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The Dirichlet-Neumann Boundary Value Problem for the Inhomogeneous Bitsadze Equation in a Ring Domain

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Abstract In this study, by using some integral representations formulas, we study solvability conditions and explicit solution of the Dirichlet-Neumann problem, an example for a combined boundary value problem, for the Bitsadze equation in a ring domain.

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1. Introduction and Preliminaries

Main boundary value problems in complex analysis are the Schwarz, the Dirichlet and the Neumann boundary value problems. Recently, these boundary problems for some model complex partial differential equations have been studied in different domains. The basic idea of these works comes with Begehr's papers (see [1, 2]). In his papers, he investigated boundary value problems for complex model equations in the unit disc D of the complex plane by aid of some integral representation formulas.

In [2], the author studied the Dirichlet-Neumann problem for the inhomogeneous Bitsadze equation in the unit disc. In this article we solve the same problem in a concentric ring domain R, by using Begehr's iteration method which were used in his papers.

In order to calculate integrals in $R = \{z \in \mathbb{C} : 0 < r < |z| < 1\}$, we need some integral formulas which are basic instruments:

Theorem 1.1 (**Gauss theorem, complex form** [1]). *Let $D \subset \mathbb{C}$ be a regular domain, $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$, then*

$$\int_D w_{\overline{z}}(z)dxdy = \frac{1}{2i}\int_{\partial D} w(z)dz \ \ and \int_D w_z(z)dxdy = -\frac{1}{2i}\int_{\partial D} w(z)d\overline{z},$$

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where the complex partial differential operators ∂_z *and* $\partial_{\overline{z}}$ *are defined by*

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ \partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right);\ z = x + iy,\ x, y\ \in \mathbb{R}.$$

In an open domain D in $\mathbb{C}$, if a complex-valued function w satisfies the differential equation

$$w_{\overline{z}} = 0 \tag{1.1}$$

then it is called an analytic function in D. For analytic functions the well-known Cauchy theorem is valid [4].

Theorem 1.2 (**Cauchy theorem**). *Let* γ *be a simple closed smooth curve and* D *be the inner domain, bounded by* γ*. If* w *is an analytic function in* D*, continuous in* $\overline{D}$*, then*

$$\int_\gamma w(z)dz = 0. \tag{1.2}$$

Then, in view of Cauchy theorem, the following representation is valid for any analytic function:

Theorem 1.3 (**Cauchy Integral Formula**). *Let* γ *be a simple closed smooth curve and* D *be the inner domain, bounded by* γ*. If* w *is an analytic function in* D *and* $z \in D$*, then*

$$w(z) = \frac{1}{2\pi i}\int_\gamma w(\zeta)\frac{d\zeta}{\zeta - z}. \tag{1.3}$$

One can obtain the Cauchy-Pompeiu representation formula from the Gauss theorem.

Theorem 1.4 (**Cauchy-Pompeiu representations**). *Let* $D \subset \mathbb{C}$ *be a regular domain of* $\mathbb{C}$*,* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$*,* $\zeta = \xi + i\eta$*. Then*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

holds for all $z \in D$*.*

2. Some Theorems on Boundary Value Problems

The following theorems about the Dirichlet and Neumann problems for the inhomogeneous Cauchy-Riemann equation are proved in [3].

Theorem 2.1 ([3]). *The Dirichlet problem for the inhomogeneous Cauchy-Riemann equation in* R

$$w_{\overline{z}} = f,\ \text{in } R;\ w = \gamma \text{ on } \partial R, \tag{2.1}$$

for $f \in L_p(R;\mathbb{C}), p > 2,\ \gamma \in C(\partial R;\mathbb{C})$ *given, is solvable in the class* $W^{1,p}_{\overline{z}}(R;\mathbb{C}) \cap C(\overline{R};\mathbb{C})$ *if and only if for* $z \in R$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{1 - \overline{z}\zeta} = \frac{1}{\pi}\int_R f(\zeta)\frac{\overline{z}}{1 - \overline{z}\zeta}d\xi d\eta, \tag{2.2}$$

$$\frac{1}{2\pi i}\int_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2 - \overline{z}\zeta} = \frac{1}{\pi}\int_R f(\zeta)\frac{\overline{z}}{r^2 - \overline{z}\zeta}d\xi d\eta. \tag{2.3}$$

