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The New Hybrid Iterative Algorithm for Numerical Reckoning Fixed Points of Suzuki's Generalized Nonexpansive Mappings with Numerical Experiments

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Abstract The purpose of this work is to introduce a new hybrid iterative algorithm to approximate fixed point of Suzuki's generalized nonexpansive mappings. We prove the convergence theorem in uniformly convex Banach spaces under the several conditions. A numerical example is also given to examine the fastness of the proposed iteration process under different control conditions and initial points with the well-known iterations in the literatures.

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1. Introduction

The first metric theoretical result for the existence and convergence of a fixed point is the Banach contraction mapping principle as follows:

Theorem 1.1 ([1]). *Let (E, d) be a complete metric space and $T : E \to E$ be a contraction mapping, i.e., a mapping for which there exists a constant $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in E$. Then T has a unique fixed point $x^* \in E$. Moreover, the Picard iteration $\{x_n\}$ associated to T with the initial point $x_1 \in X$ which is given by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, ..., \tag{P_n}$$

converges to the fixed point x^*.

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Since this principle requires only the contractive condition of the self-mapping on a complete metric space, it is easy to test such condition and so it is used to demonstrate the existence and convergence of a solution of many equations such as integral equations, ordinary differential equations, partial differential equations, matrix equations, functional equations, etc. Based on the mentioned impact, the Banach contraction principle has many applications not only in the various branches in mathematics but also in other fields. However, the Picard iteration (P_n) in the convergence part has not been successfully employed in approximating the fixed point of some mappings such as a nonexpansive self-mapping T on a metric space (X, d), that is, a mapping satisfying the following condition:

$$d(Tx, Ty) \leq d(x, y) \tag{1.2}$$

for all $x, y \in X$ even when $F(T) := \{x \in Dom(T) : Tx = x\} \neq \emptyset$. Next, we give some example showing the previous claiming.

Example 1.2. Consider a mapping $T : [0, 1] \to [0, 1]$ defined by $Tx = 1 - x$ for all $x \in [0, 1]$. Then T is a nonexpansive mapping with a usual metric and $F(T) = \{1/2\}$. If one chooses as a starting value $x = x_0$ such that $x_0 \neq \frac{1}{2}$, then the Picard iteration (P_n) of T yield that

$$\begin{aligned} x_1 = Tx_0 &= 1 - x_0, \\ x_2 = Tx_1 &= x_0, \\ x_3 = Tx_2 &= 1 - x_0, \\ &\vdots \end{aligned}$$

This implies that Picard iteration (P_n) does not converges to a fixed poiiterationnt of T.

Based on the problem in the above example, when a fixed point of nonexpansive mappings exists, other approximation techniques are needed to approximate it.

Throughout this paper, unless otherwise specified, let C be a nonempty closed convex subset of a Banach space and $T : C \to C$ be a given mapping. Next, we introduce some iterations iterationfor approximation fixed points.

In [2], the Mann iterationis defined by the following sequence $\{x_n\} \subseteq C$:

$$\left.\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tx_n \end{aligned}\right\} \tag{M_n}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ are real sequences in $[0, 1]$. If $\alpha_n = 1$ for all $n \in \mathbb{N}$, then the Mann iteration iteration reduces to the Picard iteration process. In 1974, Ishikawa [3] extended the concept of Mann iteration to the new iteration $\{x_n\} \subseteq C$, where $\{x_n\}$ construting by

$$\left.\begin{aligned} x_1 &\in C, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned}\right\} \tag{I_n}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$. This iteration becomes the Mann iteration when $\beta_n = 0$ for all $n \in \mathbb{N}$. In [4], Rhoades gave the following interesting observation related to the rate of convergence of Mann and Ishikawa iterations:

- for a decreasing function, the Mann iteration converges faster than the Ishikawa iteration, and vice versa for an increasing.

In 2000, Noor [5] introduced a new iteration $\{x_n\} \subseteq C$, where $\{x_n\}$ is defined iteratively by

$$\left.\begin{array}{rcl} x_1 & \in & C, \\ z_n & = & (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n & = & (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} & = & (1-\alpha_n)x_n + \alpha_n T y_n \end{array}\right\} \quad (N_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ are control sequences. It is easy to see that this idea covers the idea of Ishikawa [3]. From the great impact of the Picard iteration (P_n), the Mann iteration (M_n), the Ishikawa iteration (I_n), the Noor iteration (N_n), it was developed extensively by several researchers.

