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On Weakly Semiprime Ideals in Commutative Rings

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Abstract In this paper, we study weakly semiprime ideals of a commutative ring with nonzero identity. We give some properties of such ideals. Also, we investigate some results of weakly semiprime submodules of a module over a commutative ring R with nonzero identity.

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1. Introduction

Weakly prime ideals in a commutative ring with nonzero identity have been introduced and studied by D. D. Anderson and E. Smith (see [1]). Weakly primary ideals in a commutative ring with nonzero identity have been introduced and studied by S. Ebrahimi Atani and F. Farzalipour (see [2]). Also, weakly prime submodules of a module over commutative ring have been studied in [3–6]. Here we study weakly semiprime ideals of a commutative ring with nonzero identity. Also, We give some properties of weakly semiprime submodules of an R-module. For example, we show that weakly semiprime submodules of secondary modules are secondary. Throughout this work R will denote a commutative ring with nonzero identity and all modules are unitary.

Before we state some results let us introduce some notation and terminology. A proper ideal P of R is said to be semiprime if $a^k b \in P$ where $a, b \in R$ and $k \in \mathbb{Z}^+$, then $ab \in P$. A proper ideal P of R is said to be weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. A proper submodule N of a module M over a commutative ring R is said to be semiprime if whenever $r^k m \in N$, for some $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$, then $rm \in N$. A proper submodule N of a module M over a commutative ring R is said to be weakly prime if $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. An R-module M is called a secondary module provided that for every element $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent.

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2. Main Results

A proper ideal P of a commutative ring with nonzero identity is said to be weakly semiprime if $0 \neq a^k b \in P$ where $a, b \in R$ and $k \in \mathbb{Z}^+$, then $ab \in P$.

It is clear that every semiprime ideal is a weakly semiprime ideal. However, since 0 is always weakly semiprime (by definition), a weakly semiprime ideal need not be semiprime, but if R be an integral domain, then every weakly semiprime is semiprime.

Also, every weakly prime ideal is a weakly semiprime ideal, but the converse is not true in general. For example, let $R = \mathbb{Z}_{30}$ be the ring of integers modulo 30 and $I =< 6 >$. The ideal I is weakly semiprime, but it is not weakly prime. Because $0 \neq 2.3 \in I$, but $2 \notin I$ and $3 \notin I$.

Proposition 2.1. *Let R be a commutative ring and let P be a proper ideal of R. Then the following assertion are equivalent.*

(i) P is a weakly semiprime ideal of R.
(ii) For $a \in R$ and $k \in \mathbb{Z}^+$; $(P : Ra^k) = (P : Ra) \cup (0 : Ra^k)$.
(iii) For $a \in R$ and $k \in \mathbb{Z}^+$; $(P : Ra^k) = (P : Ra)$ or $(P : Ra^k) = (0 : Ra^k)$.

Proof. $(i) \Longrightarrow (ii)$ Let P is a weakly semiprime ideal of R. It is clear that $(P : Ra) \cup (0 : Ra^k) \subseteq (P : Ra^k)$. Let $x \in (P : Ra^k)$, then $xa^k \in P$. If $xa^k = 0$, then $x \in (0 : Ra^k)$. Let $0 \neq xa^k \in P$, then $xa \in P$ since P is weakly semiprime, so $x \in (P : Ra)$. Thus $(P : Ra^k) \subseteq (P : Ra) \cup (0 : Ra^k)$. Therefore $(P : Ra^k) = (P : Ra) \cup (0 : Ra^k)$.
$(ii) \Longrightarrow (iii)$ is clear.
$(iii) \Longrightarrow (i)$ Let $0 \neq a^k b \in P$ where $a, b \in P$ and $k \in \mathbb{Z}^+$. So by hypothesis, $b \in (P : Ra)$, hence $ab \in P$, as needed. ■

Proposition 2.2. *Let $R = R_1 \times R_2$ where R_i is a commutative ring with identity. Then the following hold:*

(i) P_1 is a semiprime ideal of R_1 if and only if $P_1 \times R_2$ is a semiprime ideal of R.
(ii) P_2 is a semiprime ideal of R_2 if and only if $R_1 \times P_2$ is a semiprime ideal of R.
(iii) If $P_1 \times R_2$ is a weakly semiprime ideal of R, then P_1 is a weakly semiprime ideal of R_1.
(iv) If $R_1 \times P_2$ is a weakly semiprime ideal of R, then P_2 is a weakly semiprime ideal of R_2.

