Let $\psi_x^i(t)=(t-x)^i, i=0,1,2$. From the operators (1.5), we have*

$$\begin{aligned}
T_{n,a}^{\alpha,\beta}(\psi_x^0;x) &= 1,\\
T_{n,a}^{\alpha,\beta}(\psi_x^1;x) &= \frac{-(1+\frac{1}{a-1})+\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}}{2(n+\beta)}-x,\\
T_{n,a}^{\alpha,\beta}(\psi_x^2;x) &= 2x^2+\frac{(1+\frac{1}{a-1})}{n+\beta}x-\frac{x\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}}{n+\beta}.
\end{aligned}$$

Lemma 2.3. *For the operators $T_{n,a}^{\alpha,\beta}$*

$$\lim_{n\to\infty}(n+\beta)T_{n,a}^{\alpha,\beta}(\psi_x^1;x) = -\frac{\left(1+\frac{1}{a-1}\right)}{2},$$
$$\lim_{n\to\infty}(n+\beta)T_{n,a}^{\alpha,\beta}(\psi_x^2;x) = \left(1+\frac{1}{a-1}\right)x.$$

Proof. From Lemma 2.2

$(n+\beta)T_{n,a}^{\alpha,\beta}(\psi_x^1;x)$

$$\begin{aligned}
&= \frac{-\left(1+\frac{1}{a-1}\right)}{2}+\frac{\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}-2(n+\beta)x}{2}\\
&= \frac{-\left(1+\frac{1}{a-1}\right)}{2}+\frac{1+4\alpha+\frac{4\alpha+6}{a-1}+\frac{1}{(a-1)^2}}{2(\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}+2(n+\beta)x)}
\end{aligned}$$

as $n \to \infty$, we find

$$\lim_{n\to\infty}(n+\beta)T_{n,a}^{\alpha,\beta}(\psi_x^1;x) \quad = \quad \frac{-\left(1+\frac{1}{a-1}\right)}{2}.$$

Similarly, we prove

$$\lim_{n\to\infty}(n+\beta)T_{n,a}^{\alpha,\beta}(\psi_x^2;x) = \left(1+\frac{1}{a-1}\right)x.$$

■

Lemma 2.4. *The operators $T_{n,a}^{\alpha,\beta}$ satisfy the following inequality*

$$T_{n,a}^{\alpha,\beta}(\psi;x) \quad < \quad \frac{1+\alpha}{n+\beta}$$

$$T_{n,a}^{\alpha,\beta}(\psi^2;x) \quad < \quad x\frac{\left(1+\frac{1}{a-1}\right)}{n+\beta}.$$

Proof. We observe that for $a>1$, $0\le\alpha\le\beta$ and $x\ge 0$

$$\sqrt{\left(3+2\alpha+\frac{1}{a-1}\right)^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)} < \left(3+2\alpha+\frac{1}{a-1}\right)+2(n+\beta)x$$

$$\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)} \quad > \quad 2(n+\beta)x \tag{2.1}$$

using Lemma 2.2, we prove the Lemma 2.4. ■

Lemma 2.5. *For the operators* (1.5)*, we have*

$$\begin{aligned}
T_{n,a}^{\alpha,\beta}(t^3;x) \quad &= \quad \frac{1}{(n+\beta)^3}\Bigg(n^3r_{n,a}^3(x;\alpha,\beta)+\left(6+3\alpha+3\frac{1}{a-1}\right)n^2r_{n,a}^2(x;\alpha,\beta)\\
&+ \quad \left(11+9\alpha+3\alpha^2+\frac{3\alpha+6}{a-1}+\frac{2}{(a-1)^2}\right)\\
&\qquad \times nr_{n,a}(x;\alpha,\beta)+6+6\alpha+3\alpha^2+\alpha^3\Bigg)
\end{aligned}$$

$$\begin{aligned}
T_{n,a}^{\alpha,\beta}(t^4;x) \quad &= \quad \frac{1}{(n+\beta)^4}\Bigg(n^4r_{n,a}^4(x;\alpha,\beta)+\left(10+6\alpha+6\frac{1}{a-1}\right)n^3r_{n,a}^3(x;\alpha,\beta)\\
&+ \quad \left(35+24\alpha+6\alpha^2+\frac{20+12\alpha}{a-1}+\frac{11}{(a-1)^2}\right)n^2r_{n,a}^2(x;\alpha,\beta)\\
&+ \quad \left(50+44\alpha+18\alpha^2+4\alpha^3+\frac{35+24\alpha++6\alpha^2}{a-1}+\frac{20+8\alpha}{(a-1)^2}+\frac{6}{(a-1)^3}\right)\\
&\times \quad nr_{n,a}(x;\alpha,\beta)+24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4\Bigg)
\end{aligned}$$

