



Modified Szász Operators Involving Charlier Polynomials Based on Two Parameters

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Abstract The aim of this article is to introduce a Stancu type generalization of modified Szász operators using Charlier polynomials. We establish a recursive relation between Szász-type operators defined in [S. Varma, F. Taşdelen, Szász type operators involving Charlier polynomials, Math. Comput. Modeling 56 (5–6) (2012) 118–122] and Stancu-type generalization of these operators. Further, we discuss Korovkin type theorem, rate of convergence in terms of modulus of continuity and simultaneous approximation. Moreover, we study Local approximation results using second order modulus of smoothness, Peetre's K-functional and Lipschitz class. In the last of this manuscript, we give weighted Korovkin type theorem and statistical approximation result in polynomial weighted space.

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1. INTRODUCTION

In 1950, Szász [1] introduced an extension of Bernstein operators [2] on $[0, \infty)$

$$S_n(f; x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}. \quad (1.1)$$

He studied uniform and pointwise approximations for continuous functions on $(0, \infty)$ by the operators defined by (1.1). On the other hand in 1968, Stancu [3] gave a generalization of Bernstein operators [2] using two real parameters α and β satisfying condition $0 \leq \alpha \leq \beta$ as follows

$$(P_m^{\alpha, \beta} f)(x) = \sum_{k=0}^m p_{m,k} f\left(\frac{k + \alpha}{n + \beta}\right), \quad (1.2)$$

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where $p_{m,k}(x) = x^k(1-x)^{m-k}$ and $m \in \mathbb{N}$. Many researchers studied Stancu method for different type of linear positive sequences and their approximation behaviour in some functional spaces (see [4]-[16]). Charlier polynomials [17] having the generating function of the form:

$$e^t \left(1 - \frac{t}{a}\right)^u = \sum_{k=0}^{\infty} C_k^{(a)}(u) \frac{t^k}{k!}, \quad |t| < a, \quad (1.3)$$

and the explicit representation

$$C_k^{(a)}(u) = \sum_{r=0}^k \binom{k}{r} (-u)_r \left(\frac{1}{a}\right)^r,$$

where $(-u)_r$ is the Pochhammer's symbol given by

$$(-u)_0 = 1, (-u)_r = -u(-u+1)\dots(-u+r-1), \quad r \in \mathbb{N}.$$

Recently, Varma and Tasdelen [18] gave Szász-type operators using Charlier polynomials as follows

$$L_n(f; x, a) = e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)nx) f\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, \quad (1.4)$$

where $a > 1$ and $x \geq 0$. Later on, Wafi and Rao [19] studied various approximation results in several functional spaces for one variable. Motivated by the above idea, we define a Stancu variant of the operators defined by 1.4

$$T_{n,a}^{\alpha,\beta}(f; x) = e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)nx) f\left(\frac{k+\alpha}{n+\beta}\right), \quad n \in \mathbb{N}, \quad (1.5)$$

where $0 \leq \alpha \leq \beta$. We notice that for $\alpha = \beta = 0$, operators (1.5) reduce to the operators (1.4). In this article, we established recursive relation between the operators (1.5) and (1.4) and studied the Korovkin type theorem, rate of convergence in terms of modulus of continuity, simultaneous approximation result. Local approximation results using second order modulus of smoothness, Peetre's K-functional, Lipschits class and weighted Korovkin type theorem, statistical approximation result in polynomial weighted spaces are also given.

2. PRELIMINARIES

Let $e_i(t) = t^i, i \in \{0, 1, 2\}$ be the test functions.

Lemma 2.1. From [18], we have

$$\begin{aligned} T_{n,a}(e_0; x) &= 1, \\ T_{n,a}(e_1; x) &= x + \frac{1}{n}, \\ T_{n,a}(e_2; x) &= x^2 + \frac{x}{n} \left(3 + \frac{1}{a-1}\right) + \frac{2}{n^2}. \end{aligned}$$

Lemma 2.2. For the operators $T_{n,a}^{\alpha,\beta}$ defined by (1.5), we have

$$T_{n,a}^{\alpha,\beta}(e_m; x) = \frac{n^m}{(n + \beta)^m} \sum_{i=0}^m \binom{m}{i} \left(\frac{\alpha}{n}\right)^{m-i} T_{n,a}(e_i; x).$$

