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On Contact CR-Submanifolds of $(LCS)_n$-Manifolds

Shyamal Kumar Hui[1,2,*], Mehmet Atceken[3], Tanumoy Pal[4] and Lakshmi Narayan Mishra[5]

[1] *Department of Mathematics, Bankura University Bankura–722155, West Bengal, India*

[2] *Department of Mathematics, The University of Burdwan, Golapbag, Burdwan–713104, West Bengal, India*
e-mail : shyamal_hui@yahoo.co.in (S. K. Hui)

[3] *Gaziosmanpasa University, Faculty of Arts and Sciences, Department of Mathematics, Tokat–60250, Turkey*
e-mail : mehmet.atceken@gop.edu.tr (M. Atceken)

[4] *A. M. J. High School, Mankhamar, Bankura–722144, West Bengal, India*
e-mail : tanumoypalmath@gmail.com (T. Pal)

[5] *Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India*
e-mail : lakshminarayanmishra04@gmail.com (L. N. Mishra)

Abstract The object of the present paper is to study contact CR-submanifolds of $(LCS)_n$-manifolds. We obtain the integrability conditions of the distributions of contact CR-submanifolds of $(LCS)_n$-manifolds. Finally, we give an interesting example of a contact CR-submanifold of $(LCS)_7$-manifold.

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1. Introduction

In 2003, Shaikh [1] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [2] and also by Mihai and Rosca [3]. Then Shaikh and Baishya ([4, 5]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Hui et. al ([6–12]), Shaikh and his co-authors ([13–18]), Yadav et. al [19] and many others. In this connection it may be mentioned that many authors studied various spaces and spacetime such as [20–26].

The contact CR-submanifolds are rich and very much interesting subject. The study of the differential geometry of a contact CR-submanifolds as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold was initiated by Bejancu [27]. Thereafter several authors studied submanifolds as well as contact CR-submanifolds of different classes of almost contact metric manifolds such as Chen ([28, 29]), Hasegawa and Mihai [30], Jamali and Shahid [31], Khan et. al ([32, 33]), Munteanu [34], Murathan

*Corresponding author.

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et. al [35] and many others. Recently Atceken and his co-author ([36, 37]) studied contact CR-submanifolds of Kenmotsu manifolds.

Motivated by the above studies the present paper deals with the study of contact CR-submanifolds of $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of contact CR-submanifolds of $(LCS)_n$-manifolds. We obtain many integrability conditions of the distributions of contact CR-submanifolds of $(LCS)_n$-manifolds. Finally we give an interesting example of a contact CR-submanifold of $(LCS)_7$-manifold.

2. Preliminaries

An n-dimensional Lorentzian manifold $\overline{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, $\overline{M}$ admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in \overline{M}$, the tensor $g_p : T_p\overline{M} \times T_p\overline{M} \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p\overline{M}$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, $= 0$, > 0) [38].

Definition 2.1. In a Lorentzian manifold $(\overline{M}, g)$ a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(T\overline{M})$, is said to be a concircular vector field [39] if

$$(\overline{\nabla}_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form and $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let $\overline{M}$ be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\overline{\nabla}_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{2.3}$$

that is,

$$\overline{\nabla}_X \xi = \alpha\{X + \eta(X)\xi\}, \quad \alpha \neq 0 \tag{2.4}$$

for all vector fields X, Y, where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\overline{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.5}$$

ρ being a certain scalar function given by $\rho = -\xi(\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\overline{\nabla}_X \xi, \tag{2.6}$$

then from (2.3) and (2.6) we have

$$\phi X = X + \eta(X)\xi, \tag{2.7}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold $\overline{M}$ together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [1]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure [40]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold [1]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.8}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.9}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.10}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.11}$$

$$R(\xi,Y)Z = (\alpha^2 - \rho)[g(Y,Z)\xi - \eta(Z)Y], \tag{2.12}$$

$$(\overline{\nabla}_X \phi)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.13}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.14}$$

$$R(X,Y)Z = \phi R(X,Y)Z + (\alpha^2 - \rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi \tag{2.15}$$

for all $X,\ Y,\ Z \in \Gamma(T\overline{M})$.