In 2007, Agarwal et al. [6] introduced an iteration $\{x_n\} \subseteq C$, where $\{x_n\}$ is defined iteratively by

$$\left.\begin{array}{rcl} x_1 & \in & C, \\ y_n & = & (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} & = & (1-\alpha_n)T x_n + \alpha_n T y_n \end{array}\right\} \quad (ARS_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$ are control sequences. They also claimed that this process converges at a rate same as that of the Picard iteration and faster than the Mann iteration for Banach contraction mappings. In 2014, Abbas and Nazir [7] introduced an iteration process $\{x_n\} \subseteq C$, where $\{x_n\}$ is defined iteratively by iteration

$$\left.\begin{array}{rcl} x_1 & \in & C, \\ z_n & = & (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n & = & (1-\beta_n)T x_n + \beta_n T z_n, \\ x_{n+1} & = & (1-\alpha_n)T y_n + \alpha_n T z_n \end{array}\right\} \quad (AN_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0,1]$ are control sequences. They also claimed that the above iteration converges faster than the iteration (ARS_n) for some nonlinear mappings.

Recently, Thakur et al. [8] introduced a new iteration process $\{x_n\} \subseteq C$, where $\{x_n\}$ is defined iteratively by

$$\left.\begin{array}{rcl} x_1 & \in & C, \\ z_n & = & (1-\beta_n)x_n + \beta_n T x_n, \\ y_n & = & T((1-\alpha_n)x_n + \alpha_n z_n), \\ x_{n+1} & = & T y_n \end{array}\right\} \quad (TTP_n)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$. They also prove the weak and strong convergence theorems in a uniformly convex Banach space. This work is an inspiration to writ tis paper.

On the other side of the research for convergence analysis of fixed points, several types of nonlinear mappings are presented. For instance, Suzuki [9] first presented the new nonlinear which is a generalization of nonexpansive mappings as follows:

Definition 1.3 ([9]). Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Longrightarrow \|Tx - Ty\| \leq \|x - y\| \quad (1.3)$$

for all $x, y \in C$.

Lemma 1.4 ([9])**.** *Let C be a nonempty subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be a mapping. Then the following assertions hold:*

(1) If T is a nonexpansive mapping, then T satisfies the condition (C).
(2) If T satisfies the condition (C) and T has a fixed point, then T is a quasi-nonexpansive mapping, that is, $F(T) \neq \emptyset$ and T satisfies the following condition

$$\|Tx - p\| \leq \|x - p\| \tag{1.4}$$

for all $x \in C$ and $p \in F(T)$.
(3) If T satisfies the condition (C), then $\|x - Ty\| \leq 3\|Tx - x\| + \|x - y\|$ for all $x, y \in C$.

Three years later, Phuengrattana [10] used the Ishikawa iteration for proving the convergence theorems for mappings satisfying the condition (C) in uniformly convex Banach spaces and CAT(0) spaces. Nowadays, the existence and convergence results of fixed points for mappings satisfying condition (C) have been investigated by many authors (see in [8, 11–14] and references therein).

Motivated by the above inspiration and many research papers in literatures, the main goal of this paper is to introduce a new iteration process which weak and strong converging to fixed points for mapping satisfying condition (C) in a uniformly convex Banach spaces. Theoretical results for guaranteeing the trueness of convergence results are given. Finally, we provide a numerical example for comparing the convergence of the proposed iteration with many existing iteration for a mapping satisfying condition (C).

2. Notations and Useful Tools for Convergence Theorems

For the convenience of the readers and self-dependency of the paper, we include the definitions, notations and the useful tools in this section.