Proof. Using operators defined by (1.5), we have

$$\begin{aligned} T_{n,a}^{\alpha,\beta}(t^m; x) &= e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{i=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left(\frac{k + \alpha}{n + \beta}\right)^m dt \\ &= e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{i=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \frac{n^m}{(n + \beta)^m} \left(\frac{k + \alpha}{n}\right)^m dt \\ &= \frac{n^m}{(n + \beta)^m} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{i=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \sum_{i=0}^m \binom{m}{i} \left(\frac{k}{n}\right)^i \left(\frac{\alpha}{n}\right)^{m-i} \\ &= \frac{n^m}{(n + \beta)^m} \sum_{i=0}^m \binom{m}{i} \left(\frac{\alpha}{n}\right)^{m-i} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left(\frac{k}{n}\right)^i \\ &= \frac{n^m}{(n + \beta)^m} \sum_{i=0}^m \binom{m}{i} \left(\frac{\alpha}{n}\right)^{m-i} T_{n,a}(e_i; x). \end{aligned}$$

■

Lemma 2.3. For all $x, t \geq 0$, we have

$$\begin{aligned} T_{n,a}^{\alpha,\beta}(e_0; x) &= 1, \\ T_{n,a}^{\alpha,\beta}(e_1; x) &= \frac{nx}{n + \beta} + \frac{\alpha + 1}{n + \beta}, \\ T_{n,a}^{\alpha,\beta}(e_2; x) &= \frac{n^2}{(n + \beta)^2} x^2 + \left(3 + 2\alpha + \frac{1}{a - 1}\right) \frac{nx}{(n + \beta)^2} + \frac{\alpha^2 + 2\alpha + 2}{(n + \beta)^2}. \end{aligned}$$

Proof. From Lemma 2.2, we have

$$T_{n,a}^{\alpha,\beta}(e_m; x) = \frac{n^m}{(n + \beta)^m} \sum_{i=0}^m \binom{m}{i} \left(\frac{\alpha}{n}\right)^{m-i} T_{n,a}(e_i; x).$$

For $i = 0$,

$$T_{n,a}^{\alpha,\beta}(e_0; x) = T_{n,a}(e_0; x) = 1.$$

For $i = 1$,

$$\begin{aligned} T_{n,a}^{\alpha,\beta}(e_1; x) &= \frac{n}{n + \beta} \left(\frac{\alpha}{n} T_{n,a}(e_0; x) + T_{n,a}(e_1; x)\right) \\ &= \frac{nx}{n + \beta} + \frac{\alpha + 1}{n + \beta}. \end{aligned}$$

Similarly, it can be proved for $i = 2$.

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Lemma 2.4. Let $\psi_x^i(t) = (t - x)^i, i = 0, 1, 2$ be the moments. Then, we have

$$\begin{aligned} T_{n,a}^{\alpha,\beta}(\psi_x^0; x) &= 1, \\ T_{n,a}^{\alpha,\beta}(\psi_x^1; x) &= \frac{\alpha + 1}{n + \beta} - \beta x, \\ T_{n,a}^{\alpha,\beta}(\psi_x^2; x) &= \frac{\beta^2}{(n + \beta)^2} x^2 + \left(\frac{n}{a - 1} + n - 2\alpha\beta - 2\beta \right) \frac{x}{(n + \beta)^2} + \frac{\alpha^2 + 2\alpha + 2}{(n + \beta)^2}. \end{aligned}$$

Remark 2.5. All the basic estimates proved in this section reduce to the basic estimates proved in [18] for the special case ($\alpha = \beta = 0$).

3. RATE OF CONVERGENCE OF THE OPERATORS $T_{n,a}^{\alpha,\beta}$

Definition 3.1. Let $C_B[0, \infty)$ be the set of all continuous and bounded functions defined on $[0, \infty)$ and $\delta > 0$. Then, the modulus of continuity $\omega(f; \delta)$ is defined as

$$\omega(f; \delta) := \sup_{|t-y|\leq\delta} |f(t) - f(y)|, \quad t, y \in [0, \infty).$$

Theorem 3.2. Let $f \in C[0, \infty) \cap E$. Then $T_{n,a}^{\alpha,\beta}(f; x) \rightarrow f$ uniformly on each compact subset of $[0, \infty)$ where $C[0, \infty)$ is the space of continuous functions and $E := \{f : x \geq 0, \frac{f(x)}{1+x^2}$ is convergent as $x \rightarrow \infty\}$.

