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Multiplicative Fourier Transform

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Abstract In this paper, we present a new integral transformation related to the Fourier transform, called the *multiplicative Fourier transform*. We also study some interesting properties and the inverse transformation. Moreover, we give some applications to linear and nonlinear multiplicative differential equations.

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1. Introduction

Calculus is a branch of mathematics that plays an integral role in the development and improvement of the world from the past to the present, continuously. Actually, calculus has several practical applications in real life. For instance, in sciences, especially in physics, calculus is used to define, explain and calculate motion, electricity, forces and dynamics. In engineerings, some applications of calculus appear in aerospace engineering and electrical engineering. In addition, calculus is used in economics to predict supply, demand, maximum potential profits and minimum costs. It is well known that the concept of calculus was first introduced by Sir Isaac Newton. Consequently, such concept of calculus is considered as a classical calculus or Newtonian calculus. However, basic knowledge of calculus is still improved, subsequently. Thus some relevant definitions and prominent theorems always occur and they are different from the concept of Newtonian calculus.

In 1972, Grossman and Katz [1] presented various new types of calculus such as geometric calculus, harmonic calculus, quadratic calculus, bigeometric calculus, biharmonic calculus and biquadratic calculus. Those types of calculus are called *Non-Newtonian calculus* which can be found in [2, 3]. Moreover, such new types are applied for solving problems in economics, actuarial science, finance, digital image processing and demographics etc.

Later, the concept of geometric calculus was studied spreadily and related to the multiplication, so called *multiplicative calculus* which can be seen in [4–6] for more informations.

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There are two crucial definitions consisting of multiplicative derivatives and multiplicative integrals defined as follows.

Let f be a positive function. The *multiplicative derivative* of $f(x)$ is defined by

$$\frac{d^* f(x)}{dx} = f^*(x) = \lim_{h\to 0}\left[\frac{f(x+h)}{f(x)}\right]^{\frac{1}{h}}. \tag{1.1}$$

The *multiplicative integral* of a positive bounded integrable function $f(x)$ on $[a,b]$ is defined by

$$\int_a^b f(x)^{dx} = \exp\left[\int_a^b \ln f(x)dx\right] \tag{1.2}$$

where $\ln f(x)$ is the natural logarithm of $f(x)$ with respect to the base of the mathematical constant e.

From (1.1), we obtain the relation between the first order classical derivative and the first order multiplicative derivative of $f(x)$ which is expressed as follows:

$$f^*(x) = e^{\frac{f'(x)}{f(x)}} = e^{[\ln f(x)]'}. \tag{1.3}$$

In addition, the relation between the n^{th} order classical derivative and the n^{th} order multiplicative derivative of $f(x)$ satisfies that

$$f^{*(n)}(x) = e^{[\ln f(x)]^{(n)}}, \ \forall n \in \mathbb{N}. \tag{1.4}$$

Furthermore, we obtain some prominent properties from (1.1) and (1.2) with respect to the multiplicative derivative and multiplicative integral of the multiplication of $f(x)$ and $g(x)$, respectively, as follows:

$$(f\cdot g)^*(x) = f^*(x)\cdot g^*(x) \text{ and}$$

$$\int_a^b [f(x)\cdot g(x)]^{dx} = \int_a^b f(x)^{dx}\cdot \int_a^b g(x)^{dx}.$$

Note that, from the above statement, using of the terminology "multiplicative calculus" would be more suitable and reasonable than using of "geometric calculus".

In 2016, Yalcin et al. [7] studied the multiplicative calculus related to the multiplicative Laplace transforms and their applications. In this paper, we present multiplicative Fourier transforms of functions concerned with the Fourier transform. Moreover, we purpose certain interesting properties such as the shifting property and multiplicative property. Besides the multiplicative case, the inverse multiplicative Fourier transform is also considered together with illustrating some examples. In addition, the results will be applied to solve linear and nonlinear multiplicative differential equations.

2. Preliminaries

We start this part with the definition of the multiplicative Fourier transform as follows.