Definition 2.1 ([15])**.** A Banach space X is called uniformly convex if for each $\epsilon \in (0, 2]$ there is a $\delta > 0$ such that for $x, y \in X$, the following condition holds:

$$\left.\begin{array}{rcl} \|x\| & \leq & 1, \\ \|y\| & \leq & 1, \\ \|x-y\| & > & \epsilon \end{array}\right\} \Longrightarrow \left\|\frac{x+y}{2}\right\| \leq \delta. \tag{2.1}$$

Definition 2.2 ([16])**.** A Banach space $(X, \|\cdot\|)$ is said to satisfy the Opial property if for each weakly convergent sequence $\{x_n\}$ in X with weak limit x, we get

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$.

Lemma 2.3 ([9])**.** *Let T be a mapping on a subset C of a Banach space X with the Opial property. If T satisfies the condition (C) and $\{x_n\} \subseteq X$ converges weakly to $z \in X$ such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then z is a fixed point of T.*

Lemma 2.4 ([9])**.** *Let C be a weakly compact convex subset of a uniformly convex Banach space X and $T : C \to C$ be a mapping. If T satisfies the condition (C), then T has a fixed point.*

Lemma 2.5 ([9]). *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r,$$

$$\limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\limsup_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Definition 2.6. Let C be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$ and $\{x_n\}$ be a bounded sequence in a C.

(1) The asymptotic radius of $\{x_n\}$ relative to C is denoted by $r(C, \{x_n\})$ and it is given by

$$r(C, \{x_n\}) := \inf_{x\in C} \{r(x, \{x_n\})\},$$

where $r(x, \{x_n\}) := \limsup_{n\to\infty} \|x - x_n\|$.

(2) The asymptotic center of $\{x_n\}$ relative to C is denoted by $A(C, \{x_n\})$ and it is given by

$$A(C, \{x_n\}) := \{x \in C : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Remark 2.7. A set $A(C, \{x_n\})$ consists of exactly one point in a uniformly convex Banach space.

In [17], Senter and Dostan introduced the concept of special self mapping as follows:

Definition 2.8 ([17]). Let C be a nonempty subset of a Banach space $(X, \|\cdot\|)$. A mapping $T : C \to C$ is said to satisfy condition (I) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T))) \tag{2.2}$$

for all $x \in C$, where $d(x, F(T)) := \inf_{y\in F(T)} \|x - y\|$.

3. Convergence Theoretical Results

Throughout this section, unless otherwise specified, C is a nonempty subset of a Banach space X. First, a new hybrid iterative algorithm for seeking fixed points for a mapping $T : C \to C$ satisfying condition (C) is introduce as follows:

$$\left.\begin{array}{l} x_1 \in C, \\ x_{n+1} = T^n y_n, \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ z_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n = 1, 2, ..., \end{array}\right\} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real control sequences in the interval $[0, 1]$.

Next, we establish the following useful results for helping the convergence theorem.

Lemma 3.1. *Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by (3.1). If C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ satisfies the condition (C) with $F(T) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Since $F(T) \neq \emptyset$, we may choose a point p in $F(T)$ and $z \in C$. It is easy to see that the left hand side of the implication (1.3) holds. By using the condition C of T, we obtain

$$\|Tp - Tz\| \leq \|p - z\|.$$

From the defining of z_n and y_n in the algorithm (3.1), we obtain

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{3.2}$$

and then

$$\begin{aligned} \|y_n - p\| &= \|T((1-\alpha_n)x_n + \alpha_n z_n) - p\| \\ &\leq \|(1-\alpha_n)x_n + \alpha_n z_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned} \|x_{n+1} - p\| &= \|T^n y_n - p\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\| \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$. It follows that $\{\|x_n - p\|\}$ is bounded and nonincreasing and thus $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$. ■

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a mapping satisfying condition (C) with $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in C$, let the sequence $\{x_n\}$ be generated by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are control sequences in the interval $[a, b] \subseteq (0, 1)$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. ($\Longrightarrow$) Suppose that $F(T) \neq \emptyset$ and $p \in F(T)$. From Lemma 3.1, we get $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$ is bounded. Assume that

$$\lim_{n\to\infty} \|x_n - p\| = r. \tag{3.5}$$

From (3.2), we obtain

$$\limsup_{n\to\infty} \|z_n - p\| \leq r. \tag{3.6}$$

By using Lemma 3.1, we have

$$\limsup_{n\to\infty} \|Tx_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = r. \tag{3.7}$$