Proof. Differentiating (1.3), we get

$\sum_{k=0}^{\infty} k^3 \frac{C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))}{k!}$

$$= e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x;\alpha,\beta)}\left(6+\left(11+\frac{6}{a-1}+\frac{2}{(a-1)^2}\right)nr_{n,a}(x;\alpha,\beta)\right.$$
$$\left.+\left(6+\frac{3}{a-1}\right)n^2r_n^2(x;\alpha,\beta)+n^3r_{n,a}^3(x;\alpha,\beta)\right),$$

$\sum_{k=0}^{\infty} k^4 \frac{C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))}{k!}$

$$= e\left(1-\frac{1}{a}\right)^{-(a-1)nr_{n,a}(x;\alpha,\beta)}\left(24+\left(50+\frac{35}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)nr_{n,a}(x;\alpha,\beta)\right.$$
$$+\left(35+\frac{30}{a-1}+\frac{11}{a-1}^2\right)n^2r_{n,a}^2(x;\alpha,\beta)$$
$$\left.+(10+\frac{6}{a-1})n^3r_{n,a}^3(x;\alpha,\beta)+n^4r_{n,a}^4(x;\alpha,\beta)\right).$$

Using linearity property and these equalities, Lemma 2.5 can easily be proved. ∎

Lemma 2.6. *For operators $T_{n,a}^{\alpha,\beta}$, we have*

$$\lim_{n\to\to\infty}(n+\beta)^2T_{n,a}^{\alpha,\beta}((t-x)^4;x)=0.$$

Proof. By the linearity property of the operators $T_{n,a}^{\alpha,\beta}$, we have

$$T_{n,a}^{\alpha,\beta}(\psi_x^4(t);x) = T_{n,a}^{\alpha,\beta}(t^4;x)-4xT_{n,a}^{\alpha,\beta}(t^3;x)+6x^2T_{n,a}^{\alpha,\beta}(t^2;x)-4x^3T_{n,a}^{\alpha,\beta}(t;x)$$
$$+\ x^4T_{n,a}^{\alpha,\beta}(1;x).$$

Using Lemma 2.5 and Lemma 2.2, we obtain

$T_{n,a}^{\alpha,\beta}(\psi_x^4(t);x)$

$$= \frac{1}{(n+\beta)^4}\left(n^4r_{n,a}^4(x;\alpha,\beta)+\left(10+6\alpha+6\frac{1}{a-1}\right)n^3r_{n,a}^3(x;\alpha,\beta)\right.$$
$$+\left(35+24\alpha+6\alpha^2+\frac{20+12\alpha}{a-1}+\frac{11}{(a-1)^2}\right)n^2r_{n,a}^2(x;\alpha,\beta)$$
$$+\left(50+44\alpha+18\alpha^2+4\alpha^3+\frac{35+24\alpha+6\alpha^2}{a-1}+\frac{20+8\alpha}{(a-1)^2}+\frac{6}{(a-1)^3}\right)nr_{n,a}(x;\alpha,\beta)$$

$$+ \quad 24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4\Bigg) - \frac{4x}{(n+\beta)^3}\Bigg(\Bigg(n^3r^3_{n,a}(x;\alpha,\beta)+\Bigg(6+3\alpha+3\frac{1}{a-1}\Bigg)$$

$$\times \quad n^2r^2_{n,a}(x;\alpha,\beta)+\Bigg(11+9\alpha+3\alpha^2+\frac{3\alpha+6}{a-1}+\frac{2}{(a-1)^2}\Bigg)nr_{n,a}(x;\alpha,\beta)$$

$$+ \quad 6+6\alpha+3\alpha^2+\alpha^3\Bigg)+\frac{6x^2}{(n+\beta)^2}\Bigg(n^2r^2_{n,a}(x;\alpha,\beta)+\Bigg(3+2\alpha+\frac{1}{a-1}\Bigg)nr_{n,a}(x;\alpha,\beta)$$

$$+ \quad 1+2\alpha+\alpha^2\Bigg)-\frac{4x^3}{(n+\beta)}\Bigg(nr_{n,a}(x;\alpha,\beta+R)\alpha+1\Bigg)+x^4. \tag{2.2}$$

