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Tauberian Conditions under which Statistical Convergence Follows from Statistical Summability by Weighted Means in Non-Archimedean Fields

Vaithinathasamy Srinivasan* **and D. Eunice Jemima**

Department of Mathematics, Faculty of Engineering & Technology, SRM Institute of Science and Technology, Kattankulathur- 603203, India
e-mail : drvsrinivas.5@gmail.com (V. Srinivasan); eunicejem@gmail.com (D. E. Jemima)

Abstract In this paper, K denotes a complete, non-trivially valued, non-archimedean field. Sequences and infinite matrices have entries in K. The weighted statistical convergence and statistical summability were enunciated along with the notion of $(\overline{N}, p_n)$ - summability method in [K. Suja, V. Srinivasan, Weighted statistical convergence in ultrametric fields, International Journal of Pure and Applied Mathematics 116 (4) (2017) 813–817]. We have proved here, the necessary and sufficient Tauberian conditions under which statistical convergence follows from statistical summability by weighted means over non-archimedean fields (an analogous and further extension of these concepts proved by F. Moricz and C. Orhan [F. Moricz, C. Orhan, Tauberian conditions under which statistical convergence follows from statistical summability by weighted means, Studia Sci. Math. Hung. 41 (4) (2004) 391–403], in the classical context).

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1. Introduction

The concept of statistical convergence was introduced by Fast [1] in 1951. It plays an important role in Summability Theory and Functional Analysis. Schoenberg introduced the relationship between Summability Theory and statistical convergence [2]. A little later, many researchers like Fridy [3], Kolk, Fridy and Miller, Mursaleen, Fridy and Orhan, Freedman et al. and Savas studied statistical convergence as a summability method. In general, statistical convergence of weighted means is studied as a class of regular matrix transformations. Ferenc Moricz and Cihan Orhan [4] established the Tauberian conditions [5] under which statistical convergence follows from statistical summability by weighted

*Corresponding author.

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means [6, 7] in classical analysis. These conditions are studied over non-archimedean fields in this paper.

Let K be a complete, non-trivially valued, non-archimedean field [8–10]. (It may be recalled that a valued field $(K, |.|)$ is non-archimedean if $|a + b| \leq \max\{|a|, |b|\}$, for all $a, b \in K$). A sequence $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \dots$ is said to be statistically convergent to a limit 'L' if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where the outer vertical bars indicate the cardinality of the set. In this case we write,

$$\text{st} - \lim_{k\to\infty} x_k = L \tag{1.1}$$

Let $p = (p_k)$, $k = 0, 1, 2, \dots$ be a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_n = \sum_{k=0}^{n} p_k, \;\; n = 0, 1, 2, \dots$$

and let

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k, \;\; n = 0, 1, 2, \dots$$

The sequence (x_k) is said to be statistically summable to L by the weighted mean method [11] determined by the sequence $p = (p_k)$, or, statistically summable $(\overline{N}, p)$ if

$$\text{st} - \lim_{n\to\infty} t_n = L. \tag{1.2}$$

K. Suja and V. Srinivasan [12] have proved that, if a sequence (x_k) is weighted statistically convergent to L, then (x_k) is $(\overline{N}, p)$ summable to L.

2. Main Results

Theorem 2.1. *Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$; let (λ_k) be a sequence in K such that $\lim_{k\to\infty} \lambda_k = 0$ and*

$$st - \lim \frac{P_n}{P_{\lambda_n}} < 1 \textit{ for every } \; 0 < \lambda_n < 1. \tag{2.1}$$

Let $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \dots$ be a sequence which is statistically summable $(\overline{N}, p)$ to a limit L. Then $\{x_k\}$ is statistically convergent to L if and only if for every $\epsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \left| \left\{ n \leq N : \left| (P_n - P_{\lambda_n})^{-1} \sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k) \right| \geq \epsilon \right\} \right| = 0. \tag{2.2}$$

The following Lemma is required in proving the theorem.

Lemma 2.2. *Let $p = (p_k)$ be a sequence of nonnegative numbers such that $p_0 > 0$ and*

$$st - \lim \frac{P_n}{P_{\lambda_n}} < 1 \textit{ for every } \; 0 < \lambda_n < 1,$$

and let $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \ldots$ be a sequence which is statistically summable $(\overline{N}, p)$ to a limit L. Then for every $0 < \lambda_n < 1$,

$$st-\lim_{n\to\infty} t_{\lambda_n} = L, \tag{2.3}$$

where $\{P_n\}$ and $\{t_{\lambda_n}\}$ are non-decreasing sequences of positive numbers.