Let M be a submanifold of a $(LCS)_n$-manifold $\overline{M}$ with induced metric g. Also let ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.16}$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{2.17}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\overline{M}$. The second fundamental form h and the shape operator A_V are related by

$$g(h(X,Y),V) = g(A_V X, Y) \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where g is the Riemannian metric on $\overline{M}$ as well as on M.
For any submanifold M of a Riemannian metric on $\overline{M}$, the equation of Gauss is given by

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &+ (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) \end{aligned} \tag{2.19}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\overline{R}$ and R denote the Riemannian curvature tensors of $\overline{M}$ and M respectively. The covariant derivative $\overline{\nabla}h$ of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y) \tag{2.20}$$

and the covariant derivative $\overline{\nabla}A$ is defined by

$$(\overline{\nabla}_X A_V)Y = \nabla_X(A_V Y) - A_{\nabla^{\perp}_X V}Y - A_V \nabla_X Y \tag{2.21}$$

for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. The normal part $\big(\overline{R}(X,Y)Z\big)^{\perp}$ of $\overline{R}(X,Y)Z$ from (2.19) is given by

$$\big(\overline{R}(X,Y)Z\big)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z), \tag{2.22}$$

which is known as Codazzi equation.
In particular, if $\big(\overline{R}(X,Y)Z\big)^{\perp} = 0$ then M is said to be curvature-invariant submanifold of $\overline{M}$.
The Ricci equation is given by

$$g\big(\overline{R}(X,Y)V,U\big) = g\big(R^{\perp}(X,Y)V,U\big) + g\big([A_U, A_V]X, Y\big) \tag{2.23}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where $R^{\perp}$ denotes the Riemannian curvature tensor of the normal vector bundle $T^{\perp}M$ and if $R^{\perp} = 0$ then the normal connection of M is called flat [41].

3. Contact CR-Submanifolds of $(LCS)_n$-Manifolds

Let M be an isometrically immersed submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$\phi X = EX + FX, \tag{3.1}$$

where EX is the tangential component and FX is the normal component of ϕX.
Also for any $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi V = BV + CV, \tag{3.2}$$

where BV and CV are the tangential and normal components of ϕV respectively. Also B is an endomorphism of the normal bundle $T^{\perp}M$ of TM and C is an endomorphism of the sub-bundle of the normal bundle $T^{\perp}M$.
The covariant derivatives of the tensor fields of E and F are defined as

$$(\nabla_X E)Y = \nabla_X EY - E(\nabla_X Y), \tag{3.3}$$

and

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F(\nabla_X Y) \tag{3.4}$$

for all $X, Y \in \Gamma(TM)$. The canonical structures E and F on a submanifold M are said to be parallel if $\nabla E = 0$ and $\nabla F = 0$, respectively.
Also the covariant derivatives of B and C are defined by

$$(\nabla_X B)V = \nabla_X BV - B(\nabla^{\perp}_X V), \tag{3.5}$$

and

$$(\nabla_X C)V = \nabla^{\perp}_X CV - C(\nabla^{\perp}_X V). \tag{3.6}$$

Also for any $X, Y \in \Gamma(TM)$, we have $g(EX, Y) = g(X, EY)$ and for any $U, V \in \Gamma(T^{\perp}M)$, we have $g(U, CV) = g(CU, V)$. This shows that E and C are also symmetric tensor fields. Moreover for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have the relation between F and B, given by

$$g(FX, V) = g(X, BV). \tag{3.7}$$

Definition 3.1 ([37]). Let M be an isometrically immersed submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then M is called a contact CR-submanifold of $\overline{M}$ if there is a differential distribution $D : p \to D_p \subseteq T_pM$ on M satisfying the following conditions:
(i) $\xi \in D$,
(ii) D is invariant with respect to ϕ, i.e., $\phi(D_p) \subseteq D_p$ for each $p \in M$ and
(iii) the orthogonal complementary distribution $D^{\perp} : p \to D^{\perp}_p \subseteq T_pM$ satisfying $\phi(D^{\perp}_p) \subseteq T^{\perp}_p M$ for each $p \in M$.

