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Strong convergence of modified Mann iteration method for an infinite family of nonexpansive mappings in a Banach space

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Abstract : In this paper we introduce a new modified Mann iteration for a W-mapping generated by $T_n, T_{n-1}, ..., T_1$ and $\lambda_n, \lambda_{n-1}, ..., \lambda_1$. The iteration is defined as follows:

$$\begin{cases} x_1 = x \in C, \text{ arbitrarily;} \\ y_n = \alpha_n x_n + (1-\alpha_n)W_n x_n, \quad n \geq 1 \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n, \quad n \geq 1, \end{cases}$$

where W_n is a W-mapping, C a nonempty closed convex subset of a Banach space E with uniformly Gâteaux differentiable. Then we prove that under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ converges strongly to a common fixed point of $T_n, n \in \mathbb{N}$.

Keywords : Strong convergence: nonexpansive mappings: uniformly Gâteaux differentiable: Halpern type
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1 Introduction

Let C be a closed convex subset of a Banach space E. Recall that a self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad x, y \in C.$$

We use Π_C to denote the collection of all contraction on C. That is

$$\Pi_C = \{f : f : C \to C \text{ a contraction}\}.$$

Note that each $f \in \Pi_C$ has a unique fixed point in C. Let now T be a *nonexpansive mapping* of C into itself, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Halpern [4] introduced the following iterative scheme for approximating a fixed point of T:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n \tag{1.1}$$

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0, 1]$. This iteration process is called a Halpern type iteration. Strong convergence of this type iterative sequence has been widely studied: Wittmann [17] discussed such a sequence in a Hilbert space. Shioji and Takahashi [14] extented Wittmann's result; they prove strong converge of $\{x_n\}$ defined by (1.1) in a Banach space; see also Kamimura and Takahashi [7] and Iiduka and Takahashi [5]. On the other hand, Bauschke [3] used a Halpern type iterative scheme to find a common fixed point of a finite family of nonexpansive mappings in a Hilbert space. Kimura et. al. [8] generalized the result of Shioji and Takahashi [14] and studied strong convergence to a common fixed point of a finite family of nonexpansive mappings in a Banach space; see also [13, 10]. In 2007, Aoyama, Kimura, Takahashi and Toyoda [1] introduce the following iterative sequence: Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n x_n \tag{1.2}$$

for all $n \in \mathbb{N}$, where C is nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence of $[0, 1]$, and $\{T_n\}$ is a sequence of nonexpansive mappings. Then they prove that $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of $\{T_n\}$.

In 2003, Kikkawa and Takahashi [9], introduce an iterative scheme for finding a common fixed point of infinite nonexpansive mappings in a Hilbert space by using the hybrid method:

$$\begin{cases} y_n = W_n x_n, \\ C_n = \{z \in C; \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C; (x_n - z, x_1 - x_n) \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \end{cases} \tag{1.3}$$

for every $n \in \mathbb{N}$. Then we prove that $\{x_n\}$ converges strongly to $P_{F(U)}(x_1)$ where $F(U) = \cap_{i=1}^{\infty} F(T_i)$.

Algorithm 1.1 Let $T_1, T_2, ...$ be an infinite family of nonexpansive mappings of H into itself and let $\lambda_1, \lambda_2, ...$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i \in \mathbb{N}$, we define a mapping W_n of H into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \end{cases} \tag{1.4}$$

such mapping W_n is called the $W-$ *mapping generated by* $T_n, T_{n-1}, ..., T_1$ and $\lambda_n, \lambda_{n-1}, ..., \lambda_1$.

In this paper, we introduce the following iterative sequence as follows;

$$\begin{cases} x_1 = x \in C, \text{ arbitrarily;} \\ y_n = \alpha_n x_n + (1-\alpha_n) W_n x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \end{cases} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}$ are sequences of $[0,1]$, and W_n is a W-nonexpansive mappings. Then we prove that $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $T_n, n \in \mathbb{N}$.

2 Preliminaries

Throughout this paper, we assume that E is a reflexive Banach space, C is a nonempty closed convex subset of E. E^* is the dual space of E and $J : E \to 2^{E^*}$ is the noemalized mapping defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad x \in E$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued normalized duality mapping J by j.

Let $S = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. Recall that E is said to have a G$\widehat{a}$teaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t},$$

exists for each $x, y \in E$, and E is said to have a uniformly G$\widehat{a}$teaux differentiable norm if for each $y \in S$, the limit is attained uniformly for $x \in S$.

Recall that a Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \text{ implies } \frac{\|x+y\|}{2} < 1.$$

Lemma 2.1. *[2] Let E be a Banach space and J the normalized duality mapping. Then for all $x, y \in E$*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$ for all $j(x+y) \in J(x+y)$;

(ii) $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, j(x)\rangle$ for all $j(x) \in J(x)$.