The unique solution then is expressed by the formula

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta-z}-\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{d\xi d\eta}{\zeta-z}. \tag{2.4}$$

■

We note that in R, the Neumann boundary vale problem on the boundary of the circular ring domain is related to the outward normal derivative

$$\partial_{\nu_z}=\begin{cases} z\partial_z+\overline{z}\partial_{\overline{z}}, & |z|=1,\\ -\frac{z}{r}\partial_z-\frac{\overline{z}}{r}\partial_{\overline{z}}, & |z|=r.\end{cases}$$

Theorem 2.2 ([3]). *The Neumann problem for the inhomogeneous Cauchy-Riemann equation in R,*

$$w_{\overline{z}}=f,\ \lambda|z|\partial_{\nu_z}w|_{\partial R}=\gamma,\ w(z_0)=c,\ \lambda=\begin{cases}1, & |z|=1,\\ -1, & |z|=r,\end{cases}$$

for $f\in C^{\alpha}(\overline{R};\mathbb{C})$, $0<\alpha<1$, $\gamma\in C(\partial R;\mathbb{C})$, $c\in\mathbb{C}$, $z_0\in R$ given, is solvable by a function from $W^{1+\alpha}_{\overline{z}}(\overline{R};\mathbb{C})$ with continous weak $z-$derivative on $\overline{R}$ if and only if for $z\in R$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{1-\overline{z}\zeta}+\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2}=0,$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{r^2-\overline{z}\zeta}+\frac{r^2}{\pi}\int\limits_{R}f(\zeta)\frac{d\xi d\eta}{(r^2-\overline{z}\zeta)^2}=0.$$

Moreover if γ and f satisfy the condition

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{\zeta}=0$$

then the solution is a unique, single valued function represented by

$$\begin{aligned} w(z) &= c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left[\gamma(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{1-z\overline{\zeta}}{1-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\\ &\quad+\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{r^2-z\overline{\zeta}}{r^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\\ &\quad-\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{z-z_0}{(\zeta-z_0)(\zeta-z)}d\xi d\eta. \end{aligned} \tag{2.5}$$

■

3. Dirichlet-Neumann Problem for Bitsadze Equation

Theorem 3.1. *The Dirichlet-Neumann boundary value problem for the inhomogeneous Bitsadze equation in R*

$$w_{\overline{z}\overline{z}}=f, w|_{\partial R}=\gamma_0, (\lambda|z|\partial_{\nu_z}w_{\overline{z}})|_{\partial R}=\gamma_1, w_{\overline{z}}(z_0)=c, \tag{3.1}$$

for $f \in C^{\alpha}(\overline{R};\mathbb{C}), 0<\alpha<1,\ z_0 \in R,\ c \in \mathbb{C},\ \gamma_0, \gamma_1 \in C(\partial R;\mathbb{C})$ *given, is solvable if and only if for* $z \in R$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{1-\overline{z}\zeta}+\frac{1}{\pi}\int\limits_{R} f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2}=0, \tag{3.2}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{r^2-\overline{z}\zeta}+\frac{r^2}{\pi}\int\limits_{R} f(\zeta)\frac{d\xi d\eta}{(r^2-\overline{z}\zeta)^2}=0, \tag{3.3}$$

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} &= (1-r^2)\overline{z}\Bigg\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\\ &\quad+\frac{z_0}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z_0}+\frac{1}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{(1-|\zeta|^2)}{(1-r^2)}\frac{d\xi d\eta}{1-\overline{z}\zeta}\Bigg\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} &= \left(\frac{1-r^2}{r^2}\right)\overline{z}\Bigg\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\\ &\quad+\frac{z_0}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z_0}\Bigg\}\\ &\quad-\frac{\overline{z}}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{(|\zeta|^2-1)}{r^2-\overline{z}\zeta}d\xi d\eta.\end{aligned} \tag{3.5}$$