Proof. Given that the sequence $\{x_k\}$ is statistically summable $(\overline{N}, p)$ to a limit L.

$$\text{i.e., st} - \lim_{n\to\infty} t_n = L.$$

$$\text{i.e., } \lim_{N\to\infty} \frac{1}{N}|\{n \le N : |t_n - L| \ge \epsilon\}| = 0.$$

$$\text{i.e., } \lim_{N\to\infty} \frac{1}{N}\left|\left\{n \le N : \left|\frac{1}{P_n}\sum_{k=0}^{n} p_k x_k - L\right| \ge \epsilon\right\}\right| = 0. \tag{2.4}$$

To prove, $\text{st} - \lim_{n\to\infty} t_{\lambda_n} = L$

i.e., to prove $\lim_{N\to\infty} \frac{1}{N}|\{\lambda_n \le N : |t_{\lambda_n} - L| \ge \epsilon\}| = 0$,

i.e., to prove $\lim_{N\to\infty} \frac{1}{N}\left|\left\{\lambda_n \le N : \left|\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k x_k - L\right| \ge \epsilon\right\}\right| = 0$,

consider

$$\begin{aligned}
&\frac{1}{N}\left|\left\{\lambda_n \le N : \left|\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k x_k - L\right| \ge \epsilon\right\}\right| \\
&\quad = \frac{1}{N}\left|\left\{\lambda_n \le N : \left|\left(\frac{P_n}{P_{\lambda_n}}\right)\frac{1}{P_n}\sum_{k=0}^{\lambda_n} p_k x_k - L\right| \ge \epsilon\right\}\right| \\
&\quad \le \frac{1}{N}\left|\left\{n \le N : \left|\frac{1}{P_n}\sum_{k=0}^{n} p_k x_k - L\right| \ge \epsilon\right\}\right| \quad \text{(using (2.1))} \\
&\qquad\qquad \to 0 \quad \text{as } N \to \infty \quad \text{(using (2.4))}
\end{aligned}$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{\lambda_n \le N : \left|\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k x_k - L\right| \ge \epsilon\right\}\right| = 0.$$

or, $\text{st} - \lim_{n\to\infty} t_{\lambda_n} = L$, which proves the lemma. ∎

We shall next prove that, for $0 < \lambda_n < 1$,

$$(P_n - P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n} p_k x_k = t_n + P_{\lambda_n}(P_n - P_{\lambda_n})^{-1}(t_n - t_{\lambda_n})$$

provided $P_n > P_{\lambda_n}$. To this end, consider the right-hand side.

$$\begin{aligned} t_n + \frac{P_{\lambda_n}}{P_n - P_{\lambda_n}}(t_n - t_{\lambda_n}) &= \frac{P_n t_n - P_{\lambda_n} t_n + P_{\lambda_n} t_n - P_{\lambda_n} t_{\lambda_n}}{P_n - P_{\lambda_n}} \\ &= \frac{1}{P_n - P_{\lambda_n}}\left[P_n\left(\frac{1}{P_n}\sum_{k=0}^{n} p_k x_k\right) - P_{\lambda_n}\left(\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_k x_k\right)\right] \\ &= \frac{1}{P_n - P_{\lambda_n}}\left[\sum_{k=0}^{n} p_k x_k - \sum_{k=0}^{\lambda_n} p_k x_k\right] \\ &= (P_n - P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n} p_k x_k \end{aligned}$$

Thus, $(P_n - P_{\lambda_n})^{-1}\sum\limits_{k=\lambda_n+1}^{n} p_k x_k = t_n + P_{\lambda_n}(P_n - P_{\lambda_n})^{-1}(t_n - t_{\lambda_n})$.

Now, rearranging the terms and adding x_n on both sides we have,

$$\begin{aligned} x_n - t_n &= P_{\lambda_n}(P_n - P_{\lambda_n})^{-1}(t_n - t_{\lambda_n}) - (P_n - P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n} p_k x_k + x_n \\ &= P_{\lambda_n}(P_n - P_{\lambda_n})^{-1}(t_n - t_{\lambda_n}) + (P_n - P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n} p_k (x_n - x_k) \end{aligned} \tag{2.5}$$

Proof of Theorem 2.1.