Definition 2.1. Let $f(x)$ be a positive function on $\mathbb{R}$. The *multiplicative Fourier transform* of $f(x)$ is defined by

$$\mathcal{F}_m(s) = \mathcal{F}_m\{f(x)\} = \int_{-\infty}^{\infty}\left[f(x)^{e^{-ixs}}\right]^{dx}, \ i=\sqrt{-1}.$$

From the definition of the multiplicative integral in (1.2) and by applying the above definition, we consequently obtain that

$$\begin{aligned}\mathcal{F}_m(s) = \mathcal{F}_m\{f(x)\} &= \int_{-\infty}^{\infty} \left[(f(x))^{e^{-ixs}}\right]^{dx} \\ &= \exp\left[\int_{-\infty}^{\infty} \ln\left((f(x))^{e^{-ixs}}\right) dx\right] \\ &= \exp\left[\int_{-\infty}^{\infty} \ln f(x) \cdot e^{-ixs} dx\right] \\ &= e^{\mathcal{F}\{\ln f(x)\}} \qquad (2.1)\end{aligned}$$

where $\mathcal{F}\{g(x)\}$ is the Fourier transform of $g(x)$ defined by $\int_{-\infty}^{\infty} g(x) \cdot e^{-ixs} dx$. It follows that (2.1) is the equation which shows the relation between the multiplicative Fourier transform and the Fourier transform of functions.

Next, we investigate the existence condition of the multiplicative Fourier transform of functions which is shown in the following theorem.

Theorem 2.2. (The existence of multiplicative Fourier transform) *Let $f(x)$ be a positive function on $\mathbb{R}$. If $\ln f(x)$ is a bounded integrable function on $\mathbb{R}$ and $\lim_{x\to\pm\infty} f(x) = 1$, then the multiplicative Fourier transform of $f(x)$ exists.*

Proof. We prove that $|\mathcal{F}_m\{f(x)\}| < \infty$. We now consider

$$\begin{aligned}|\mathcal{F}_m\{f(x)\}| &= \left|e^{\mathcal{F}\{\ln f(x)\}}\right| \\ &\leq e^{|\mathcal{F}\{\ln f(x)\}|}\end{aligned}$$

and

$$\begin{aligned}|\mathcal{F}\{\ln f(x)\}| &= \left|\int_{-\infty}^{\infty} \ln f(x) \cdot e^{-ixs} dx\right| \\ &\leq \int_{-\infty}^{\infty} \left|\ln f(x) \cdot e^{-ixs}\right| dx \\ &\leq \int_{-\infty}^{\infty} |\ln f(x)|\, dx.\end{aligned}$$

Since $\ln f(x)$ is a bounded integrable function on $\mathbb{R}$ and $\lim_{x\to\pm\infty} f(x) = 1$, we have

$$\int_{-\infty}^{\infty} |\ln f(x)|\, dx < \infty.$$

Therefore, $|\mathcal{F}_m\{f(x)\}| < \infty$. This completely proves the existence of the multiplicative Fourier transform of f. ■

Then, we mention about the inverse multiplicative Fourier transform according to the following definition.

Definition 2.3. Let $\mathcal{F}_m(s)$ be the multiplicative Fourier transform of $f(x)$, that is, $\mathcal{F}_m(s) = \mathcal{F}_m\{f(x)\}$. We define the *inverse multiplicative Fourier transform* of $\mathcal{F}_m(s)$ by

$$\begin{aligned}\mathcal{F}_m^{-1}(x) &= \left(\mathcal{F}_m^{-1}\{\mathcal{F}_m(s)\}\right)(x) \\ &= f(x).\end{aligned}$$

3. Results

In this section, we study important properties of the multiplicative Fourier transform and the inverse multiplicative Fourier transform of functions. We first start with the following theorem related to the multiplication of two functions.