Lemma 2.2. *[18] Let $\{a_n\}$ be a sequence of nonnegative real numbers, satisfying the property,*

$$a_{n+1} \le (1-\gamma_n)a_n + b_n, \;\; n \ge 0,$$

where $\{\gamma_n\} \subset (0,1)$, and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that:

i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

ii) $\limsup_{n\to\infty} \frac{b_n}{\gamma_n} \le 0$ or $\Sigma_{n=1}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. *[6] Let E be a real reflexive and strictly convex Banach with uniformly Gâteaux differentiable norm. Suppose C is a nonempty closed convex subset of E. Suppose that $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to fixed point of T such that p is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (f-I)p, j(x^* - p)\rangle \leq 0$$

for all $x^* \in F(T)$.

Let μ be a continuous linear functional on l^∞ and $s = (a_0, a_1, ...) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(s)$. We call μ a *Banach limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) \leq \mu_n(a_n)$ for all $(a_0, a_1, ...) \in l^\infty$. If μ is a Banach limit, then we have the following:

(i) for all $n \geq 1, a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$,
(ii) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r,
(iii) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, ...) \in l^\infty$.

Remark 2.4. *If $s = (a_0, a_1, ...) \in l^\infty$ with $a_n \to a$, then $\mu(s) = \mu_n(a_n) = a$ for any Banach limit μ by (iii). For more details on Banach limits, we refer readers to [16].*

Lemma 2.5. *[19] Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \subset l^\infty$ satisfying the condition $\mu_n(a_n) \leq a$ for all Banach limits. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

Lemma 2.6. *[15] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7. *[12] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^\infty F(T_n)$ is nonempty, and let $\lambda_1, \lambda_2, ...$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.7, one can define mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for every $x \in C$. Such a W is called the *$W-$mapping generated by* $T_1, T_2, ...$ and $\lambda_1, \lambda_2, ...$. Throughout this paper we will assume that $0 < \lambda_n \leq b < 1$ for every $n \geq 1$.

Lemma 2.8. *[12] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^\infty F(T_n)$ is nonempty, and let $\lambda_1, \lambda_2, ...$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, $F(W) = \cap_{n=1}^\infty F(T_n)$.*

First we give our implicit iterative scheme as follows: For each $k \geq 1$ define a mapping $S_k : H \to H$ by

$$S_k(x) = \frac{1}{k}f(x) + (1 - \frac{1}{k})Wx, \quad \forall k \geq 1, x \in H.$$

It is easy to see that for each $k \geq 1$, S_k is a contraction on C. Indeed, we note that

$$\begin{aligned}\|S_k(x) - S_k(y)\| &= \|\tfrac{1}{k}f(x) + (1 - \tfrac{1}{k})Wx - (\tfrac{1}{k}f(y) + (1 - \tfrac{1}{k})Wy)\| \\ &\leq \tfrac{1}{k}\|f(x) - f(y)\| + (1 - \tfrac{1}{k})\|Wx - Wy\| \\ &\leq \tfrac{1}{k}\alpha\|x - y\| + (1 - \tfrac{1}{k})\|x - y\| \\ &\leq (1 - \tfrac{1}{k}(1 - \alpha))\|x - y\|.\end{aligned}$$

By Banach contraction principle, there exist a unique fixed point $u_k \in C$ of S_k such that

$$u_k = \frac{1}{k}f(u_k) + (1 - \frac{1}{k})Wu_k, \quad \forall k \geq 1. \tag{2.1}$$

3 Main Results

In this section, we To obtain our result, we need some Lemmas.

Lemma 3.1. *Let E be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose C be a nonempty closed convex subset of E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $f \in \Pi_C$. Let $\{x_n\}$ be a sequence in (1.5) with $\lim_{n\to\infty} \beta_n = 0$, then*

$$\mu_n\langle f(p) - p, j(x_n - p)\rangle \leq 0,$$

for $p \in \cap_{n=1}^{\infty} F(T_n)$.

Proof. First we show that $\{x_n\}$ is bounded. Let $p \in \cap_{n=1}^{\infty} F(T_n)$. By the definition of y_n and x_n, we have

$$\begin{aligned}\|y_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n)W_n x_n - p\| \\ &= \|\alpha_n(x_n - p) + (1 - \alpha_n)(W_n x_n - p)\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|W_n x_n - p\| \\ &\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= \|x_n - p\|\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1 - \beta_n)y_n - p\| \\ &= \|\beta_n(f(x_n) - p) + (1 - \beta_n)(y_n - p)\| \\ &\leq \beta_n\|f(x_n) - p\| + (1 - \beta_n)\|y_n - p\| \\ &\leq \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|y_n - p\| \\ &\leq \beta_n\alpha\|x_n - p\| + \beta_n\|f(p) - p\| + (1 - \beta_n)\|x_n - p\| \\ &\leq (1 - \beta_n(1 - \alpha))\|x_n - p\| + \beta_n(1 - \alpha)\tfrac{\|f(p) - p\|}{(1 - \alpha)} \\ &\leq \max\{\|x_n - p\|, \tfrac{\|f(p) - p\|}{(1 - \alpha)}\}.\end{aligned}$$

By induction on n, we obtain

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{(1-\alpha)}\}$$

for every $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded. So are $\{y_n\}$, $\{f(x_n)\}$ and $\{W_n x_n\}$.