For each $n \in \mathbb{N}$, we get

$$\begin{aligned} \|x_{n+1}-p\| &\leq \|y_n - p\| \\ &= \|T((1-\alpha_n)x_n + \alpha_n z_n) - p\| \\ &\leq \|(1-\alpha_n)x_n + \alpha_n z_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|z_n - p\|. \end{aligned} \tag{3.8}$$

This implies that

$$\frac{\|x_{n+1}-p\| - \|x_n - p\|}{\alpha_n} \leq \|z_n - p\| - \|x_n - p\|$$

and so

$$\|x_{n+1}-p\| - \|x_n - p\| \leq \frac{\|x_{n+1}-p\| - \|x_n - p\|}{\alpha_n} \leq \|z_n - p\| - \|x_n - p\|$$

for all $n \in \mathbb{N}$. Therefore,

$$\|x_{n+1}-p\| \leq \|z_n - p\|$$

for all $n \in \mathbb{N}$. Taking limit inferior as $n \to \infty$ in the above inequality, we get

$$r \leq \liminf_{n\to\infty}\|z_n - p\|. \tag{3.9}$$

It follows from (3.6) and (3.9) that

$$\lim_{n\to\infty}\|z_n - p\| = r \tag{3.10}$$

and hence

$$\lim_{n\to\infty}\|\beta_n(Tx_n - p) - (1-\beta_n)(x_n - p)\| = r. \tag{3.11}$$

By using Lemma 2.5 with (3.5), (3.7) and (3.11), we obtain $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$.

($\Longleftarrow$) Suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Let $p \in A(C, \{x_n\})$. From Lemma 3.1, we get

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty}\|x_n - Tp\| \\ &\leq \limsup_{n\to\infty}(3\|Tx_n - x_n\| + \|x_n - p\|) \\ &= \limsup_{n\to\infty}\|x_n - p\| \\ &= r(p, \{x_n\}). \end{aligned}$$

It yields that $Tp \in A(C, \{x_n\})$. Since X is uniformly convex, we get $A(C, \{x_n\})$ is a singleton set and hence $p = Tp$.

This completes the proof. ■

Here we give the (weak/strong) convergence theorem under several conditions.

Theorem 3.3. *Let C be a nonempty convex subset of a uniformly convex Banach space X and $T : C \to C$ be a mapping satisfying the condition (C). Suppose that $x_1 \in C$ and the sequence $\{x_n\}$ is generated by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are control sequences in the interval $[a, b] \subseteq (0, 1)$. Then the following assertions hold.*

(W1) If C is closed, X satisfies the Opial property and $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of T.

(S1) If C is compact, then $\{x_n\}$ converges strongly to a fixed point of T.

(S2) If C is closed, T satisfies thecondition (I) and $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. (W1) First, Theorem 3.2 implies the boundedness of a sequence $\{x_n\}$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. It follows from the uniform convexity of X that X is reflexive. Now, we can use the Eberlin's theorem to guarantee the existence of a weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$. We may assume that $\{x_{n_j}\}$ converges weakly to some $z \in X$. By the closedness and the convexity of C, the Mazur's theorem claims that $z \in C$. Using Lemma 2.3, $z \in F(T)$. Next, we will show that $\{x_n\}$ converges weakly to z. We may assume this to contrary that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $w \in C$ such that $z \neq w$. By Lemma 2.3, we obtain w is a fixed point of $F(T)$. By mixing Theorem 3.2 together with the Opial property, we get

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z\| &= \lim_{j\to\infty} \|x_{n_j} - z\| \\
&< \lim_{j\to\infty} \|x_{n_j} - w\| \\
&= \lim_{n\to\infty} \|x_n - w\| \\
&= \lim_{k\to\infty} \|x_{n_k} - w\| \\
&< \lim_{k\to\infty} \|x_n - z\| \\
&= \lim_{n\to\infty} \|x_n - z\|,
\end{aligned}$$

which is a contradiction. So $z = w$ and then $\{x_n\}$ converges weakly to a fixed point of T.