Proof. In the light of Lemma 2.3, we have $T_{n,a}^{\alpha,\beta}(e_i; x) \rightarrow x^i$ as $n \rightarrow \infty$ for $i \in \{0, 1, 2\}$. Using universal Korovkin-type property (vi) of Theorem 4.1.4 in [20], we prove Theorem 3.2. ■

Theorem 3.3. (See [21]) Let $L : C([a, b]) \rightarrow B([a, b])$ be a linear and positive operator and let φ_x be the function defined by

$$\varphi_x(t) = |t - x|, \quad (x, t) \in [a, b] \times [a, b].$$

If $f \in C_B([a, b])$ for any $x \in [a, b]$ and any $\delta > 0$, the operator L verifies:

$$|(Lf)(x) - f(x)| \leq |f(x)| |(Le_0)(x) - L\{(Le_0)(x) + \delta^{-1} \sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\} \omega_f(\delta).$$

Theorem 3.4. For $f \in C_B[0, \infty)$, the relation

$$|T_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq 2\omega(f; \delta),$$

holds uniformly, where $\delta = \sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)}$.

Proof. Using Lemma 2.3, Lemma 2.4 and Theorem 3.3, we get

$$|T_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq \{1 + \delta^{-1} \sqrt{T_{n,a}^{\alpha,\beta}(\varphi(x)^2; x)\} \omega(f; \delta),$$

which prove Theorem 3.4. ■

4. SIMULTANEOUS APPROXIMATION

Theorem 4.1. For $f \in C^r[0, \infty) \cap E$, we have

$$\begin{aligned} \left| \frac{d^r}{dx^r} T_{n,a}^{\alpha,\beta}(f;x) - \frac{d^r}{dx^r} f(x) \right| &\leq \left(\frac{n}{n+\beta} \right)^r \omega \left(\frac{d^r}{dx^r} f; \frac{1}{\sqrt{n+\beta}} + \frac{r}{n+\beta} \right) \\ &\quad \times \left\{ 1 + \sqrt{\frac{\beta^2}{(n+\beta)} x^2 + \left(\frac{an}{a-1} - 2\alpha\beta - 2\beta \right) \frac{x}{(n+\beta)} + \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)}} \right\} \\ &\quad + \left(\frac{n}{n+\beta} \right)^r \omega \left(\frac{d^r}{dx^r} f; \frac{r}{n+\beta} \right). \end{aligned}$$

Proof. By simple calculation, we obtained

$$\frac{d^r}{dx^r} T_{n,a}^{\alpha,\beta}(f;x) = e^{-1} \left(1 - \frac{1}{a} \right)^{(a-1)nx} \left(\frac{n}{n+\beta} \right)^r r! \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \frac{\Delta_{\frac{1}{n+\beta}}^r f \left(\frac{k+\alpha}{n+\beta} \right)}{r! \left(\frac{1}{n+\beta} \right)^r},$$

where $\Delta_{\frac{1}{n+\beta}}^r f \left(\frac{k+\alpha}{n+\beta} \right)$ is the r order difference with the step length $\frac{1}{n+\beta}$.

$$\begin{aligned} \frac{d^r}{dx^r} T_{n,a}^{\alpha,\beta}(f;x) &= e^{-1} \left(1 - \frac{1}{a} \right)^{(a-1)nx} \left(\frac{n}{n+\beta} \right)^r r! \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \\ &\quad \times \left[\frac{k+\alpha}{n+\beta}, \frac{k+\alpha+1}{n+\beta}, \frac{k+\alpha+2}{n+\beta}, \dots, \frac{k+\alpha+r}{n+\beta}; f \right] \\ &= e^{-1} \left(1 - \frac{1}{a} \right)^{(a-1)nx} \left(\frac{n}{n+\beta} \right)^r r! \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} g \left(\frac{k+\alpha}{n+\beta} \right) \\ &= \left(\frac{n}{n+\beta} \right)^r r! T_{n,a}^{\alpha,\beta}(g;x), \end{aligned}$$

where $g(t) = \left[t, t + \frac{1}{n+\beta}, t + \frac{2}{n+\beta}, \dots, t + \frac{r}{n+\beta} \right]$. Now,

$$\begin{aligned} \left| \frac{d^r}{dx^r} T_{n,a}^{\alpha,\beta}(f;x) - \frac{d^r}{dx^r} f(x) \right| &\leq \left(\frac{n}{n+\beta} \right)^r \left| r! T_{n,a}^{\alpha,\beta}(g;x) - r!g(x) + r!g(x) - \frac{d^r}{dx^r} f(x) \right| \\ &\leq \left(\frac{n}{n+\beta} \right)^r \left\{ \left| r! T_{n,a}^{\alpha,\beta}(g;x) - r!g(x) \right| + \left| r!g(x) - \frac{d^r}{dx^r} f(x) \right| \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{n}{n+\beta}\right)^r \left\{ \left| r!g(x) - \frac{d^r}{dx^r} f(x) \right| \right. \\
&+ r! \left[1 + \delta^{-1} \sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)} \right] \omega(f; \delta) \left. \right\} \\
&\leq \left(\frac{n}{n+\beta}\right)^r \left\{ \left| r!g(x) - \frac{d^r}{dx^r} f(x) \right| \right. \\
&+ r! \left[1 + \sqrt{\frac{\beta^2}{(n+\beta)} x^2 + \left(\frac{an}{a-1} - 2\alpha\beta - 2\beta\right) \frac{x}{(n+\beta)} + \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)}} \right] \\
&\times \omega\left(g; \frac{1}{\sqrt{n+\beta}}\right) \left. \right\}. \tag{4.1}
\end{aligned}$$