Moreover if γ_1 *anf* f *satisfy the condition*

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{\zeta}=0, \tag{3.6}$$

then the unique solution is given by

$$\begin{aligned}w(z) &= \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{\zeta-z}-\frac{c}{z}(r^2-|z|^2)\\ &\quad+\frac{1}{2\pi i z}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\left[(1-|z|^2)\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)-(1-r^2)\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\right]\frac{d\zeta}{\zeta}\\ &\quad+\frac{1}{\pi}\int\limits_{R} f(\zeta)\left[\frac{|\zeta|^2-z_0(\overline{\zeta-z})-|z|^2}{\zeta-z}-\frac{r^2z_0}{z\zeta}\right]\frac{d\xi d\eta}{\zeta-z_0}.\end{aligned} \tag{3.7}$$

Proof. The problem (3.1) is equivalent to the system

$$w_{\overline{z}}=g,\ \ w|_{\partial R}=\gamma_0, \tag{3.8}$$

$$g_{\overline{z}}=f,\ \ (\lambda|z|\partial_\nu g)|_{\partial R}=\gamma_1, g(z_0)=c. \tag{3.9}$$

According to Theorem 2.1 the solvability conditions of (3.8) are

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta, \tag{3.10}$$

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} = \frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{\overline{z}}{r^2-\overline{z}\zeta}d\xi d\eta. \tag{3.11}$$

The unique solution then is expressed by the formula

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{d\xi d\eta}{\zeta - z}. \tag{3.12}$$

On the other hand, by using Theorem 2.2, (3.9) is solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{1-\overline{z}\zeta}+\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{d\xi d\eta}{(1-\overline{z}\zeta)^2}=0 \tag{3.13}$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{r^2-\overline{z}\zeta}+\frac{r^2}{\pi}\int\limits_{R}f(\zeta)\frac{d\xi d\eta}{(r^2-\overline{z}\zeta)^2}=0. \tag{3.14}$$

Moreover if γ_1 and f satisfy the condition

$$\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{\zeta}=0, \tag{3.15}$$

then the solution is a unique, single valued function represented by

$$\begin{aligned} g(z) &= c-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{1-z\overline{\zeta}}{1-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta} \\ &\quad +\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{r^2-z\overline{\zeta}}{r^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta} \\ &\quad -\frac{1}{\pi}\int\limits_{R}f(\zeta)\frac{z-z_0}{(\zeta-z_0)(\zeta-z)}d\xi d\eta. \end{aligned} \tag{3.16}$$

For convenience in further calculations, (3.16) can be written with $t = t_1 + it_2$ as

$$g(\zeta) = c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(t)-\overline{t}f(t)\right]\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{dt}{t}-\frac{1}{\pi}\int\limits_{R}f(t)\frac{\zeta-z_0}{(t-z_0)(t-\zeta)}dt_1dt_2. \tag{3.17}$$

If (3.17) is plugged into the right-hand side of (3.10), we get

$$\begin{aligned} \frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta &= \frac{1}{\pi}\int\limits_{R}\left\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(t)-\overline{t}f(t)\right]\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{dt}{t}\right\}\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta \\ &\quad -\frac{1}{\pi}\int\limits_{R}\left\{\frac{1}{\pi}\int\limits_{R}f(t)\frac{\zeta-z_0}{(t-z_0)(t-\zeta)}dt_1dt_2\right\}\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta. \end{aligned}$$