Theorem 3.1. *Let a, b be arbitrary constants and $f(x), g(x)$ positive functions on $\mathbb{R}$. If the multiplicative Fourier transforms of $f(x)$ and $g(x)$ exist, then*

$$\mathcal{F}_m\{(f(x))^a\,(g(x))^b\} = (\mathcal{F}_m\{f(x)\})^a \cdot (\mathcal{F}_m\{g(x)\})^b\,. \tag{3.1}$$

Proof. From (2.1), we obtain that

$$\begin{aligned}
\mathcal{F}_m\{(f(x))^a\,(g(x))^b\} &= \exp\left[\mathcal{F}\left\{\ln\left((f(x))^a\,(g(x))^b\right)\right\}\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \ln\left((f(x))^a\,(g(x))^b\right)\cdot e^{-ixs}dx\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \left(\ln\,(f(x))^a + \ln\,(g(x))^b\right)\cdot e^{-ixs}dx\right] \\
&= \exp\left[\int_{-\infty}^{\infty} (a\ln f(x) + b\ln g(x))\cdot e^{-ixs}dx\right] \\
&= \exp\left[a\int_{-\infty}^{\infty} \ln f(x)\cdot e^{-ixs}dx + b\int_{-\infty}^{\infty} \ln g(x)\cdot e^{-ixs}dx\right] \\
&= \left(\exp\left[\int_{-\infty}^{\infty} \ln f(x)\cdot e^{-ixs}dx\right]\right)^a \cdot \left(\exp\left[\int_{-\infty}^{\infty} \ln g(x)\cdot e^{-ixs}dx\right]\right)^b \\
&= (\mathcal{F}_m\{f(x)\})^a \cdot (\mathcal{F}_m\{g(x)\})^b\,.
\end{aligned}$$

This completes the proof of our assertion. ■

The following theorem presents the shifting properties of the multiplicative Fourier transforms of functions.

Theorem 3.2. *Let $f(x)$ be a positive function on $\mathbb{R}$ and a an arbitrary constant. If $\mathcal{F}_m(s)$ is the multiplicative Fourier transform of $f(x)$, then the following statements hold:*

(1) $\mathcal{F}_m\{(f(x))^{e^{ax}}\} = \mathcal{F}_m(s-a)$;

(2) $\mathcal{F}_m\{f(x-a)\} = (\mathcal{F}_m(s))^{e^{-ias}}$.

Proof. Consider

$$\begin{aligned}
\mathcal{F}_m\{(f(x))^{e^{ax}}\} &= \exp\left[\mathcal{F}\{\ln\,(f(x))^{e^{ax}}\}\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \ln\,(f(x))^{e^{ax}}\cdot e^{-ixs}dx\right] \\
&= \exp\left[\int_{-\infty}^{\infty} e^{ax}\cdot \ln f(x)\cdot e^{-ixs}dx\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \ln f(x)\cdot e^{-ix(s-a)}dx\right] \\
&= \mathcal{F}_m(s-a)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_m\{f(x-a)\} &= \exp\left[\mathcal{F}\{\ln f(x-a)\}\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \ln f(x-a)\cdot e^{-ixs}dx\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \ln f(x-a)\cdot e^{-i(x-a)s}\cdot e^{-ias}d(x-a)\right] \\
&= \exp\left[\int_{-\infty}^{\infty} \ln f(y)\cdot e^{-iys}\cdot e^{-ias}dy\right], \quad y=x-a \\
&= \exp\left[e^{-ias}\int_{-\infty}^{\infty} \ln f(y)\}\cdot e^{-iys}dy\right] \\
&= \left(\exp\int_{-\infty}^{\infty} \ln f(y)\}\cdot e^{-iys}dy\right)^{e^{-ias}} \\
&= \left(\mathcal{F}_m(s)\right)^{e^{-ias}}.
\end{aligned}$$

The theorem is completely proved. ■

We next verify the multiplicative Fourier transform of the first order multiplicative derivative of $f(x)$ as follows.

Theorem 3.3. *Let $f(x)$ be a positive function on $\mathbb{R}$. If $\ln f(x)$ is a bounded integrable function on $\mathbb{R}$, $\lim_{x\to\pm\infty} f(x)=1$ and $f(x)$ has the first order multiplicative derivative, then*

$$\mathcal{F}_m\{f^*(x)\} = \left(\mathcal{F}_m(s)\right)^{si} \tag{3.2}$$

where $\mathcal{F}_m(s)$ is the multiplicative Fourier transform of $f(x)$.