From [22], we have

$$\omega\left(g; \frac{1}{\sqrt{n+\beta}}\right) \leq \frac{1}{r!} \omega\left(\frac{d^r}{dx^r} f; \frac{1}{\sqrt{n+\beta}} + \frac{r}{n+\beta}\right) \tag{4.2}$$

and

$$\left| r!g(x) - \frac{d^r}{dx^r} f(x) \right| \leq \frac{1}{r!} \omega\left(\frac{d^r}{dx^r} f; \frac{r}{n+\beta}\right). \tag{4.3}$$

From (4.1), (4.2) and (4.3), we have

$$\begin{aligned}
\left| \frac{d^r}{dx^r} T_{n,a}^{\alpha,\beta}(f; x) - \frac{d^r}{dx^r} f(x) \right| &\leq \left(\frac{n}{n+\beta}\right)^r \omega\left(\frac{d^r}{dx^r} f; \frac{1}{\sqrt{n+\beta}} + \frac{r}{n+\beta}\right) \\
&\times \left\{ 1 + \sqrt{\frac{\beta^2}{(n+\beta)} x^2 + \left(\frac{an}{a-1} - 2\alpha\beta - 2\beta\right) \frac{x}{(n+\beta)} + \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)}} \right\} \\
&+ \left(\frac{n}{n+\beta}\right)^r \omega\left(\frac{d^r}{dx^r} f; \frac{r}{n+\beta}\right).
\end{aligned}$$

■

Consequently, we obtain

Corollary 4.1. Let $L_n = T_{n,a}^{0,0}$ be the operators defined by (1.6) in [18]. For $f \in C^r[0, \infty) \cap E$, we have

$$\begin{aligned}
\left| \frac{d^r}{dx^r} L_n(f; x, a)(f; x) - \frac{d^r}{dx^r} f(x) \right| &\leq \omega\left(\frac{d^r}{dx^r} f; \frac{1}{\sqrt{n}} + \frac{r}{n}\right) \\
&\times \left\{ 1 + \sqrt{\left(1 + \frac{1}{a-1}\right)x + \frac{2}{n}} \right\} \\
&+ \omega\left(\frac{d^r}{dx^r} f; \frac{r}{n}\right).
\end{aligned}$$

5. LOCAL APPROXIMATION RESULTS

In this section, we deal with order of approximation locally in $C_B[0, \infty)$ (space of real valued continuous and bounded functions f on $[0, \infty)$) with the norm $\|f\| = \sup_{0 \leq x < \infty} |f(x)|$.

For any $f \in C_B[0, \infty)$ and $\delta > 0$, Peeter’s K-functional is defined as

$$K_2(f, \delta) = \inf\{\|f - g\| + \delta\|g''\| : g \in C_B^2[0, \infty)\},$$

where $C_B^2[0, \infty) = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By DeVore and Lorentz [23, p.177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(f; \delta) \leq C\omega_2(f; \sqrt{\delta}),$$

where $\omega_2(f; \delta)$ is the second order modulus of continuity is defined as

$$\omega_2(f; \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x + 2h) - 2f(x + h) + f(x)|.$$

Theorem 5.1. *Let $f \in C_B^2[0, \infty)$. Then, there exists a constant $C > 0$ such that*

$$|T_{n,a}^{\alpha,\beta}(f; x) - f(x)| \leq C\omega_2(f; \sqrt{\xi_{n,a}^{\alpha,\beta}(x)}) + \omega(f; T_{n,a}^{\alpha,\beta}(\psi_x; x)),$$

where $\xi_{n,a}^{\alpha,\beta}(x) = T_{n,a}^{\alpha,\beta}(\psi_x^2; x) + (T_{n,a}^{\alpha,\beta}(\psi_x; x))^2$.