(S1) From Lemma 2.4, we get $F(T) \neq \emptyset$. By Theorem 3.2, we obtain $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since C is compact, there exists a subsequence x_{n_k} of $\{x_n\}$ such that x_{n_k} converges strongly to p for some $p \in C$. From Lemma 1.4, we have

$$\|x_{n_k} - Tp\| \leq 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$, we have x_{n_k} converges to Tp. By the uniqueness of the limit for x_{n_k}, we get $p = Tp$ and so $p \in F(T)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we get $\lim_{n\to\infty} \|x_n - p\| = \lim_{k\to\infty} \|x_{n_k} - p\| = 0$. This yields that $\{x_n\}$ converges strongly to a fixed point of T.

(S2) Suppose that $p \in F(T)$. From Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists and hence $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} \|x_n - p\| = r$ for some $r \geq 0$. If $r = 0$, we have nothing to prove. Next, we may assume that $r > 0$. Since T satisfies the condition (I), there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T))) \tag{3.12}$$

for all $x \in C$. From (3.12), we get

$$f(d(x_n, F(T))) \leq \|x_n - Tx_n\| \tag{3.13}$$

for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequality and using Theorem 3.2, we get

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{3.14}$$

This implies that $\lim_{n\to\infty} d(x_n, F(T)) = 0$ because f is nondecreasing. Thus we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subseteq F(T)$ such that

$$\|x_{n_k} - y_k\| < \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}.$$

It implies that

$$\|x_{n_{k+1}} - y_k\| \le \|x_{n_k} - y_k\| < \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}.$$

Then

$$\|y_{k+1} - y_k\| \le \|y_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - y_k\| \le \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}$$

for all $k \in \mathbb{N}$. This implies that $\{y_k\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed in a Banach space X, we get $\{y_k\}$ converges to some point $p \in F(T)$. It follows that $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we get $\lim_{n\to\infty} \|x_n - p\| = \lim_{k\to\infty} \|x_{n_k} - p\| = 0$. This yields that $\{x_n\}$ converges strongly to a fixed point of T.

This completes the proof. ∎

4. A Numerical Result

In this section, using Example 4.1, we will compare the convergence of new hybrid iteration (3.1) with the other iterations.

Example 4.1. Let $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$ be a usual normed space and $C = [0, 1]$. We see that C is a compact convex subset of X. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} 1 - x & \text{if } x \in [0, 0.2); \\ \frac{x+4}{5} & \text{if } x \in (0.2, 1], \end{cases}$$

Recently, Thakur et al. [8] showed that T satisfies condition (C) and T has a unique fixed point $x^* = 1$. Also, he claimed that T is not a nonexpansive mapping.

Next, we will present the comparison of the convergence behavior of the proposed iteration (3.1) and several iterations in the literatures. We set

$$\alpha_n = \tfrac{n}{n+1},\ \beta_n = \tfrac{n}{n+5} \text{ and } \gamma_n = \tfrac{1}{\sqrt{2n+9}}.$$

Table 1 shows the computational results for several initial values with the stopping criterion that $\|x_n - x^*\| \le 10^{-15}$.

To investigate the influence of parameters α_n, β_n and γ_n in our agorithm, the following set of parameters are taken:

Case-A:: $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{n}{n+5}$ and $\gamma_n = \frac{1}{\sqrt{2n+9}}$

Case-B:: $\alpha_n = \frac{2n}{5n+2}$, $\beta_n = \frac{1}{n+5}$ and $\gamma_n = \frac{1}{(7n+9)^{3/2}}$

Case-C:: $\alpha_n = \frac{2n}{7n+9}$, $\beta_n = \frac{1}{(3n+7)^{3/2}}$ and $\gamma_n = \frac{1}{(n+1)^{5/2}}$

Case-D:: $\alpha_n = \frac{n}{4n+5}$, $\beta_n = \frac{1}{(3n+7)^{5/2}}$ and $\gamma_n = \frac{1}{(n+5)^{3/2}}$

Initial value	Number of iterations required to obtain fixed point							
	Picard iteration	Mann iteration	Ishikawa iteration	Agarwal iteration	Noor iteration	Abbas iteration	Thakur iteration	New iteration
0.10	22	27	21	16	21	18	10	6
0.15	22	27	21	16	21	18	10	6
0.20	22	27	21	16	21	18	10	6
0.50	22	27	21	16	21	18	10	6
0.80	21	26	21	16	20	18	10	6
0.90	21	26	20	16	20	17	9	5

TABLE 1. Influence of initial point.