A contact CR-submanifold is called anti-invariant (or totally real) if $D_p = \{0\}$ and invariant (or holomorphic) if $D^{\perp}_p = \{0\}$, respectively for any $p \in M$. It is called proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D^{\perp}_p = \{0\}$.
Now for $\xi \in \Gamma(D) \subseteq \Gamma(TM)$ we have from (3.1) that $\phi\xi = E\xi + F\xi = 0$, which is equivalent to

$$E\xi = F\xi = 0. \tag{3.8}$$

Applying ϕ to (3.1) and using (2.9) and (3.2), we get

$$E^2 + BF = I + \eta \oplus \xi \text{ and } FE + CF = 0. \tag{3.9}$$

Similarly applying ϕ to (3.2) and using (2.9) and (3.1), we get

$$C^2 + FB = I \text{ and } EB + BC = 0. \tag{3.10}$$

We now prove the following:

Theorem 3.2. *Let M be an isometrically immersed submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then M is a contact CR-submanifold if and only if $FE = 0$.*

Proof. Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Let us denote the orthogonal projections on D and $D^{\perp}$ by R and S, respectively. Then we have

$$R + S = I, R^2 = R, S^2 = S \text{ and } RS = SR = 0. \tag{3.11}$$

For any $X \in \Gamma(TM)$, we can write

$$X = RX + SX \text{ and } \phi X = \phi RX + \phi SX = ERX + FRX + ESX + FSX. \tag{3.12}$$

Since D is invariant distribution, it follows that

$$FR = 0 \text{ and } ES = 0. \tag{3.13}$$

Also we have

$$RE = E = ER \;\; and \;\; FS = SF = F. \tag{3.14}$$

From (3.9) we get

$$FER + CFR = 0. \tag{3.15}$$

In view of (3.13), (3.15) yields

$$FE = 0. \tag{3.16}$$

From (3.9) and (3.16), we obtain

$$CF = 0. \tag{3.17}$$

Conversely, let M be a submanifold of a $(LCS)_n$-manifold $\overline{M}$ such that $FE = 0$. For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$\begin{aligned} g(X, \phi^2 V) &= g(\phi^2 X, V), g(X, \phi BV) = g(\phi FX, V), \\ g(X, EBV) &= g(CFX, V) = 0. \end{aligned}$$

Consequently we get

$$EB = 0. \tag{3.18}$$

By virtue of (3.9), (3.10), (3.17) and (3.18) we obtain $E^3 - E = 0$ and $C^3 - C = 0$, which shows that E and C are f-structures on TM and $T^{\perp}M$, respectively.
If we take $R = E^2 + \eta \otimes \xi$ and $S = I - E^2 - \eta \otimes \xi$ then we can easily see that

$$R + S = I, R^2 = R, S^2 = S \text{ and } RS = SR = 0, \tag{3.19}$$

that is, R and S are orthogonal projections and they define orthogonal complementary distributions such as D and $D^{\perp}$. Since $R = E^2 + \eta \otimes \xi$ and $E^3 - E = 0$, we get $ER = E$ and $ES = 0$.
For any $X, Y \in \Gamma(TM)$, we have $g(SEX, Y) = g(EX, SY) = g(X, ESY) = 0$. Thus we have $SE = 0$, which implies that $SER = 0$.
Using (3.8), (3.16) and the relation $R = E^2 + \eta \otimes \xi$, we get

$$FR = 0. \tag{3.20}$$

From (3.19) and (3.20), we can say that D and $D^{\perp}$ are invariant and anti-invariant distributions on M, respectively. Also from (3.11), we have $R\xi = \xi$ and $S\xi = 0$, that is, the distribution D contains ξ.
On the other hand, setting $R = E^2$ and $S = I - E^2$, we can easily see that projections R and S define orthogonal distributions such as D and $D^{\perp}$, respectively.
Thus we have $ER = E$, $SE = 0$, $FR = 0$ and $ES = 0$, that is D is an invariant distribution, $D^{\perp}$ is an anti-invariant distribution and $R\xi = 0$ and $S\xi = \xi$, which implies that $\xi \in D^{\perp}$. Hence the theorem is proved. ∎