Proof. Consider the auxiliary operators as follows

$$\widehat{T}_{n,a}^{\alpha,\beta}(f; x) = T_{n,a}^{\alpha,\beta}(f; x) + f(x) - f(\eta_{n,a}(x)),$$

where $\eta_{n,a}(x) = T_{n,a}(\psi_x; x) + x$. Using Lemma 2.4, we have

$$\begin{aligned} \widehat{T}_{n,a}^{\alpha,\beta}(1; x) &= 1, \\ \widehat{T}_{n,a}^{\alpha,\beta}(\psi_x(t); x) &= 0, \\ |\widehat{T}_{n,a}^{\alpha,\beta}(f; x)| &\leq 3\|f\|. \end{aligned} \tag{5.1}$$

For any $g \in C_B^2[0, \infty)$ and by the Taylor’s theorem, we get

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - v)g''(v)dv. \tag{5.2}$$

$$\begin{aligned} \widehat{T}_{n,a}^{\alpha,\beta}(g; x) - g(x) &= g'(x)\widehat{T}_{n,a}^{\alpha,\beta}(t - x; x) + \widehat{T}_{n,a}^{\alpha,\beta}\left(\int_x^t (t - v)g''(v)dv; x\right) \\ &= \widehat{T}_{n,a}^{\alpha,\beta}\left(\int_x^t (t - v)g''(v)dv; x\right) \\ &= T_{n,a}^{\alpha,\beta}\left(\int_x^t (t - v)g''(v)dv; x\right) - \int_x^{\eta_{n,a}(x)} (\eta_{n,a}(x) - v)g''(v)dv. \end{aligned}$$

$$|\widehat{T}_{n,a}^{\alpha,\beta}(g; x) - g(x)| \leq \left| T_{n,a}^{\alpha,\beta}\left(\int_x^t (t - v)g''(v)dv; x\right) \right| + \left| \int_x^{\eta_{n,a}(x)} (\eta_{n,a}(x) - v)g''(v)dv \right|. \tag{5.3}$$

Since,

$$\left| \int_x^t (t-v)g''(v)dv \right| \leq (t-x)^2 \|g''\|. \quad (5.4)$$

Then

$$\left| \int_x^{\eta_{n,a}(x)} (\eta_{n,a}(x)-v)g''(v)dv \right| \leq (\eta_{n,a}(x)-x)^2 \|g''\|. \quad (5.5)$$

$$\begin{aligned} |\widehat{T}_{n,a}^{\alpha,\beta}(g;x) - g(x)| &\leq \left\{ T_{n,a}^{\alpha,\beta}((t-x)^2;x) + (\eta_{n,a}(x)-x)^2 \right\} \|g''\| \\ &= \xi_{n,a}^{\alpha,\beta}(x) \|g''\|. \end{aligned} \quad (5.6)$$

Now, we have

$$\begin{aligned} |T_{n,a}^{\alpha,\beta}(f;x) - f(x)| &\leq |\widehat{T}_{n,a}^{\alpha,\beta}(f-g;x)| + |(f-g)(x)| + |\widehat{T}_{n,a}^{\alpha,\beta}(g;x) - g(x)| \\ &\quad + |f(\eta_{n,a}(x)) - f(x)|, \end{aligned}$$

$$\begin{aligned} |T_{n,a}^{\alpha,\beta}(f;x) - f(x)| &\leq 4\|f-g\| + |\widehat{T}_{n,a}^{\alpha,\beta}(g;x) - g(x)| + |f(\eta_{n,a}(x)) - f(x)| \\ &\leq 4\|f-g\| + \xi_{n,a}^{\alpha,\beta}(x) \|g''\| + \omega\left(f; T_{n,a}^{\alpha,\beta}(\psi_x;x)\right). \end{aligned}$$

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq C\omega_2\left(f; \sqrt{\xi_{n,a}^{\alpha,\beta}(x)}\right) + \omega\left(f; T_{n,a}^{\alpha,\beta}(\psi_x;x)\right).$$

■

Now, we shall discuss rate of convergence of the operators defined by (1.5) for the functions which belong to Lipschitz class

$$Lip_M^*(\gamma) = \left\{ f \in C[0, \infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\gamma}{(t+x)^{\frac{\gamma}{2}}} : x, t \in (0, \infty) \right\}, \quad (5.7)$$

where M is a constant and $0 < \gamma \leq 1$.

Theorem 5.2. *Let $f \in Lip_M^*(\gamma)$. Then for $x \geq 0$ and $0 < \gamma \leq 1$, we have*

$$|T_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq M \left[\frac{\zeta_{n,a}(x)}{x} \right]^{\frac{\gamma}{2}},$$

where $\zeta_{n,a}(x) = T_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)$.