Let $k_{4,3}=\Bigg(10+6\alpha+6\frac{1}{a-1}\Bigg)$, $k_{4,2}=\Bigg(35+24\alpha+6\alpha^2+\frac{20+12\alpha}{a-1}+\frac{11}{(a-1)^2}\Bigg)$,
$k_{4,1}=\Bigg(50+44\alpha+18\alpha^2+4\alpha^3+\frac{35+24\alpha++6\alpha^2}{a-1}+\frac{20+8\alpha}{(a-1)^2}+\frac{6}{(a-1)^3}\Bigg)$,
$k_{4,0}=24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4$,
$k_{3,2}=\Bigg(6+3\alpha+3\frac{1}{a-1}\Bigg)$, $k_{3,1}=\Bigg(11+9\alpha+3\alpha^2+\frac{3\alpha+6}{a-1}+\frac{2}{(a-1)^2}\Bigg)$,
$k_{3,0}=6+6\alpha+3\alpha^2+\alpha^3$,
$k_{2,1}=\Bigg(3+2\alpha+\frac{1}{a-1}\Bigg)$,$k_{2,0}=1+2\alpha+\alpha^2$, $k_{1,0}=\alpha+1$.

From (2.2), we get

$$T^{\alpha,\beta}_{n,a}(\psi^4_x(t);x)$$

$$< \quad \frac{1}{(n+\beta)^4}\Bigg((n+\beta)^4x^4+\frac{k_{4,3}(n+\beta)^3x^3}{2n}+\frac{k_{4,2}(n+\beta)^2x^2}{4n^2}+\frac{k_{4,1}(n+\beta)x}{8n^3}$$

$$+ \quad \frac{k_{4,0}}{16n^4}-4(n+\beta)x\Bigg((n+\beta)^3x^3+\frac{k_{3,2}(n+\beta)^2x^2}{2n}+\frac{k_{3,1}(n+\beta)x}{4n^2}+\frac{k_{3,0}}{8n^3}\Bigg)$$

$$+ \quad 6x^2(n+\beta)^2\Bigg(x^2(n+\beta)^2+\frac{k_{2,1}(n+\beta)^2x^2}{2n}++\frac{k_{2,0}}{4n^2}\Bigg)$$

$$- \quad 4x^3(n+\beta)^3\Bigg((n+\beta)x++\frac{k_{1,0}}{2n}\Bigg)\Bigg),$$

$$= \quad \frac{1}{(n+\beta)^2}\Bigg(\frac{k_{4,2}-4k_{3,1}+6k_{2,0}}{4n^2}\Bigg)+\frac{1}{(n+\beta)^3}\Bigg(\frac{k_{4,1}-4k_{3,0}}{8n^3}\Bigg)+\frac{k_{4,0}}{16n^4(n+\beta)^4}$$

$$(n+\beta)^2T^{\alpha,\beta}_{n,a}(\psi^4_x(t);x)<\Bigg(\frac{k_{4,2}-4k_{3,1}+6k_{2,0}}{4n^2}\Bigg)+\frac{1}{(n+\beta)}\Bigg(\frac{k_{4,1}-4k_{3,0}}{8n^3}\Bigg)+\frac{k_{4,0}}{16n^4(n+\beta)^2}$$

as $n\to\infty$, right hand side tends to 0. ∎

3. Main Results

$$E=\{f:[0,\infty)\to R, |f(x)\leq Me^{Ax}, A\in R \text{ and } M\in R^{+}|\}.$$

Theorem 3.1. *Let $f\in C[0,\infty)\cap E$ and $x\geq 0$. Then from operators $T_{n,a}^{\alpha,\beta}$ defined by (1.5), we have*

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)|$$

$$\leq\left\{1+\sqrt{x\left(1+\frac{1}{a-1}\right)-\frac{x\left(1+4\alpha+\frac{4\alpha+6}{a-1}+\frac{1}{(a-1)^2}\right)}{\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)+2(n+\beta)x}}}\right\}\omega\left(f;\delta_n^{\beta}\right)$$

where $\delta_n^{\beta}=\frac{1}{\sqrt{(n+\beta)}}$.