Proof. (i) Let P_1 is a weakly semiprime ideal of R_1. Suppose $(a, b)^k (c, d) \in P_1 \times R_2$ where $(a, b), (c, d) \in R$ and $k \in \mathbb{Z}^+$. So $a^k b \in P_1$, and $ab \in P_1$. Hence $(a, b)(c, d) \in P_1 \times R_2$, as required. Let $P_1 \times R_2$ is a semiprime ideal of R. Let $a^k b \in P_1$ where $a, b \in R_1$ and $k \in \mathbb{Z}^+$. So $(a, 1)^k (b, 1) \in P_1 \times R_2$, thus $(a, 1)(b, 1) \in P_1 \times R_2$. Hence $ab \in P_1$, as needed.
(ii) The proof is similar to that in case (i) and we omit it.
(iii) Let $P_1 \times R_2$ is a weakly semiprime ideal of R. Suppose $0 \neq a^k b \in P_1$. So $0 \neq (a, 1)^k (b, 1) \in P_1 \times R_2$, then $(a, 1)(b, 1) \in P_1 \times R_2$. Hence $ab \in P_1$, so P_1 is a weakly semiprime ideal of R_1.
(iv) The proof is similar to that in case (iii). ■

Theorem 2.3. *Let I be a secondary ideal of a commutative ring R. Then if Q is a weakly semiprime subideal of I, then Q is a secondary ideal of R.*

Proof. Let I be a P-secondary ideal and let $a \in R$. If $a \in P$, then there exists $n \in \mathbb{N}$ such that $a^nQ \subseteq a^nI = 0$. If $a \notin P$, so $aI = I$. We show that $aQ = Q$. Let $q \in Q$. We may assume that $q \neq 0$. So there exists $b \in I$ such that $q = ab$. Hence $b = ac$ for some $c \in I$, thus $0 \neq ab = a^2c \in Q$. So $b = ac \in Q$ since Q is a weakly semiprime ideal. Therefore $q = ab \in aQ$, as required. ■

Theorem 2.4. *Let I be a secondary ideal of a commutative ring R. Then if Q is a weakly semiprime ideal of R, then $Q \cap I$ is a secondary ideal of R.*

Proof. The proof is straightforward by Theorem 2.3. ■

Proposition 2.5. *Let R be a commutative ring and S be a multiplication closed subset of R. If P is a weakly semiprime ideal of R, then $S^{-1}P$ is a weakly semiprime ideal of $S^{-1}R$.*

Proof. Let $0/1 \neq (r/s)^k \cdot a/t \in S^{-1}P$ where $r/s, a/t \in S^{-1}R$ and $k \in \mathbb{Z}$. So $0/1 \neq r^ka/s^kt = b/t'$ for some $b \in P$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'r^ka = s's^ktb \in P$ (because if $s't'r^ka = 0$, $r^ka/s^kt = s't'r^ka/s't's^kt = 0/1$, a contradiction), so P weakly semiprime gives $ras't' \in P$. Hence $ra/st = ras't'/sts't' \in S^{-1}P$, as needed. ■

Proposition 2.6. *Let $I \subseteq P$ be proper ideals of a commutative ring R. Then the following hold:*

(i) If P is a weakly semiprime ideal of R, then P/I is a weakly semiprime ideal of R/I.
(ii) If I and P/I are weakly semiprime ideals of R and R/I respectively, then P is weakly semiprime.