Proof. By (1.3), we get that

$$\begin{aligned}
\mathcal{F}_m\left\{f^*(x)\right\} &= \mathcal{F}_m\left\{e^{\frac{f'(x)}{f(x)}}\right\} \\
&= \exp\left[\mathcal{F}\left\{\ln e^{\frac{f'(x)}{f(x)}}\right\}\right] \\
&= \exp\left[\mathcal{F}\left\{\frac{f'(x)}{f(x)}\right\}\right] \\
&= \exp\left[\int_{-\infty}^{\infty}\frac{f'(x)}{f(x)}\cdot e^{-ixs}dx\right].
\end{aligned}$$

Using the integration by parts by setting $u=e^{-ixs}$ and $dv=\frac{f'(x)}{f(x)}dx$ implies that $du=-ise^{-ixs}dx$ and $v=\ln f(x)$ and

$$\mathcal{F}_m\{f^*(x)\} = \exp\left[\ln f(x)e^{-ixs}|_{-\infty}^{\infty} + si\int_{-\infty}^{\infty}\ln f(x)\cdot e^{-ixs}dx\right].$$

Since $\lim_{x\to\pm\infty} f(x) = 1$ and $\left|e^{-ixs}\right| \le 1$, we obtain that $\ln f(x) \cdot e^{-ixs}|_{-\infty}^{\infty} = 0$. It follows that

$$\begin{aligned}\mathcal{F}_m\{f^*(x)\} &= \exp\left[si\int_{-\infty}^{\infty} \ln f(x) \cdot e^{-ixs}dx\right] \\ &= \left(\exp\left[\int_{-\infty}^{\infty} \ln f(x) \cdot e^{-ixs}dx\right]\right)^{si}.\end{aligned}$$

By our assumption and Theorem 2.2, we can conclude that $\mathcal{F}_m(s)$ exists. Consequently,

$$\mathcal{F}_m\{f^*(x)\} = (\mathcal{F}_m(s))^{si}.$$

Hence, the result follows. ■

We now extend the concept in Theorem 3.3 to find the multiplicative Fourier transform of the multiplicative derivative of order n of $f(x)$.

Theorem 3.4. *Let $f(x)$ be a positive function on $\mathbb{R}$ and $k, n \in \mathbb{N}$. If $f(x)$ has the multiplicative derivative of order n, $(\ln f(x))^{(k)}$ is a bounded integrable function on $\mathbb{R}$ and $\lim_{x\to\pm\infty} (\ln f(x))^{(k)} = 0$ for all $k \le n$, then*

$$\mathcal{F}_m\{f^{*(n)}(x)\} = (\mathcal{F}_m(s))^{(si)^n}. \tag{3.3}$$

Proof. We will prove the theorem by using the mathematical induction. For $n = 1$, the result holds by Theorem 3.3. We now assume that, for $k \in \mathbb{N}$,

$$\mathcal{F}_m\{f^{*(k)}(x)\} = (\mathcal{F}_m(s))^{(si)^k}. \tag{3.4}$$

We need to show that $\mathcal{F}_m\{f^{*(k+1)}(x)\} = (\mathcal{F}_m(s))^{(si)^{k+1}}$. From (1.4), we obtain that

$$\begin{aligned}\mathcal{F}_m\{f^{*(k+1)}(x)\} &= \mathcal{F}_m\{e^{(\ln f(x))^{(k+1)}}\} \\ &= \exp\left[\mathcal{F}\{\ln e^{(\ln f(x))^{(k+1)}}\}\right] \\ &= \exp\left[\mathcal{F}\{(\ln f(x))^{(k+1)}\}\right] \\ &= \exp\left[\int_{-\infty}^{\infty} (\ln f(x))^{(k+1)} \cdot e^{-ixs}dx\right] \\ &= \exp\left[\int_{-\infty}^{\infty} \left((\ln f(x))^{(k)}\right)' \cdot e^{-ixs}dx\right].\end{aligned}$$