Proof. We have the difference

$$|T_{n,a}^{\alpha,\beta}(f;x)-f(x)|$$

$$\leq e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\left|f\left(\frac{k+\alpha}{n+\beta}\right)-f(x)\right|$$

$$\leq\left\{1+\frac{1}{\delta_n^{\beta}}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\left|\frac{k+\alpha}{n+\beta}-x\right|\right\}\omega\left(f;\delta_n^{\beta}\right)$$

$$\leq\left\{1+\frac{1}{\delta_n^{\beta}}\sqrt{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nr_{n,a}(x;\alpha,\beta)}\sum_{k=0}^{\infty}C_k^{(u)}(-(a-1)nr_{n,a}(x;\alpha,\beta))\left(\frac{k+\alpha}{n+\beta}-x\right)^2}\right\}$$

$$\times\,\omega\left(f;\delta_n^{\beta}\right)$$

$$=\left\{1+\frac{1}{\delta_n^{\beta}}\sqrt{2x^2+\frac{(1+\frac{1}{a-1})}{n+\beta}x-\frac{x\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}}{n+\beta}}\right\}\omega\left(f;\delta_n^{\beta}\right)$$

$$=\left\{1+\sqrt{x\left(1+\frac{1}{a-1}\right)-\frac{x\left(1+4\alpha+\frac{4\alpha+6}{a-1}+\frac{1}{(a-1)^2}\right)}{\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)+2(n+\beta)x}}}\right\}\omega\left(f;\delta_n^{\beta}\right)$$

where $\delta_n^{\beta}=\frac{1}{\sqrt{n+\beta}}$. ∎

Remark 3.2. For the Szász type operators T_n given by (1.4), we have, for every $f \in C[0,\infty) \cap E$,

$$|T_n(f;x,a) - f(x)| \leq \left\{1 + \sqrt{x\left(1 + \frac{1}{a-1}\right) + \frac{2}{n}}\right\}\omega\left(f; \frac{1}{\sqrt{n}}\right). \tag{3.1}$$

Here we show that our operators $T_{n,a}^{\alpha,\beta}$ has the better approximation than the operators (1.4).
Since

$$2x = \frac{\sqrt{4x^2(n+\beta)^2}}{n+\beta}$$

and $(3 + 2\alpha + \frac{1}{a-1})^2 - 8 - 8\alpha - 4\alpha^2 > 0$ for all values of $a > 1$ and $0 \leq \alpha \leq \beta$. Therefore

$2x^2 < x\frac{\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4(x^2(n+\beta)^2-2-2\alpha-\alpha^2)}}{n+\beta}$

$2x^2 - x\frac{\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4(x^2(n+\beta)^2-2-2\alpha-\alpha^2)}}{n+\beta} < 0$

$2x^2 - x\frac{\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4(x^2(n+\beta)^2-2-2\alpha-\alpha^2)}}{n+\beta} + \frac{(1+\frac{1}{a-1})}{n+\beta}x < \frac{(1+\frac{1}{a-1})}{n+\beta}x < \frac{(1+\frac{1}{a-1})}{n}x < \frac{(1+\frac{1}{a-1})}{n}x + \frac{2}{n^2}$

which implies that $\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2;x)} < \sqrt{T_n(\psi_x^2;x)}$. Hence, $\delta_n^\beta < \delta$.

Example 3.3 Let $f(x) = sin^2x \in C[0,4]$, $a = 2, \alpha = 10, \beta = 11$. There are two figure for two different values set of n which shows better approximation of $T_{n,a}^{\alpha,\beta}$ than the operators T_n.