Necessity:

Here, we assume that $\text{st} - \lim\limits_{n\to\infty} x_n = L$ and prove that, for every $0 < \lambda_n < 1$,

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{n \le N : \left|(P_n - P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k)\right| \ge \epsilon\right\}\right| = 0.$$

Now, since $\text{st} - \lim\limits_{n\to\infty} x_n = L$ and $\text{st} - \lim\limits_{n\to\infty} t_n = L$, we have

$$\text{st} - \lim_{n\to\infty}(x_n - t_n) = 0.$$

$$\text{i.e., } \lim_{N\to\infty}\frac{1}{N}|\{n \le N : |x_n - t_n| \ge \epsilon\}| = 0.$$

From equation (2.5) we have,

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{n \le N : \left|P_{\lambda_n}(P_n - P_{\lambda_n})^{-1}(t_n - t_{\lambda_n}) + (P_n - P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k)\right| \ge \epsilon\right\}\right| = 0.$$

Since the valuation is non-archimedean wherein $|a + b| = |a|$ if $|a| > |b|$ and since

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{n \le N : \left|P_{\lambda_n}(P_n - P_{\lambda_n})^{-1}(t_n - t_{\lambda_n})\right| \ge \epsilon\right\}\right| \to 0 \text{ as } N \to \infty$$

by (1.2) and (2.3),

we have,

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{n\le N:\left|(P_n-P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k)\right|\ge\epsilon\right\}\right|=0.$$

Sufficiency:

We now assume that

$$\lim_{N\to\infty}\frac{1}{N}\left|\left\{n\le N:\left|(P_n-P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k)\right|\ge\epsilon\right\}\right|=0,$$

and prove that

$$\mathrm{st}-\lim_{n\to\infty}x_n=L.$$

To this end, it is enough if we prove that

$$\mathrm{st}-\lim_{n\to\infty}(x_n-t_n)=0.$$

i.e., to prove, $\lim_{N\to\infty}|\{n\le N:|x_n-t_n|\ge\epsilon\}|=0.$
Using equation (2.5) we have,

$$\begin{aligned}&\frac{1}{N}|\{n\le N:|x_n-t_n|\ge\epsilon\}|\\&\quad=\frac{1}{N}\left|\left\{n\le N:\left|P_{\lambda_n}(P_n-P_{\lambda_n})^{-1}(t_n-t_{\lambda_n})\right.\right.\right.\\&\qquad\left.\left.\left.+(P_n-P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k)\right|\ge\epsilon\right\}\right|\\&\quad\le\max\left\{\begin{array}{l}\frac{1}{N}\left|\left\{n\le N:\left|P_{\lambda_n}(P_n-P_{\lambda_n})^{-1}(t_n-t_{\lambda_n})\right|\ge\epsilon\right\}\right|,\\ \frac{1}{N}\left|\left\{n\le N:\left|(P_n-P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k)\right|\ge\epsilon\right\}\right|\end{array}\right\}\end{aligned}$$

But, by our assumption,

$$\frac{1}{N}\left|\left\{n\le N:\left|(P_n-P_{\lambda_n})^{-1}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k)\right|\ge\epsilon\right\}\right|\to 0\quad\text{as}\quad N\to\infty.$$

Therefore,

$$\begin{aligned}&\frac{1}{N}|\{n\le N:|x_n-t_n|\ge\epsilon\}|\\&\quad\le\max\left\{\frac{1}{N}\left|\left\{n\le N:\left|P_{\lambda_n}(P_n-P_{\lambda_n})^{-1}(t_n-t_{\lambda_n})\right|\ge\epsilon\right\}\right|,0\right\}\\&\quad\le\frac{1}{N}\left|\left\{n\le N:\left|P_{\lambda_n}(P_n-P_{\lambda_n})^{-1}(t_n-t_{\lambda_n})\right|\ge\epsilon\right\}\right|\\&\qquad\to 0\quad\text{as}\quad N\to\infty\ \text{by (1.2) and (2.3)}\end{aligned}$$

which implies that

$$\lim_{N\to\infty} \frac{1}{N}|\{n \leq N : |x_n - t_n| \geq \epsilon\}| = 0.$$
$$\text{i.e., st} - \lim_{n\to\infty} (x_n - t_n) = 0.$$

Thus, $\{x_k\}$ is statistically convergent to L. This completes the proof of the theorem.

Conclusion

In this paper, we thus have proved both the necessary and sufficient Tauberian conditions under which statistical convergence follows from statistical summability by (one of the special methods of summability) weighted means over non-archimedean fields. This has given us an impetus to go for a development of this notion through weighted statistical convergence of double sequences in such fields K.

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