By using the integration by parts, let $u = e^{-ixs}$ and $dv = \left((\ln f(x))^{(k)}\right)' dx$. Then $du = -sie^{-ixs}dx$ and $v = (\ln f(x))^{(k)}$. Hence

$$\mathcal{F}_m\{f^{*(k+1)}(x)\} = \exp\left[(\ln f(x))^{(k)}\, e^{-ixs}|_{-\infty}^{\infty} + si\int_{-\infty}^{\infty} (\ln f(x))^{(k)} \cdot e^{-ixs}dx\right].$$

By the assumption of the theorem, we have

$$\begin{aligned}
\mathcal{F}_m\{f^{*(k+1)}(x)\} &= \exp\left[0 + si\int_{-\infty}^{\infty}(\ln f(x))^{(k)}\cdot e^{-ixs}dx\right] \\
&= \left(\exp\left[\int_{-\infty}^{\infty}(\ln f(x))^{(k)}\cdot e^{-ixs}dx\right]\right)^{si} \\
&= \left(\exp\left[\int_{-\infty}^{\infty}\ln e^{(\ln f(x))^{(k)}}\cdot e^{-ixs}dx\right]\right)^{si} \\
&= \left(\exp\left[\mathcal{F}\{\ln e^{(\ln f(x))^{(k)}}\}\right]\right)^{si} \\
&= \left(\exp\left[\mathcal{F}\{\ln f^{*(k)}(x)\}\right]\right)^{si} \\
&= \left(\mathcal{F}_m\{f^{*(k)}(x)\}\right)^{si} \\
&= \left((\mathcal{F}_m(s))^{(si)^k}\right)^{si} \qquad \text{(by (3.4))} \\
&= (\mathcal{F}_m(s))^{(si)^{k+1}}.
\end{aligned}$$

Hence

$$\mathcal{F}_m\{f^{*(n)}(x)\} = (\mathcal{F}_m(s))^{(si)^n}, \quad \forall n \in \mathbb{N}.$$

The theorem is proved, as required. ■

Next, we focus on the multiplicative Fourier transform of functions with the exponent x^n where $n \in \mathbb{N}$. The case $n = 1$ is first considered as follows.

Theorem 3.5. *Let $f(x)$ be a positive function on $\mathbb{R}$ with the multiplicative Fourier transform $\mathcal{F}_m(s)$. If the first order multiplicative derivative of $\mathcal{F}_m(s)$ (denoted by $\mathcal{F}_m^*(s)$) exists, then*

$$\mathcal{F}_m\{(f(x))^x\} = (\mathcal{F}_m^*(s))^i. \tag{3.5}$$

Proof. Consider

$$\begin{aligned}
\mathcal{F}_m^*(s) &= \frac{d^*}{ds}\exp\left[\int_{-\infty}^{\infty}\ln f(x)\cdot e^{-ixs}dx\right] \\
&= \exp\left[\frac{d}{ds}\ln\left(\exp\left[\int_{-\infty}^{\infty}\ln f(x)\cdot e^{-ixs}dx\right]\right)\right] \\
&= \exp\left[\frac{d}{ds}\int_{-\infty}^{\infty}\ln f(x)\cdot e^{-ixs}dx\right] \\
&= \exp\left[\int_{-\infty}^{\infty}-ix\ln f(x)\cdot e^{-ixs}dx\right] \\
&= \left(\exp\left[\int_{-\infty}^{\infty}\ln(f(x))^x\cdot e^{-ixs}dx\right]\right)^{-i} \\
&= (\mathcal{F}_m\{(f(x))^x\})^{-i}.
\end{aligned}$$

It follows that $\mathcal{F}_m\{(f(x))^x\} = (\mathcal{F}_m^*(s))^i$. ■

For the case of an arbitrary positive integer n, the result is presented in the following theorem.