Next we prove the following theorem:

Theorem 3.3. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then the anti-invariant distribution $D^{\perp}$ is completely integrable and its maximal integral submanifold is an anti-invariant submanifold of $\overline{M}$ if and only if the shape operator A satisfy*

$$A_{FX}Y = A_{FY}X,$$

for all $X, Y \in \Gamma(D^{\perp})$.

Proof. Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then from (2.13), (2.16), (2.17), (3.1) and (3.2) and the relation $(\overline{\nabla}_X\phi)Y = \overline{\nabla}_X\phi Y - \phi\overline{\nabla}_X Y$, we get

$$\begin{aligned}&\alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}\\ &= \overline{\nabla}_X EY + \overline{\nabla}_X FY - \phi\nabla_X Y - \phi h(X,Y)\end{aligned} \tag{3.21}$$

for any $X, Y \in \Gamma(TM)$.
From the tangential and normal components of (3.21), we get

$$(\nabla_X E)Y = A_{FY}X + Bh(X,Y) + \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\} \tag{3.22}$$

and

$$(\nabla_X F)Y = Ch(X,Y) - h(X,EY). \tag{3.23}$$

Similarly we have for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ that

$$(\nabla_X B)V + (\nabla_X C)V + h(X,BV) - A_{CV}X + EA_V X + FA_V X = 0. \tag{3.24}$$

Comparing the tangential and normal components of (3.24) we get

$$(\nabla_X C)V = -h(X,BV) - FA_V X, \tag{3.25}$$

and

$$(\nabla_X B)V = A_{CV}X - EA_V X. \tag{3.26}$$

Since M is tangent to ξ, we have from (2.13) and (2.18) that

$$A_V\xi = \alpha BV, \quad h(X,\xi) = \alpha FX \tag{3.27}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$. It is well known that $Bh = 0$ plays an important role in the geometry of submanifolds. Consequently (3.22) reduces to

$$(\nabla_X E)Y = \alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\} \tag{3.28}$$

for any $X, Y \in \Gamma(D)$. From (3.28) we may conclude that the induced structure E is a $(LCS)_n$- structure on M.
Again using (2.13) and (2.18) we have

$$E[X,Y] = A_{FY}X - A_{FX}Y,$$

for all $X, Y \in \Gamma(D^{\perp})$. Thus $[Z,W] \in \Gamma(D^{\perp})$ for any $Z, W \in \Gamma(D^{\perp})$, that is $D^{\perp}$ is integrable. Hence the theorem is proved. ∎

Theorem 3.4. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. If the second fundamental form of the contact CR-submanifold M is parallel, then α is a constant function on $\overline{M}$.*

Proof. Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$ and let the second fundamental form of the contact CR-submanifold M is parallel. Then from (2.20) we get

$$\nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y) = 0 \tag{3.29}$$

for any $X, Y, Z \in \Gamma(TM)$.
Putting $Y = \xi$ in (3.29) and using (2.4), (2.16) and (3.27) we get

$$\nabla^{\perp}_X h(Y,\xi) - h(\nabla_X Y, \xi) - h(\nabla_X \xi, Y) = 0.$$

By using (3.23) and (3.27), we have

$$\begin{aligned} 0 &= \nabla^{\perp}_X(\alpha.FY) - \alpha.F\nabla_X Y - \alpha.h(TX,Y) \\ &= X(\alpha)FY + \alpha\nabla^{\perp}_X FY - \alpha F\nabla_X Y - \alpha.h(TX,Y) \\ &= X(\alpha)FY + \alpha({}_X F)Y - \alpha.h(TX,Y) \\ &= X(\alpha)FY - \alpha\{h(X,TY) + h(TX,Y) - Ch(X,Y)\}. \end{aligned}$$