Proof. For $f \in Lip_M^*(1)$ and $\gamma = 1$, $x \in (0, \infty)$, we have

$$\begin{aligned}
 |T_{n,a}^{\alpha,\beta}(f; x) - f(x)| &\leq e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left(f\left(\frac{k+\alpha}{n+\beta}\right) - f(x)\right) \\
 &\leq M e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \frac{\left|\frac{k+\alpha}{n+\beta} - x\right|}{\sqrt{\frac{k+\alpha}{n+\beta} + x}} \\
 &\leq \frac{M}{\sqrt{x}} e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left|\frac{k+\alpha}{n+\beta} - x\right| \\
 &\leq \frac{M}{\sqrt{x}} T_{n,a}^{\alpha,\beta}(|t-x|; x) \\
 &\leq M \frac{\sqrt{T_{n,a}^{\alpha,\beta}(\psi_x^2(t); x)}}{\sqrt{x}} \\
 &= M \left(\frac{\zeta_{n,a}(x)}{x}\right)^{\frac{1}{2}}.
 \end{aligned}$$

Thus, the assertion hold for $\gamma = 1$. Now, we will prove for $\gamma \in (0, 1)$. From the Hölder's Inequality with $p = \frac{1}{\gamma}$ and $q = \frac{1}{1-\gamma}$, we have

$$\begin{aligned}
 |T_{n,a}^{\alpha,\beta}(f; x) - f(x)| &= \left(e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left(\left|f\left(\frac{k+\alpha}{n+\beta}\right) - f(x)\right|\right)^{\frac{1}{\gamma}}\right)^{\gamma} \\
 &\quad \times \left(e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!}\right)^{1-\gamma} \\
 &\leq \left(e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left(\left|f\left(\frac{k+\alpha}{n+\beta}\right) - f(x)\right|\right)^{\frac{1}{\gamma}}\right)^{\gamma} \\
 |T_{n,a}^{\alpha,\beta}(f; x) - f(x)| &\leq M \left(e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \frac{\left|\frac{k+\alpha}{n+\beta} - x\right|}{\sqrt{\frac{k+\alpha}{n+\beta} + x}} dt\right)^{\gamma} \\
 &\leq \frac{M}{x^{\frac{\alpha}{2}}} \left(e^{-1} \left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(- (a-1)nx)}{k!} \left|\frac{k+\alpha}{n+\beta} - x\right|\right)^{\gamma} \\
 &= \frac{M}{x^{\frac{\alpha}{2}}} (T_{n,a}^{\alpha,\beta}(|t-x|; x))^{\gamma} \\
 &\leq M \left(\frac{\zeta_{n,a}(x)}{x}\right)^{\frac{\gamma}{2}}.
 \end{aligned}$$

■

Consequently, we obtain

Corollary 5.1. Let $f \in C_B^2[0, \infty)$ and $L_n = T_{n,a}^{0,0}$ be the operators defined in (1.6) by Varma and Taşdelen [18]. Then, there exists a constant $C > 0$ such that

$$|L_n(f; x, a) - f(x)| \leq C\omega_2(f; \sqrt{\xi_{n,a}^{0,0}(x)}) + \omega(f; L_n(\psi_x; x, a)),$$

where $\xi_{n,a}^{0,0}(x) = L_n(\psi_x^2; x, a) + (L_n(\psi_x; x))^2$.

Corollary 5.2. Let $f \in Lip_M^*(\gamma)$ and $L_n = T_{n,a}^{0,0}$ be the operators defined in (1.6) by Varma and Taşdelen [18]. Then for $x \geq 0$ and $0 < \gamma \leq 1$, we have

$$|L_n(f; x, a) - f(x)| \leq M \left[\frac{\zeta_{n,a}(x)}{x} \right]^{\frac{\gamma}{2}},$$

where $\zeta_{n,a}(x) = L_n(\psi_x^2(t); x)$.

6. WEIGHTED APPROXIMATION

Let $B_{1+x^2}[0, \infty) = \{f(x) : |f(x)| \leq M_f(1+x^2), 1+x^2 \text{ is weight function, } M_f \text{ is a constant depending on } f \text{ and } x \in [0, \infty)\}$, $C_{1+x^2}[0, \infty)$ is the space of continuous function in $B_{1+x^2}[0, \infty)$ with the norm $\|f(x)\|_{1+x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}$ and $C_{1+x^2}^K = \{f \in C_{1+x^2} : \lim_{|x| \rightarrow \infty} \frac{f(x)}{1+x^2} = K, \text{ where } K \text{ is a constant depending on } f\}$.