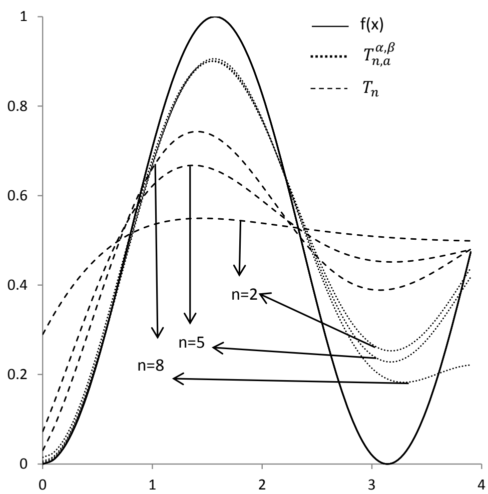

FIGURE 1. For $n = 2, 5, 8$ approximation by $T_{n,a}^{\alpha,\beta}$ and T_n

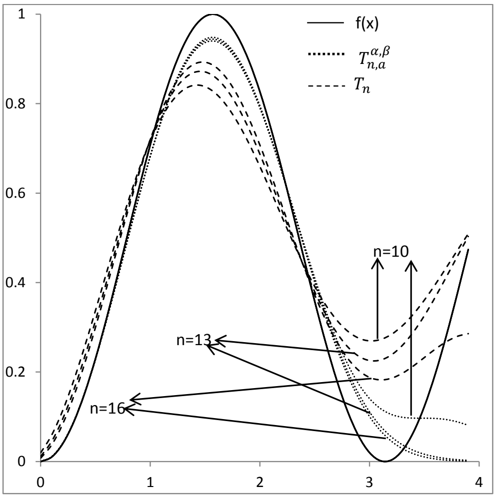

FIGURE 2. For $n = 10, 13, 16$ approximation by $T_{n,a}^{\alpha,\beta}$ and T_n

Example 3.3 For $f(x) = \frac{1}{\sqrt{1+x^2}}$, $a = 2$ and $\alpha = 100, \beta = 200$, we estimate error for different power of 3 by using modulus of continuity for $T_{n,a}^{\alpha,\beta}$ and T_n at the point $x = \frac{2}{3}$.

Value of n	Estimates by operators $T_{n,a}^{\alpha,\beta}$	Estimates by operators T_n^a
3	0.0261	0.47953
3^2	0.026017	0.276994
3^3	0.025851	0.159961
3^4	0.025187	0.092296
3^5	0.023446	0.05319
3^6	0.019881	0.030672
3^7	0.014659	0.017645
3^8	0.009521	0.010185
3^9	0.005712	0.005877
3^{10}	0.003311	0.003393

4. Weighted Korovkin Type Theorem

In this section, we introduce $T_{n,a}^{\alpha,\beta}$ in polynomial weighted spaces of continuous and unbounded functions defined on $[0, \infty)$. In [?], Gadzhiev give the weighted Korovkin-type theorems. Here we recall some symbols and notions from [?]. Let $\rho(x) = 1 + x^2$, $-\infty < x < \infty$ and $B_\rho[0, \infty) = \{f(x) : |f(x)| \leq M_f \rho(x), \rho(x)$ is weight function, M_f is a constant depending on f and $x \in [0, \infty)\}$, $C_\rho[0, \infty)$ is the space of continuous function in $B_\rho[0,\infty)$ with the norm $\|f(x)\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\rho(x)}$ and $C_\rho^k = \{f \in C_\rho : \lim_{|x|\to\infty} \frac{f(x)}{\rho(x)} = k$, where k is a constant depending on $f\}$.

Theorem 4.1. *Let $T_{n,a}^{\alpha,\beta}$ be the sequence of linear positive operators defined by (1.5). Then for $f \in C_\rho^k$,*

$$\lim_{n\to\infty} \|T_{n,a}^{\alpha,\beta}(f;x) - f(x)\|_\rho = 0.$$

Proof. To prove the theorem, it is sufficient to show that

$$\lim_{n\to\infty} ||T_{n,a}^{\alpha,\beta}(t^i;x) - x^i||_\rho = 0, \qquad for \quad i = 0,1,2.$$