Here choosing $Y=\xi$, we mean that $h(TX,\xi)-Ch(X,\xi)=\alpha FTX - Ch(X,\xi)=0$. Since M is a contact CR-submanifold, we conclude that $Ch(X,\xi)=0$. So we obtain

$$X(\alpha)FY - \alpha\{h(\xi,TY) - Ch(\xi,TY)\} = 0 \Rightarrow X(\alpha)FY = 0,$$

that is, $X(\alpha)=0$. This proves our assertion. ∎

Theorem 3.5. *Let M be a submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then M is a contact CR-submanifold if and only if the endomorphism C defines an f structure on $\Gamma(T^{\perp}M)$, that is, $C^3 - C = 0$.*

Proof. Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then from (3.10) and (3.17) we have $C^3 - C = 0$, which implies that C is an f structure on $T^{\perp}M$.
Conversely, let C be an f-structure on $\Gamma(T^{\perp}M)$. Then from (3.10), we can derive $CFB = 0$. So for any $V \in \Gamma(T^{\perp}M)$ by using (3.7), we have

$$g(BCV,BCV) = g(\phi CV,BCV) = g(CV,FBCV) = g(V,CFBCV) = 0,$$

which implies that $BC=0$ and hence $EB=0$. Thus from Theorem 3.1, we conclude that M is a contact CR-submanifold. ∎

Theorem 3.6. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. If the endomorphism E on M is parallel, then M is anti-invariant submanifold of $\overline{M}$.*

Proof. Since E is parallel then from (3.22) and (3.27), we get

$$\begin{aligned} 0 &= g(h(X,\xi),FY) + \alpha\{-g(X,Y) - 2\eta(X)\eta(Y) + \eta(X)\eta(Y)\} \\ &= \alpha g(FX,FY) - \alpha\{g(X,Y) + \eta(X,Y)\} = -\alpha g(EX,EY) \end{aligned}$$

for any $X,Y \in \Gamma(TM)$. Since $\alpha \neq 0$. This implies that M is anti-invariant submanifold. ∎

Theorem 3.7. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold. Then the endomorphism F is parallel if and only if the endomorphism B is parallel.*

Proof. From (3.23) we get

$$\begin{aligned} g((\nabla_X F)Y,V) &= g(Ch(X,Y) - h(X,EY),V) \\ &= g(h(X,Y),CV) - g(A_V X,EY) \\ &= g(A_{CV}X,Y) - g(EA_V X,Y) = g((\nabla_X B)V,Y). \end{aligned} \tag{3.30}$$

This proves our assertion. ∎

Definition 3.8. If the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesic in M then M is called contact CR-product.

Now we characterize contact CR-products in $(LCS)_n$-manifold.

Theorem 3.9. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then M is a contact CR-product if and only if the shape operator A of M satisfies the condition*

$$A_{\phi D^{\perp}}D = 0. \tag{3.31}$$

Proof. Let us take M be a contact CR-submanifold of $\overline{M}$. Then for all $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$ we have from (2.13), (2.16) and (2.18) we get

$$\begin{aligned} g(A_{\phi W}\phi X, Y) &= g\big(h(\phi X, Y), \phi W\big) \\ &= g(\overline{\nabla}_Y \phi X, \phi W) \\ &= g\big((\overline{\nabla}_Y \phi)X + \phi\overline{\nabla}_Y X, \phi W\big) \\ &= g\big(\alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(X)Y\}, \phi W\big) + g(\nabla_Y X, W) \\ &= \alpha\eta(X)g(Y, \phi W) + g(\nabla_Y X, W) \\ &= g(\nabla_Y X, W) \end{aligned}$$