Theorem 3.6. *Let $f(x)$ be a positive function on $\mathbb{R}$ with the multiplicative Fourier transform $\mathcal{F}_m(s)$. If the multiplicative derivative of order n of $\mathcal{F}_m(s)$ (denoted by $\mathcal{F}_m^{*(n)}(s)$) exists, then*

$$\mathcal{F}_m\{(f(x))^{x^n}\} = \left(\mathcal{F}_m^{*(n)}(s)\right)^{\frac{1}{(-i)^n}}. \tag{3.6}$$

Proof. The proof is done by the mathematical induction. The case $n = 1$ is true from Theorem 3.5 as shown in (3.6). Next, assume that

$$\mathcal{F}_m\{(f(x))^{x^k}\} = \left(\mathcal{F}_m^{*(k)}(s)\right)^{\frac{1}{(-i)^k}}. \tag{3.7}$$

We will show that $\mathcal{F}_m\{(f(x))^{x^{k+1}}\} = \left(\mathcal{F}_m^{*(n)}(s)\right)^{\frac{1}{(-i)^{k+1}}}$ by considering the value of $\mathcal{F}_m^{*(k+1)}(s)$ as follows:

$$\begin{aligned}
\mathcal{F}_m^{*(k+1)}(s) &= \frac{d^*}{ds}\left(\mathcal{F}_m^{*(k)}(s)\right) \\
&= \frac{d^*}{ds}\left(\mathcal{F}_m\{(f(x))^{x^k}\}\right)^{(-i)^k} \qquad \text{(from (3.7))} \\
&= \frac{d^*}{ds}\left(\exp\left[\int_{-\infty}^{\infty} \ln(f(x))^{x^k} \cdot e^{-ixs} dx\right]\right)^{(-i)^k} \\
&= \exp\left[\frac{d}{ds}\ln\left(\exp\left[\int_{-\infty}^{\infty} \ln(f(x))^{x^k} \cdot e^{-ixs} dx\right]\right)^{(-i)^k}\right] \\
&= \exp\left[(-i)^k \frac{d}{ds}\int_{-\infty}^{\infty} x^k \ln(f(x)) \cdot e^{-ixs} dx\right] \\
&= \exp\left[(-i)^k \int_{-\infty}^{\infty} (-i)x^{k+1} \ln(f(x)) \cdot e^{-ixs} dx\right] \\
&= \left(\exp\left[\int_{-\infty}^{\infty} \ln(f(x))^{x^{k+1}} \cdot e^{-ixs} dx\right]\right)^{(-i)^{k+1}} \\
&= \left(\mathcal{F}_m\{(f(x))^{x^{k+1}}\}\right)^{(-i)^{k+1}}.
\end{aligned}$$

It follows that $\mathcal{F}_m\{(f(x))^{x^{k+1}}\} = \left(\mathcal{F}_m^{*(n)}(s)\right)^{\frac{1}{(-i)^{k+1}}}$. Therefore,

$$\mathcal{F}_m\{(f(x))^{x^n}\} = \left(\mathcal{F}_m^{*(n)}(s)\right)^{\frac{1}{(-i)^n}}, \quad \forall n \in \mathbb{N}.$$

The theorem is proved. ∎

Theorem 3.7. *Let a and b be arbitrary constants, $G_m(s)$ and $H_m(s)$ the multiplicative Fourier transforms of $g(x)$ and $h(x)$, respectively, that is, $G_m(s) = \mathcal{F}_m\{g(x)\}$ and $H_m(s) = \mathcal{F}_m\{h(x)\}$, then*

$$\mathcal{F}_m^{-1}\{(G_m(s))^a (H_m(s))^b\} = \left(\mathcal{F}_m^{-1}\{G_m(s)\}\right)^a \left(\mathcal{F}_m^{-1}\{H_m(s)\}\right)^b = (g(x))^a (h(x))^b.$$

Proof. By Theorem 3.1, we have

$$\begin{aligned}\mathcal{F}_m\{(g(x))^a\,(h(x))^b\} &= (\mathcal{F}_m\{g(x)\})^a\,(\mathcal{F}_m\{h(x)\})^b \\ &= (G_m(s))^a\,(H_m(s))^b\,.\end{aligned}$$

Using Definition 2.3 implies that

$$(g(x))^a\,(h(x))^b = \mathcal{F}_m^{-1}\{(G_m(s))^a\,(H_m(s))^b\}.$$

This completes the proof, actually. ∎

4. Applications

In this section, we illustrate the multiplicative Fourier transform and the inverse multiplicative Fourier transform of some functions. After that we apply for solving linear and nonlinear multiplicative differential equations.