By changing order of integration, it becomes

$$
\begin{aligned}
\frac{1}{\pi}\int\limits_R g(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta &= \frac{c\overline{z}}{\pi}\int\limits_R \frac{d\xi d\eta}{1-\overline{z}\zeta} \\
&-\frac{\overline{z}}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(t)-\overline{t}f(t)\right]\left[\frac{1}{\pi}\int\limits_R \log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\xi d\eta}{1-\overline{z}\zeta}\right]\frac{dt}{t} \\
&-\frac{\overline{z}}{\pi}\int\limits_R f(t)\left[\frac{1}{\pi}\int\limits_R \frac{\zeta-z_0}{t-\zeta}\frac{d\xi d\eta}{1-\overline{z}\zeta}\right]\frac{dt_1dt_2}{t-z_0}.
\end{aligned}
$$

Because of

$$
\begin{aligned}
\frac{c\overline{z}}{\pi}\int\limits_R \frac{d\xi d\eta}{1-\overline{z}\zeta} &= \frac{c\overline{z}}{2\pi i}\int\limits_{\partial R}\overline{\zeta}\frac{d\zeta}{1-\overline{z}\zeta} = \frac{c\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} - \frac{c\overline{z}r^2}{2\pi i}\int\limits_{|\zeta|=r}\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} \\
&= (1-r^2)c\overline{z},
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\pi}\int\limits_R \log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\xi d\eta}{1-\overline{z}\zeta} &= \frac{1}{2\pi i}\int\limits_{\partial R}\overline{\zeta}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{1-\overline{z}\zeta} \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} \\
&\quad -\frac{r^2}{2\pi i}\int\limits_{|\zeta|=r}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{(1-\overline{z}\zeta)\zeta} \\
&= (1-r^2)\log\left(\frac{|t|^2}{|t|^2-z_0\overline{t}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\pi}\int\limits_R \frac{\zeta-z_0}{t-\zeta}\frac{d\xi d\eta}{1-\overline{z}\zeta} &= -\frac{1}{\pi}\int\limits_R \frac{\zeta-z_0}{1-\overline{z}\zeta}\frac{d\xi d\eta}{\zeta-t} = \frac{\overline{t}(t-z_0)}{1-\overline{z}t} - \frac{1}{2\pi i}\int\limits_{\partial R}\frac{\overline{\zeta}(\zeta-z_0)}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta-t} \\
&= \frac{\overline{t}(t-z_0)}{1-\overline{z}t} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta-z_0}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta(\zeta-t)} + \frac{r^2}{2\pi i}\int\limits_{|\zeta|=r}\frac{\zeta-z_0}{1-\overline{z}\zeta}\frac{d\zeta}{\zeta(\zeta-t)} \\
&= \frac{\overline{t}(t-z_0)}{1-\overline{z}t} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta-z_0}{1-\overline{z}\zeta}\left(\frac{1}{t\,(\zeta-t)}-\frac{1}{t\zeta}\right)d\zeta + \frac{r^2z_0}{t} \\
&= \frac{1}{t}\left[\frac{(|t|^2-1)(t-z_0)}{1-\overline{z}t} - z_0(1-r^2)\right],
\end{aligned}
$$

the right-hand side of (3.10) can be written as

$$\begin{aligned}\frac{1}{\pi}\int\limits_R g(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\xi d\eta &= (1-r^2)\overline{z}\left\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\right.\\ &\left.+\frac{z_0}{\pi}\int\limits_R\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z_0}\right\}\\ &-\frac{\overline{z}}{\pi}\int\limits_R\frac{f(\zeta)}{\zeta}\frac{(|\zeta|^2-1)}{1-\overline{z}\zeta}d\xi d\eta.\end{aligned} \tag{3.18}$$

Therefore, one of the solvability conditions of the given problem (3.1) can be found as

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta} &= (1-r^2)\overline{z}\left\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\right.\\ &\left.+\frac{z_0}{\pi}\int\limits_R\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z_0}\right\}\\ &-\frac{\overline{z}}{\pi}\int\limits_R\frac{f(\zeta)}{\zeta}\frac{(|\zeta|^2-1)}{1-\overline{z}\zeta}d\xi d\eta.\end{aligned} \tag{3.19}$$