Proof. (i) Let $0 \neq (a+I)^k(b+I) = a^kb+I \in P/I$ where $(a+I), (b+I) \in R/I$ and $k \in \mathbb{Z}$. So $0 \neq a^kb \in P$, P weakly semiprime gives $ab \in P$. Hence $(a+I)(b+I) \in P/I$.
(ii) Let $0 \neq a^kb \in P$ where $a, b \in R$ and $k \in \mathbb{Z}$. So $a^kb + I = (a+I)^k(b+I) \in P/I$. If $0 \neq a^kb \in I$, then $ab \in I \subseteq P$ since I is weakly semiprime, as needed. Let $0 \neq (a+I)^k(b+I) \in P/I$, then $(a+I)(b+I) \in P/I$ since P/I is weakly semiprime. Hence $ab \in P$, as required. ■

Let R be a commutative ring and M an R-module. A proper submodule N of M is said to be weakly semiprime, if $0 \neq r^km \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$, then $rm \in N$.

Proposition 2.7. *Let M be an R-module and let N be a proper submodule of M. Consider the following statements.*

(i) N is a weakly semiprime submodule of M.
(ii) For $m \in M$, $Rad(N : Rm) = (N : Rm) \cup Rad(0 : Rm)$.
(iii) For $m \in M$, $Rad(N : Rm) = (N : Rm)$ or $Rad(N : Rm) = (0 : Rm)$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$

Proof. $(i) \Longrightarrow (ii)$ Let N is a weakly semiprime submodule of M. Let $x \in Rad(N : Rm)$, then $x^km \in N$ for some $k \in \mathbb{N}$. If $x^km = 0$, then $x \in Rad(0 : Rm)$. Let $0 \neq x^km \in N$, then $xm \in N$ since N is weakly semiprime, so $x \in (N : Rm)$. Thus $Rad(N : Rm) \subseteq (N : Rm) \cup Rad(0 : Rm)$. It is clear that $(N : Rm) \cup Rad(0 : Rm) \subseteq Rad(N : Rm)$.
$(ii) \Longrightarrow (iii)$ is clear. ■

An R-module M is called prime module if the zero submodule is prime.

Remark 2.8. R-module M is prime if and only if $(0 : M) = (0 : m)$ for any $0 \neq m \in M$.

Theorem 2.9. *Let M be a prime R-module and let N be a proper submodule of M. Then the following statements are equivalent.*

(i) N is a weakly semiprime submodule of M.

(ii) For $m \in M$, $Rad(N : Rm) = (N : Rm) \cup Rad(0 : Rm)$.

(iii) For $m \in M$, $Rad(N : Rm) = (N : Rm)$ or $Rad(N : Rm) = (0 : Rm)$.

Proof. It is enough to show that $(iii) \Rightarrow (i)$. Let $0 \neq r^k m \in N$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}^+$. So $r \in Rad(N : Rm)$. If $r \in Rad(0 : Rm)$, then $r^n m = 0$ for some $n \in \mathbb{Z}^+$. Let t be an integer such that $r^t m = 0$ and $r^{t-1}m \neq 0$. If $t > k$, then $0 < t - k < t$; $r^t m = r^k(r^{t-k}m) = 0$; $r^k \in (0 : r^{t-k}m) = (0 : M)$ since M a is prime module. Hence $r^k M = 0$, so $r^k m = 0$, a contradiction. Let $k \geq t$. Thus $r^k m = r^{k-t}(r^t m) = 0$ which is a contradiction. Therefore $r \notin Rad(0 : Rm)$. So $r \in (N : Rm)$, hence $rm \in N$, as needed. ∎

Proposition 2.10. *Let $R = R_1 \times R_2$ where R_i is a commutative ring with identity. Let M_i be an R_i-module and let $M = M_1 \times M_2$ be the R-module with action $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then the following hold:*

(i) N_1 is a semiprime submodule of M_1 if and only if $N_1 \times M_2$ is a semiprime submodule of M.

(ii) N_2 is a semiprime submodule of M_2 if and only if $M_1 \times N_2$ is a semiprime submodule of M.

(iii) If $N_1 \times M_2$ is a weakly semiprime submodule of M, then N_1 is a weakly semiprime submodule of M_1.

(iv) If $M_1 \times N_2$ is a weakly semiprime submodule of M, then N_2 is a weakly semiprime submodule of M_2.