Example 4.1. Let $\mathcal{F}_m\{f(x)\}$ be the multiplicative Fourier transform of $f(x)$ and $\mathcal{F}_m^{-1}\{F(s)\}$ be the inverse multiplicative Fourier transform of $F(s)$. We obtain that

(1) $\mathcal{F}_m\{1\} = 1$ and $\mathcal{F}_m^{-1}\{1\} = 1$.
(2) $\mathcal{F}_m\{e^{e^{-|x|}}\} = e^{\frac{2}{1+s^2}}$ and $\mathcal{F}_m^{-1}\{e^{\frac{2}{1+s^2}}\} = e^{e^{-|x|}}$.
(3) $\mathcal{F}_m\{e^{e^{-x^2}}\} = e^{\sqrt{\pi}e^{-\frac{s^2}{4}}}$ and $\mathcal{F}_m^{-1}\{e^{\sqrt{\pi}e^{-\frac{s^2}{4}}}\} = e^{e^{-x^2}}$.
(4) $\mathcal{F}_m\{e^{xe^{-x^2}}\} = e^{-\frac{\sqrt{\pi}si}{2}e^{-\frac{s^2}{4}}}$ and $\mathcal{F}_m^{-1}\{e^{-\frac{\sqrt{\pi}si}{2}e^{-\frac{s^2}{4}}}\} = e^{xe^{-x^2}}$.

Solution.

(1) As the fact that $\ln 1 = 0$ we get $\mathcal{F}_m\{1\} = e^0 = 1$.
(2) By equation (2.1), we have that

$$\begin{aligned}\mathcal{F}_m\{e^{e^{-|x|}}\} &= e^{\mathcal{F}\{\ln e^{e^{-|x|}}\}} \\ &= \exp\left[\int_{-\infty}^{\infty} \ln e^{e^{-|x|}} \cdot e^{-ixs}dx\right] \\ &= \exp\left[\int_{-\infty}^{\infty} e^{-|x|}\cdot(\cos(xs) - i\sin(xs))\,dx\right].\end{aligned}$$

Since $e^{-|x|}\cos(xs)$ is an even function and $e^{-|x|}\sin(xs)$ is an odd function, it follows that

$$\begin{aligned}\mathcal{F}_m\{e^{e^{-|x|}}\} &= \exp\left[2\int_0^{\infty} e^{-x}\cos(xs)dx\right] \\ &= e^{\frac{2}{1+s^2}}.\end{aligned}$$

(3) The multiplicative Fourier transform of $f(x) = e^{e^{-x^2}}$ is

$$\begin{aligned} \mathcal{F}_m\{e^{e^{-x^2}}\} &= \exp\left[\mathcal{F}\{\ln e^{e^{-x^2}}\}\right] \\ &= \exp\left[\mathcal{F}\{e^{-x^2}\}\right] \\ &= \exp\left[\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-ixs} dx\right] \\ &= \exp\left[\int_{-\infty}^{\infty} e^{-(x+\frac{si}{2})^2+(\frac{si}{2})^2} dx\right] \\ &= \exp\left[e^{-\frac{s^2}{4}} \int_{-\infty}^{\infty} e^{-y^2} dy\right] \text{ where } y = x + \frac{si}{2}. \end{aligned}$$

It is well known that $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ which leads to $\mathcal{F}_m\{e^{e^{-x^2}}\} = e^{\sqrt{\pi}e^{-\frac{s^2}{4}}}$.