Next, we test the algorithms for different initial points with the above set of parameters and the stopping criterion $\|x_n - x^*\| \leq 10^{-15}$ are given in Table 2.

TABLE 2. Influence of parameters: comparison of the new method to other methods.

Iteration	Initial point					
	0.1	0.15	0.2	0.5	0.8	0.9
Case-A: $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{n}{n+5}$ and $\gamma_n = \frac{1}{\sqrt{2n+9}}$						
Mann	27	27	27	27	26	26
Ishikawa	21	21	21	21	21	20
Agarwal	16	16	16	16	16	16
Noor	21	21	21	21	20	20
Abbas	18	18	18	18	18	17
Thakur	10	10	10	10	10	9
New	6	6	6	6	5	5
Case-B: $\alpha_n = \frac{2n}{5n+2}$, $\beta_n = \frac{1}{n+5}$ and $\gamma_n = \frac{1}{(7n+9)^{3/2}}$						
Mann	91	91	91	90	88	86
Ishikawa	91	91	91	89	87	85
Agarwal	21	22	22	21	21	20
Noor	91	91	91	89	87	85
Abbas	15	15	15	15	15	14
Thakur	11	11	11	11	11	10
New	6	6	6	6	6	5
Case-C: $\alpha_n = \frac{2n}{7n+9}$, $\beta_n = \frac{1}{(3n+7)^{3/2}}$ and $\gamma_n = \frac{1}{(n+1)^{5/2}}$						
Mann	139	139	139	137	134	131
Ishikawa	139	139	139	137	134	131
Agarwal	21	22	22	21	21	21
Noor	139	139	139	137	134	131
Abbas	14	14	14	14	13	13
Thakur	11	11	11	11	11	11
New	6	6	6	6	6	6
Case-D: $\alpha_n = \frac{n}{4n+5}$, $\beta_n = \frac{1}{(3n+7)^{5/2}}$ and $\gamma_n = \frac{1}{(n+5)^{3/2}}$						
Mann	24	24	24	21	17	14
Ishikawa	161	161	161	158	155	151
Agarwal	22	22	22	22	21	21
Noor	161	161	161	158	155	151
Abbas	13	14	13	13	13	13
Thakur	11	11	11	11	11	11
New	6	6	6	6	6	6

The comparison in Table 2 don't have the Picard iteration since parameters α_n, β_n and γ_n do not appear in this iteration. Here, the fastness and stability of various iterations corresponding to all mentioned set of parameters is determine. In Figure 1, we give the observations of the average of number of iterations for different initial points from each sub-table of the Table 2. It shows that the proposed hybrid iteration (3.1) not only converges faster than the known iterations but also is stable.

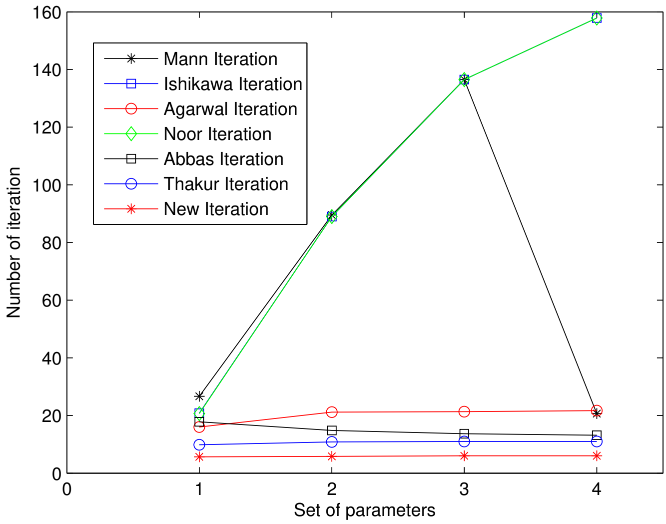

Figure 1. Average number of iterations for different set of parameters.

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