i.e.,

$$g\big(A_{\phi W}\phi X, Y\big) = g(\nabla_Y X, W) \tag{3.32}$$

and

$$\begin{aligned} g(A_{\phi W}\phi X, Z) &= g\big(h(\phi X, Z), \phi W\big) \\ &= g(\overline{\nabla}_Z \phi X, \phi W) \\ &= g\big((\overline{\nabla}_Z \phi)X + \phi\overline{\nabla}_Z X, \phi W\big) \\ &= g\big(\alpha\{g(X,Z)\xi + \eta(X)Z\}, \phi W\big) + g(\overline{\nabla}_Z X, W) \\ &= \alpha\eta(X)g(Z, \phi W) - g(\nabla_Z W, X) \\ &= -g(\nabla_Z W, X) \end{aligned}$$

i.e.,

$$g\big(A_{\phi W}\phi X, Z\big) = -g(\nabla_Z W, X) \tag{3.33}$$

Thus from (3.32) and (3.33) we get $\nabla_X Y \in \Gamma(D)$ and $\nabla_Z W \in \Gamma(D^{\perp})$ if and only if the relation (3.31) holds. ∎

From 3.9, we can state the following:

Corollary 3.10. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then M is a contact CR-product if and only if $Bh(D, TM) = 0$.*

Theorem 3.11. *Let M be a contact CR-submanifold of a $(LCS)_n$-manifold $\overline{M}$. Then the structure C is parallel if and only if the shape operator A_V of M satisfies the condition*

$$A_V BU + A_U BV = 0 \tag{3.34}$$

for all $U, V \in \Gamma(T^{\perp}M)$.

Proof. From (2.18), (3.7) and (3.25), we have

$$\begin{aligned} g\big((\nabla_X C)V, U\big) &= -g\big(h(X, BV), U\big) - g(FA_V X, U) \\ &= -g(A_U X, BV) - g(A_V X, BU) \\ &= -g(A_V BU + A_U BV, X) \end{aligned} \tag{3.35}$$

for all $X \in \Gamma(TM)$. From (3.35) it follows that the structure C is parallel if and only if the relation (3.34) holds. ∎

Example 3.12. Let $\overline{M} = \mathbb{R}^7$ be the semi-Euclidean space endowed with the semi-Euclidean metric $g = \big[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2\big]$ with coordinate $(t, x_1, x_2, x_3, x_4, x_5, x_6)$. Define

$$\eta = dt, \xi = \frac{\partial}{\partial t}, \phi(\frac{\partial}{\partial t}) = 0, \phi(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_4}, \phi(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_5},$$
$$\phi(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_6}, \phi(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_1}, \phi(\frac{\partial}{\partial x_5}) = \frac{\partial}{\partial x_2}, \phi(\frac{\partial}{\partial x_6}) = -\frac{\partial}{\partial x_3}.$$

Then it can be easily seen that the structure (ϕ, ξ, η, g) is a $(LCS)_7$-manifold on $\overline{M} = \mathbb{R}^7$. Now we define a submanifold M of $\overline{M}$ by $M = \{(x_1, 0, x_3, x_4, 0, x_6, t) \in \mathbb{R}^7\}$ endowed with the global vector fields

$$e_1 = \xi = \frac{\partial}{\partial t}, e_2 = \frac{\partial}{\partial x_4}, e_3 = \frac{\partial}{\partial x_2} + x_6\frac{\partial}{\partial t}, e_4 = \frac{\partial}{\partial x_6}, e_5 = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial t}.$$

Then the distributions $D_T = \text{span}\{e_2, e_5\}$ and $D^\perp = \text{span}\{e_3, e_4\}$ are respectively invariant and anti-invariant distributions on $\overline{M}$. Thus we can write $TM = D_T(= \{e_2, e_5\}) \oplus D^\perp(= \{e_3, e_4\}) \oplus < e_1(= \{\xi\}) >$. Consequently M is a contact CR-submanifold of $\overline{M} = \mathbb{R}^7$.

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