It is obvious that $\lim_{n\to\infty} \|T_{n,a}^{\alpha,\beta}(1;x) - 1\|_\rho = 0$ and $\lim_{n\to\infty} \|T_{n,a}^{\alpha,\beta}(x^2;x) - x^2\|_\rho = 0$. Now, from the Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{|T_{n,a}^{\alpha,\beta}(t;x) - x|}{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\left|\frac{-(1+\frac{1}{a-1})+\sqrt{(3+2\alpha+\frac{1}{a-1})^2+4((n+\beta)^2x^2-2-2\alpha-\alpha^2)}}{2(n+\beta)} - x\right|}{1+x^2} \\
&< \frac{1+\alpha}{n+\beta} \sup_{x\in[0,\infty)} \frac{1}{(1+x)2}.
\end{aligned}$$

which shows that as $n \to \infty$, $||T_{n,a}^{\alpha,\beta}(t;x) - x||_\rho \to 0$.
Hence, we prove the theorem. ∎

Consequently, we have for the operators defined by Varma and Taşdélen * Let L_n be the sequence of linear positive operators defined by (1.4). Then for $f \in C_\rho^k$,

$$\lim_{n\to\infty} \|L_n(f;x,a) - f(x)\|_\rho = 0.$$

5. Voronovskaya Type Theorem

Voronovskaya type theorem is the study of rate of convergence for at least two times differentiable functions. We prove the following

Theorem 5.1. *For $f \in C^2[0,b]$, $0 < b < \infty$ and the operators defined by (1.5), we have*

$$\lim_{n\to\infty} (n+\beta)\{T_{n,a}^{\alpha,\beta}(f;x) - f(x)\} = -\left(1+\frac{1}{a-1}\right)\frac{f'(x)}{2} + \left(1+\frac{1}{a-1}\right)xf''(x).$$

Proof. Let $x,t \in [0,b]$, $f \in C^2[0,b]$. By Taylor's formula, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2}f''(x) + \eta(t,x)(t-x)^2,$$

where the function $\eta(t,x) \in C[0,b]$ and $\lim_{t\to x} \eta(t,x) = 0$. Applying the operators $T_{n,a}^{\alpha,\beta}$ both sides, we get

$$\begin{aligned}
&T_{n,a}^{\alpha,\beta}(f;x) \\
&= f(x)T_{n,a}^{\alpha,\beta}(1;x) + f'(x)T_{n,a}^{\alpha,\beta}(t-x;x) + \frac{f''(x)}{2}T_{n,a}^{\alpha,\beta}((t-x)^2;x) + T_{n,a}^{\alpha,\beta}(\eta(t,x)(t-x);x).
\end{aligned}$$

Using Lemma 2.3, we obtain

$$\begin{aligned}\lim_{n\to\infty}(n+\beta)\{T_{n,a}^{\alpha,\beta}(f;x)-f(x)\} &= -\left(1+\frac{1}{a-1}\right)\frac{f'(x)}{2}+\left(1+\frac{1}{a-1}\right)xf''(x)\\ &+ \lim_{n\to\infty}(n+\beta)T_{n,a}^{\alpha,\beta}(\eta(t;x)(t-x)^2;x). \end{aligned}\tag{5.1}$$

Using Holder's inequality for the last term

$$(n+\beta)T_{n,a}^{\alpha,\beta}(\eta(t;x)(t-x)^2;x)\le(n+\beta)^2T_{n,a}^{\alpha,\beta}((t-x)^4;x)T_{n,a}^{\alpha,\beta}(\eta(t;x)^2;x). \tag{5.2}$$

Let $\varphi(t;x)=\eta^2(t;x)$ which implies that $\lim_{t\to x}\varphi(t;x)=0$. Therefore

$$\lim_{t\to x}T_{n,a}^{\alpha,\beta}(\varphi(t;x);x)=0. \tag{5.3}$$

From (5.2), (5.3) and Lemma 2.6

$$\lim_{n\to\infty}(n+\beta)T_{n,a}^{\alpha,\beta}(\eta(t;x)(t-x)^2;x)=0. \tag{5.4}$$

On substituting (5.4) in (5.1), we arrive at the result.

■

Consequently, we get for the operators defined by Varma and Taşdélen [14] in (1.4) as

* For $f\in C^2[0,b]$, $0<b<\infty$ and the operators defined by (1.4), we have

$$\lim_{n\to\infty}n\{L_n(f;x,a)-f(x)\}=-\left(1+\frac{1}{a-1}\right)\frac{f'(x)}{2}+\left(1+\frac{1}{a-1}\right)xf''(x).$$

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