Proof. (i) Let N_1 is a weakly semiprime submodule of M_1. Suppose $(a, b)^k(m, n) \in N_1 \times M_2$ where $(a, b) \in R$, $(m, n) \in M$ and $k \in \mathbb{Z}^+$. So $a^k m \in N_1$, and $am \in N_1$. Hence $(a, b)(m, n) \in N_1 \times M_2$, as required. Let $N_1 \times M_2$ is a semiprime submodule of M. Let $a^k m \in N_1$ where $a \in R_1$, $m \in M_1$ and $k \in \mathbb{Z}^+$. So $(a, 1)^k(m, 0) \in N_1 \times M_2$, thus $(a, 1)(m, 0) \in N_1 \times M_2$. Hence $am \in N_1$, as needed.
(ii) The proof is similar to that in case (i) and we omit it.
(iii) Let $N_1 \times M_2$ is a weakly semiprime submodule of M. Suppose $0 \neq a^k m \in N_1$ where $r \in R$, $m \in M$ and $k \in \mathbb{Z}$. So $0 \neq (a, 1)^k(m, 0) \in N_1 \times M_2$, then $(a, 1)(m, 0) \in N_1 \times M_2$. Hence $am \in N_1$, so N_1 is a weakly semiprime submodule of M_1.
(iv) The proof is similar to that in case (iii). ∎

Theorem 2.11. *Let M be a secondary R-module and N a non-zero weakly semiprime R-submodule of M. Then N is secondary.*

Proof. Let $r \in R$. If $r^n M = 0$ for some $n \in \mathbb{N}$, then $r^n N \subseteq r^n M = 0$, so r is nilpotent on N. Suppose that $rM = M$; we show that r divides N. Let $n \in N$. We may assume that $0 \neq n$. So $n = rm$ for some $m \in M$. We have $rm' = m$ for some $m' \in M$, hence $0 \neq rm = r^2 m' \in N$, so $m = rm' \in N$ since N is weakly semiprime. Thus $rN = N$, as needed. ∎

Corollary 2.12. *Let M be an R-module, N a secondary R-submodule of M and K a weakly semiprime submodule of M. Then $N \cap K$ is secondary.*

Proof. The proof is straightforward. ∎

Proposition 2.13. *Let R be a commutative ring and S be a multiplication closed subset of R. If N is a weakly semiprime submodule of M, then $S^{-1}N$ is a weakly semiprime submodule of $S^{-1}M$.*

Proof. Let $0/1 \neq (r/s)^k \cdot m/t \in S^{-1}N$ where $r/s \in S^{-1}R, m/t \in S^{-1}M$ and $k \in \mathbb{Z}$. So $0/1 \neq r^k m/s^k t = n/t'$ for some $n \in N$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'r^k m = s's^k tn \in N$ (because if $s't'r^k m = 0$, $r^k m/s^k t = s't'r^k m/s't's^k t = 0/1$, a contradiction), so N weakly semiprime gives $rms't' \in N$. Hence $rm/st = rms't'/sts't' \in S^{-1}N$, as needed. ∎

Proposition 2.14. *Let $K \subseteq N$ be proper submodules of an R-module M. Then the following hold:*

(i) If N is a weakly semiprime submodule of M, then N/K is a weakly semiprime submodule of M/N.

(ii) If K and N/K are weakly semiprime submodules of M and M/K respectively, then N is weakly semiprime.

Proof. (i) Let $0 \neq r^k(m+K) =\in N/K$ where $r \in R$, $m+K \in M/K$ and $k \in \mathbb{Z}$. So $0 \neq r^k m \in N$, N weakly semiprime gives $rm \in N$. Hence $r(m+K) \in N/K$.
(ii) Let $0 \neq r^k m \in N$ for some $r \in R$, $m \in M$ and $k \in \mathbb{Z}$. So $r^k m + K = r^k(m+K) \in N/K$. If $0 \neq r^k m \in K$, then $rm \in K \subseteq N$ since K is weakly semiprime, as needed. Let $0 \neq r^k(m+K) \in N/K$, then $r(m+K) \in N/K$ since N/K is weakly semiprime. Hence $rm \in P$, as required. ∎

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