In a similar way, by (3.17) plugging into in (3.11), after changing order of integration, we get

$$\begin{aligned}\frac{1}{\pi}\int\limits_R g(\zeta)\frac{\overline{z}}{r^2-\overline{z}\zeta}d\xi d\eta &= \frac{c\overline{z}}{\pi}\int\limits_R\frac{d\xi d\eta}{r^2-\overline{z}\zeta}\\ &-\frac{\overline{z}}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(t)-\overline{t}f(t)\right]\left\{\frac{1}{\pi}\int\limits_R\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\xi d\eta}{r^2-\overline{z}\zeta}\right\}\frac{dt}{t}\\ &-\frac{\overline{z}}{\pi}\int\limits_R f(t)\left\{\frac{1}{\pi}\int\limits_R\frac{\zeta-z_0}{t-\zeta}\frac{d\xi d\eta}{r^2-\overline{z}\zeta}\right\}\frac{dt_1dt_2}{t-z_0}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{c\overline{z}}{\pi}\int\limits_R\frac{d\xi d\eta}{r^2-\overline{z}\zeta} &= \frac{c\overline{z}}{2\pi i}\int\limits_{\partial R}\overline{\zeta}\frac{d\xi d\eta}{r^2-\overline{z}\zeta}=\frac{c\overline{z}}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(r^2-\overline{z}\zeta)\zeta}-\frac{c\overline{z}r^2}{2\pi i}\int\limits_{|\zeta|=r}\frac{d\zeta}{(r^2-\overline{z}\zeta)\zeta}\\ &= \left(\frac{1-r^2}{r^2}\right)c\overline{z},\end{aligned}$$

$$
\begin{aligned}
\frac{1}{\pi}\int\limits_{R}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\xi d\eta}{r^2-\overline{z}\zeta} &= \frac{1}{2\pi i}\int\limits_{\partial R}\overline{\zeta}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{r^2-\overline{z}\zeta} \\
&= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{(r^2-\overline{z}\zeta)\zeta} \\
&\quad -\frac{r^2}{2\pi i}\int\limits_{|\zeta|=r}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{(r^2-\overline{z}\zeta)\zeta} \\
&= \left(\frac{1-r^2}{r^2}\right)\log\left(\frac{|t|^2}{|t|^2-z_0\overline{t}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\pi}\int\limits_{R}\frac{\zeta-z_0}{t-\zeta}\frac{d\xi d\eta}{r^2-\overline{z}\zeta} &= -\frac{1}{\pi}\int\limits_{R}\frac{\zeta-z_0}{r^2-\overline{z}\zeta}\frac{d\xi d\eta}{\zeta-t} = \frac{\overline{t}(t-z_0)}{r^2-\overline{z}t} - \frac{1}{2\pi i}\int\limits_{\partial R}\frac{\overline{\zeta}(\zeta-z_0)}{r^2-\overline{z}\zeta}\frac{d\zeta}{\zeta-t} \\
&= \frac{\overline{t}(t-z_0)}{r^2-\overline{z}t} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta-z_0}{r^2-\overline{z}\zeta}\frac{d\zeta}{\zeta(\zeta-t)} \\
&\quad +\frac{r^2}{2\pi i}\int\limits_{|\zeta|=r}\frac{\zeta-z_0}{r^2-\overline{z}\zeta}\frac{d\zeta}{\zeta(\zeta-t)} \\
&= \frac{\overline{t}(t-z_0)}{r^2-\overline{z}t} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta-z_0}{r^2-\overline{z}\zeta}\left(\frac{1}{t(\zeta-t)}-\frac{1}{t\zeta}\right)d\zeta + \frac{z_0}{t} \\
&= \frac{1}{t}\left[\frac{(|t|^2-1)(t-z_0)}{r^2-\overline{z}t} - z_0\left(\frac{1-r^2}{r^2}\right)\right],
\end{aligned}
$$

the right-hand side of (3.11) can be written as

$$
\begin{aligned}
\frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{\overline{z}}{r^2-\overline{z}\zeta}d\xi d\eta &= \left(\frac{1-r^2}{r^2}\right)\overline{z}\left\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\right\} \\
&\quad +\left(\frac{1-r^2}{r^2}\right)\overline{z}\frac{z_0}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z_0} \\
&\quad -\frac{\overline{z}}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{(|\zeta|^2-1)}{r^2-\overline{z}\zeta}d\xi d\eta. \qquad (3.20)
\end{aligned}
$$