(4) Rewrite the function $f(x) = e^{xe^{-x^2}}$ as $f(x) = \left(e^{e^{-x^2}}\right)^x$. Then

$$\mathcal{F}_m\{e^{xe^{-x^2}}\} = \mathcal{F}_m\left\{\left(e^{e^{-x^2}}\right)^x\right\}.$$

Hence we can conclude by Theorem 3.5 that

$$\begin{aligned} \mathcal{F}_m\{e^{xe^{-x^2}}\} &= \left[\frac{d^*}{ds}\left(e^{\sqrt{\pi}e^{-\frac{s^2}{4}}}\right)\right]^i \\ &= e^{-\frac{\sqrt{\pi}si}{2}e^{-\frac{s^2}{4}}}. \end{aligned}$$

All of these results, together with Definition 2.3, provide the desired inverse multiplicative Fourier transform.

Example 4.2. The inverse multiplicative Fourier transform of $Z_m(s) = e^{-\frac{(2-si)\sqrt{\pi}}{2}e^{-\frac{s^2}{4}}}$ is $z(x) = e^{e^{-x^2}} \cdot e^{xe^{-x^2}}$.

Solution. Since $e^{-\frac{(2-si)\sqrt{\pi}}{2}e^{-\frac{s^2}{4}}} = e^{\sqrt{\pi}e^{-\frac{s^2}{4}}} \cdot e^{-\frac{\sqrt{\pi}si}{2}e^{-\frac{s^2}{4}}}$, we obtain that

$$z(x) = \mathcal{F}_m^{-1}\{e^{-\frac{(2-si)\sqrt{\pi}}{2}e^{-\frac{s^2}{4}}}\} = \mathcal{F}_m^{-1}\{e^{\sqrt{\pi}e^{-\frac{s^2}{4}}} \cdot e^{-\frac{\sqrt{\pi}si}{2}e^{-\frac{s^2}{4}}}\}.$$

By Theorem 3.7, we get

$$z(x) = \mathcal{F}_m^{-1}\{e^{\sqrt{\pi}e^{-\frac{s^2}{4}}}\} \cdot \mathcal{F}_m^{-1}\{e^{-\frac{\sqrt{\pi}si}{2}e^{-\frac{s^2}{4}}}\}.$$

From Example 4.1, we conclude that

$$\begin{aligned} z(x) &= e^{e^{-x^2}} \cdot e^{xe^{-x^2}} \\ &= e^{(1+x)e^{-x^2}}. \end{aligned}$$

Example 4.3. Find a solution of the linear multiplicative differential equation

$$y^{**}(x) = y(x). \tag{4.1}$$

Solution. Applying the multiplicative Fourier transform to both sides of (4.1) yields that

$$[Y_m(s)]^{-s^2} = Y_m(s).$$

Then

$$[Y_m(s)]^{s^2+1} = 1,$$

that is,

$$Y_m(s) = 1.$$

By applying the inverse multiplicative Fourier transform to both sides of $Y_m(s) = 1$, we have $y(x) = 1$ which is the solution of (4.1), certainly.

Example 4.4. Find a solution of the nonlinear multiplicative differential equation

$$\sqrt{y^*(x)} = e^{-xe^{-x^2}}. \tag{4.2}$$

Solution. We rewrite (4.2) as

$$[y^*(x)]^{\frac{-1}{2}} = e^{xe^{-x^2}}. \tag{4.3}$$

Applying the multiplicative Fourier transform to both sides of (4.3) implies that

$$[Y_m(s)]^{\frac{-is}{2}} = e^{-\frac{\sqrt{\pi}is}{2}e^{-\frac{s^2}{4}}}.$$

Thus

$$Y_m(s) = e^{\sqrt{\pi}e^{-\frac{s^2}{4}}}.$$

By applying the inverse multiplicative Fourier transform to both sides of the above equation, we have

$$y(x) = e^{e^{-x^2}},$$

which is the solution of (4.3).

5. Conclusions

In this paper, the definition of a new type of the integral transform of functions, called the multiplicative Fourier transform, has been introduced which is related to the Fourier transform. In addition, the existence of such transform and some interesting properties consisting of the shifting property and multiplicative derivative property have been studied. Moreover, the inverse multiplicative Fourier transform has been considered and some relevant examples have been illustrated. Finally, applications to linear and nonlinear multiplicative differential equations have been presented to find their solutions.

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