Theorem 6.1. If the operators $T_{n,a}^{\alpha,\beta}$ defined by (1.5) from $C_{1+x^2}^k[0, \infty)$ to $B_{1+x^2}[0, \infty)$ satisfying the conditions

$$\lim_{n \rightarrow \infty} \|T_{n,a}^{\alpha,\beta}(e_i; x) - x^i\|_{1+x^2} = 0, \quad i = 0, 1, 2.$$

Then for each $C_{1+x^2}^k[0, \infty)$

$$\lim_{n \rightarrow \infty} \|T_{n,a}^{\alpha,\beta}(f; x) - f\|_{1+x^2} = 0.$$

Proof. To prove Theorem 7.1, it is enough to show that

$$\lim_{n \rightarrow \infty} \|T_{n,a}^{\alpha,\beta}(e_i; x) - x^i\|_{1+x^2} = 0, \quad i = 0, 1, 2.$$

From Lemma 2.4, we have

$$\|T_{n,a}^{\alpha,\beta}(e_0; x) - x^0\|_{1+x^2} = \sup_{x \in [0, \infty)} \frac{|T_{n,a}^{\alpha,\beta}(1; x) - 1|}{1+x^2} = 0 \text{ for } i = 0.$$

For $i = 1$

$$\begin{aligned} \|T_{n,a}^{\alpha,\beta}(e_1; x) - x^1\|_{1+x^2} &= \sup_{x \in [0, \infty)} \frac{|\frac{nx}{n+\beta} + \frac{\alpha+1}{n+\beta} - x|}{1+x^2} \\ &= \frac{\beta}{n+\beta} \sup_{x \in [0, \infty)} \frac{x}{1+x^2} + \frac{\alpha+1}{n+\beta} \sup_{x \in [0, \infty)} \frac{1}{1+x^2}. \end{aligned}$$

This implies that $\|T_{n,a}^{\alpha,\beta}(e_1; x) - x^1\|_{1+x^2} \rightarrow 0$ as $n \rightarrow \infty$. For $i = 2$

$$\begin{aligned} \|T_{n,a}^{\alpha,\beta}(e_2; x) - x^2\|_{1+x^2} &= \sup_{x \in [0, \infty)} \frac{\left| \frac{n^2}{(n+\beta)^2}x^2 + \left(3 + 2\alpha + \frac{1}{a-1}\right) \frac{nx}{(n+\beta)^2} + \frac{\alpha^2+2\alpha+2}{(n+\beta)^2} - x^2 \right|}{1+x^2} \\ &\leq \left(\left| \frac{n^2}{(n+\beta)^2} - 1 \right| \right) \sup_{x \in [0, \infty)} \frac{x^2}{1+x^2} \\ &\quad + \left(3 + 2\alpha + \frac{1}{a-1} \right) \frac{n}{(n+\beta)^2} \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \\ &\quad + \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)^2} \sup_{x \in [0, \infty)} \frac{1}{1+x^2}. \end{aligned}$$

Which shows that $\|T_{n,a}^{\alpha,\beta}(e_2; x) - x^2\|_{1+x^2} \rightarrow 0$ as $n \rightarrow \infty$. ■

Consequently, we obtain

Corollary 6.1. If the operators $L_n = T_{n,a}^{0,0}$ be defined in (1.6) by Varma and Taşdelen [18]. Then from $C_{1+x^2}^k[0, \infty)$ to $B_{1+x^2}[0, \infty)$ satisfying the conditions

$$\lim_{n \rightarrow \infty} \|L_n(e_i; x, a) - x^i\|_{1+x^2} = 0, \quad i = 0, 1, 2.$$

Then for each $C_{1+x^2}^k[0, \infty)$

$$\lim_{n \rightarrow \infty} \|L_n(f; x, a) - f\|_{1+x^2} = 0.$$

7. STATISTICAL APPROXIMATION

Statistical approximation theorems in operators theory were introduced by Gadjiev et al [24]. It is known that every convergent sequence is statically convergent but conversely need not be true. Many researchers have studied Statistical approximation results for different positive linear operators (see [25, 26]). Let $A = (a_{nk})$ be a non-negative infinite suitability matrix. For a given sequence $x := (x_k)$, the A -transform of x denoted by $Ax : ((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$$

provided the series converges for each n . A is said to be regular if $\lim(Ax)_n = L$ whenever $\lim x = L$. Then $x = (x_n)$ is said to be a A -statistically convergent to L i.e. $st_A - \lim x = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k-L| \geq \epsilon} a_{nk} = 0$.