Finally the other solvability condition of the given problem (3.1) can be written as

$$
\begin{aligned}
\frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta} &= \left(\frac{1-r^2}{r^2}\right)\overline{z}\left\{c-\frac{1}{2\pi i}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\frac{d\zeta}{\zeta}\right. \\
&\quad \left.+\frac{z_0}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z_0}\right\} - \frac{\overline{z}}{\pi}\int\limits_{R}\frac{f(\zeta)}{\zeta}\frac{(|\zeta|^2-1)}{r^2-\overline{z}\zeta}d\xi d\eta. \qquad (3.21)
\end{aligned}
$$

It can be said that (3.19), (3.21) with (3.13), (3.14) are solvability conditions of the given problem (3.1). If these conditions are satisfied, the problem is solvable. Furthermore, if γ_1 and f satisfy the condition

$$\frac{1}{2\pi i}\int_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\frac{d\zeta}{\zeta}=0 \tag{3.22}$$

then the solution is unique.

In order to determine the solution of Problem (3.1), we take account of (3.12). By rewritting (3.17) in related term of (3.12), one can get

$$\begin{aligned}\frac{1}{\pi}\int_R g(\zeta)\frac{d\xi d\eta}{\zeta-z} &= \frac{1}{\pi}\int_R\left\{c-\frac{1}{2\pi i}\int_{\partial R}\left[\gamma_1(t)-\overline{t}f(t)\right]\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{dt}{t}\right\}\frac{d\xi d\eta}{\zeta-z}\\ &\quad -\frac{1}{\pi}\int_R\left\{\frac{1}{\pi}\int_R f(t)\frac{\zeta-z_0}{(t-z_0)(t-\zeta)}dt_1dt_2\right\}\frac{d\xi d\eta}{\zeta-z}. \end{aligned}\tag{3.23}$$

By calculating the following integrals

$$\begin{aligned}\frac{c}{\pi}\int_R\frac{d\xi d\eta}{\zeta-z} &= \frac{c}{2\pi i}\int_{\partial R}\frac{\overline{\zeta}d\zeta}{\zeta-z}-c\overline{z}\\ &= \frac{c}{2\pi i}\int_{|\zeta|=1}\frac{d\zeta}{\zeta(\zeta-z)}-\frac{cr^2}{2\pi i}\int_{|\zeta|=r}\frac{d\zeta}{\zeta(\zeta-z)}-c\overline{z}\\ &= \frac{c}{z}(r^2-|z|^2),\end{aligned}$$

$$\begin{aligned}\frac{1}{\pi}\int_R\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\xi d\eta}{\zeta-z} &= \frac{1}{2\pi i}\int_{\partial R}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{\overline{\zeta}d\zeta}{\zeta-z}-\overline{z}\log\left(\frac{|t|^2-z\overline{t}}{|t|^2-z_0\overline{t}}\right)\\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{\zeta(\zeta-z)}\\ &\quad -\frac{r^2}{2\pi i}\int_{|\zeta|=r}\log\left(\frac{|t|^2-\zeta\overline{t}}{|t|^2-z_0\overline{t}}\right)\frac{d\zeta}{\zeta(\zeta-z)}-\overline{z}\log\left(\frac{|t|^2-z\overline{t}}{|t|^2-z_0\overline{t}}\right)\\ &= \left(\frac{1-|z|^2}{z}\right)\log\left(\frac{|t|^2-z\overline{t}}{|t|^2-z_0\overline{t}}\right)-\left(\frac{1-r^2}{z}\right)\log\left(\frac{|t|^2}{|t|^2-z_0\overline{t}}\right)\end{aligned}$$