Theorem 7.1. Let $A = (a_{nk})$ be a non-negative regular suitability matrix and $x \geq 0$. Then, we have

$$st_A - \lim_n \|T_{n,a}^{\alpha,\beta}(f; x) - f\|_{1+x^{2+\lambda}} = 0, \text{ for all } f \in C_{1+x^{2+\lambda}} \text{ and } \lambda > 0.$$

Proof. From ([27], p. 191, Th. 3), it is sufficient to show that for $\lambda = 0$

$$st_A - \lim_n \|T_{n,a}^{\alpha,\beta}(e_i; x) - e_i\|_{1+x^2} = 0, \text{ for } i \in \{0, 1, 2\}.$$

In view of Lemma 2.3, we get

$$\|T_{n,a}^{\alpha,\beta}(e_1; x) - x\|_{1+x^2} = \sup_{x \in [0, \infty)} \frac{1}{1+x^2} \left| \left(\frac{n}{n+\beta} - 1 \right) x + \frac{\alpha+1}{n+\beta} \right| \leq \left(\frac{n}{n+\beta} - 1 \right) + \frac{\alpha+1}{n+\beta}.$$

Now, for a given $\epsilon > 0$, we define the following sets

$$\begin{aligned} D_1 &: = \left\{ n : \|T_{n,a}^{\alpha,\beta}(e_1; x) - x\| \geq \epsilon \right\} \\ D_2 &: = \left\{ n : \left(\frac{n}{n+\beta} - 1 \right) \geq \frac{\epsilon}{2} \right\} \\ D_3 &: = \left\{ n : \frac{\alpha+1}{n+\beta} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

This implies that $D_1 \subseteq D_2 \cup D_3$, which shows that $\sum_{k \in D_1} a_{nk} \leq \sum_{k \in D_2} a_{nk} + \sum_{k \in D_3} a_{nk}$. Therefore, we get

$$st_A - \lim_n \|T_{n,a}^{\alpha,\beta}(e_1; x) - x\|_{1+x^2} = 0.$$

For $i = 2$ and using Lemma 2.3, we have

$$\begin{aligned} \|T_{n,a}^{\alpha,\beta}(e_2; x) - x^2\|_{1+x^2} &= \sup_{x \in [0, \infty)} \frac{1}{1+x^2} \left| \left(\frac{n^2}{(n+\beta)^2} - 1 \right) x^2 + \left(3 + 2\alpha + \frac{1}{a-1} \right) \frac{nx}{(n+\beta)^2} + \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)^2} \right| \\ &\leq \left(\frac{n^2}{(n+\beta)^2} - 1 \right) + \left(3 + 2\alpha + \frac{1}{a-1} \right) \frac{n}{(n+\beta)^2} + \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)^2}. \end{aligned}$$

For a given $\epsilon > 0$, we have the following sets

$$\begin{aligned} E_1 &: = \left\{ n : \left\| T_{n,a}^{\alpha,\beta}(e_2; x) - x^2 \right\| \geq \epsilon \right\} \\ E_2 &: = \left\{ n : \left(\frac{n^2}{(n+\beta)^2} - 1 \right) \geq \frac{\epsilon}{3} \right\} \\ E_3 &: = \left\{ n : \left(3 + 2\alpha + \frac{1}{a-1} \right) \frac{n}{(n+\beta)^2} \geq \frac{\epsilon}{3} \right\} \\ E_4 &: = \left\{ n : \frac{\alpha^2 + 2\alpha + 2}{(n+\beta)^2} \geq \frac{\epsilon}{3} \right\}. \end{aligned}$$

This implies that $E_1 \subseteq E_2 \cup E_3 \cup E_4$. By which, we obtained

$$\sum_{k \in E_1} a_{nk} \leq \sum_{k \in E_2} a_{nk} + \sum_{k \in E_3} a_{nk} + \sum_{k \in E_4} a_{nk}.$$

As $n \rightarrow \infty$, we get

$$st_A - \lim_n \|T_{n,a}^{\alpha,\beta}(e_2; x) - x^2\|_{1+x^2} = 0.$$

Hence, the proof of Theorem 7.1 is completed. ■

Consequently, we obtain

Corollary 7.1. Let $A = (a_{nk})$ be a non-negative regular suitability matrix and $x \geq 0$ and $L_n = T_{n,a}^{0,0}$ be the operators defined by (1.4). Then, we have

$$st_A - \lim_n \|L_n(f; x, a) - f\|_{1+x^{2+\lambda}} = 0, \text{ for all } f \in C_{1+x^{2+\lambda}} \text{ and } \lambda > 0.$$

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