and

$$\frac{1}{\pi}\int_R\frac{\zeta-z_0}{(t-\zeta)}\frac{d\xi d\eta}{\zeta-z}=-\frac{1}{t-z}\frac{1}{\pi}\int_R\frac{\zeta-z_0}{\zeta-t}d\xi d\eta+\frac{1}{t-z}\frac{1}{\pi}\int_R\frac{\zeta-z_0}{\zeta-z}d\xi d\eta$$

with

$$\begin{aligned}\frac{1}{\pi}\int_R\frac{\zeta-z_0}{\zeta-t}d\xi d\eta &= \frac{1}{2\pi i}\int_{\partial R}(\zeta-z_0)\frac{\overline{\zeta}d\zeta}{\zeta-t}-\overline{t}\,(t-z_0)\\ &= 1-(|t|^2)+\frac{z_0}{t}(|t|^2-r^2),\end{aligned}$$

$$\frac{1}{\pi}\int\limits_{R}\frac{\zeta-z_0}{\zeta-z}d\xi d\eta=1-(|z|^2)+\frac{z_0}{z}(|z|^2-r^2),$$

for the last term of (3.23)

$$\begin{aligned}\frac{1}{\pi}\int\limits_{R}\frac{\zeta-z_0}{(t-\zeta)}\frac{d\xi d\eta}{\zeta-z} &= -\frac{1}{t-z}\left(1-(|t|^2)+\frac{z_0}{t}(|t|^2-r^2)\right)\\ &\quad+\frac{1}{t-z}\left(1-(|z|^2)+\frac{z_0}{z}(|z|^2-r^2)\right)\\ &= \frac{|t|^2-|z|^2}{t-z}-\frac{z_0(\overline{t}-\overline{z})}{t-z}-\frac{r^2z_0}{zt},\end{aligned}$$

it can be found that

$$\begin{aligned}\frac{1}{\pi}\int\limits_{R}g(\zeta)\frac{d\xi d\eta}{\zeta-z} &= \frac{c}{z}(r^2-|z|^2)\\ &\quad-\frac{1}{2\pi iz}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\left[(1-|z|^2)\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)\right]\frac{d\zeta}{\zeta}\\ &\quad+\frac{1}{2\pi iz}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\left[(1-r^2)\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\right]\frac{d\zeta}{\zeta}\\ &\quad-\frac{1}{\pi}\int\limits_{R}f(\zeta)\left[\frac{\overline{\zeta}(\zeta-z_0)-\overline{z}(z-z_0)}{\zeta-z}-\frac{r^2z_0}{z\zeta}\right]\frac{d\xi d\eta}{\zeta-z_0}.\end{aligned}$$

Finally solution of the problem can be given as

$$\begin{aligned}w(z) &= \frac{1}{2\pi i}\int\limits_{\partial R}\gamma_0(\zeta)\frac{d\zeta}{\zeta-z}-\frac{c}{z}(r^2-|z|^2)\\ &\quad+\frac{1}{2\pi iz}\int\limits_{\partial R}\left[\gamma_1(\zeta)-\overline{\zeta}f(\zeta)\right]\left[(1-|z|^2)\log\left(\frac{|\zeta|^2-z\overline{\zeta}}{|\zeta|^2-z_0\overline{\zeta}}\right)-(1-r^2)\log\left(\frac{|\zeta|^2}{|\zeta|^2-z_0\overline{\zeta}}\right)\right]\frac{d\zeta}{\zeta}\\ &\quad+\frac{1}{\pi}\int\limits_{R}f(\zeta)\left[\frac{\overline{\zeta}(\zeta-z_0)-\overline{z}(z-z_0)}{\zeta-z}-\frac{r^2z_0}{z\zeta}\right]\frac{d\xi d\eta}{\zeta-z_0}.\end{aligned}$$

This completes the proof. ■

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