For each $k \in \mathbb{N}$, let u_k be a unique element of C such that

$$u_k = \frac{1}{k} f(u_k) + (1 - \frac{1}{k}) W u_k. \tag{3.1}$$

From Lemma 2.3, and Lemma 2.8, we obtain that

$$u_k \to p \in F(W) = \cap_{n=1}^{\infty} F(T_n) \quad as\ k \to \infty.$$

For every $n, k \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{n+1} - Wu_k\| &= \|\beta_n f(x_n) + (1-\beta_n) y_n - W u_k\| \\
&\leq \beta_n \|f(x_n) - Wu_k\| + (1-\beta_n)\|y_n - Wu_k\| \\
&\leq \beta_n \|f(x_n) - Wu_k\| + (1-\beta_n)\|\alpha_n x_n + (1-\alpha_n) W_n x_n - Wu_k\| \\
&\leq \beta_n \|f(x_n) - Wu_k\| + (1-\beta_n)\alpha_n \|x_n - Wu_k\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|W_n x_n - Wu_k\| \\
&\leq \beta_n \|f(x_n) - Wu_k\| + (1-\beta_n)\alpha_n \|x_n - Wu_k\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|W_n x_n - W_n u_k\| + (1-\beta_n)(1-\alpha_n)\|W_n u_k - W u_k\| \\
&\leq \beta_n \|f(x_n) - Wu_k\| + \alpha_n \|x_n - Wu_k\| + (1-\alpha_n)\|x_n - u_k\| \\
&\quad +(1-\beta_n)(1-\alpha_n)\|W_n u_k - Wu_k\| \\
&= \alpha_n \|x_n - Wu_k\| + (1-\alpha_n)\|x_n - u_k\| + b_n, \qquad (3.2)
\end{aligned}$$

where $b_n = \beta_n \|f(x_n) - Wu_k\| + (1-\beta_n)(1-\alpha_n)\|W_n u_k - Wu_k\|$. From $\lim_{n\to\infty} \beta_n = 0$ and Lemma 2.7, we have $\lim_{n\to\infty} b_n = 0$. From (3.2), we obtain

$$\begin{aligned}
\|x_{n+1} - Wu_k\|^2 &\leq (\alpha_n \|x_n - Wu_k\| + (1-\alpha_n)\|x_n - u_k\| + b_n)^2 \\
&= (\alpha_n \|x_n - Wu_k\| + (1-\alpha_n)\|x_n - u_k\|)^2 + 2(\alpha_n \|x_n - Wu_k\| \\
&\quad +(1-\alpha_n)\|x_n - u_k\|) b_n + b_n^2 \\
&= \alpha_n^2 \|x_n - Wu_k\|^2 + (1-\alpha_n)^2 \|x_n - u_k\|^2 \\
&\quad +2(1-\alpha_n)\alpha_n \|x_n - Wu_k\| \|x_n - u_k\| \\
&\quad +b_n(2(\alpha_n \|x_n - Wu_k\| + (1-\alpha_n)\|x_n - u_k\|) + b_n) \\
&\leq \alpha_n^2 \|x_n - Wu_k\|^2 + (1-\alpha_n)^2 \|x_n - u_k\|^2 + (1-\alpha_n)\alpha_n(\|x_n - Wu_k\|^2 \\
&\quad +\|x_n - u_k\|^2) + r_n \\
&= \alpha_n \|x_n - Wu_k\|^2 + (1-\alpha_n)\|x_n - u_k\|^2 + r_n \qquad (3.3)
\end{aligned}$$

where $r_n = b_n(2(\alpha_n \|x_n - Wu_k\| + (1-\alpha_n)\|x_n - u_k\|) + b_n) \to 0$ as $n \to \infty$.

For any Banach limit μ, from (3.3), we obtain

$$\mu_n \|x_n - Wu_k\|^2 = \mu_n \|x_{n+1} - Wu_k\|^2 \leq \mu_n \|x_n - u_k\|^2. \tag{3.4}$$

From (3.1), we have

$$u_k - x_n = \frac{1}{k}(f(u_k) - x_n) + (1 - \frac{1}{k})(Wu_k - x_n),$$

that is

$$(1-\frac{1}{k})(x_n - Wu_k) = (x_n - u_k) + \frac{1}{k}(f(u_k) - x_n).$$

It follows from Lemma 2.1 (ii), that

$$\begin{aligned}\|(1-\tfrac{1}{k})(x_n - Wu_k)\|^2 &= \|(x_n - u_k) + \tfrac{1}{k}(f(u_k) - x_n)\|^2\\ &\geq \|x_n - u_k\|^2 + \tfrac{2}{k}\langle f(u_k) - x_n, j(x_n - u_k)\rangle\\ &= \|x_n - u_k\|^2 + \tfrac{2}{k}\langle f(u_k) - u_k - (x_n - u_k), j(x_n - u_k)\rangle\\ &= \|x_n - u_k\|^2 + \tfrac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k)\rangle - \tfrac{2}{k}\langle x_n - u_k, j(x_n - u_k)\rangle\\ &= \|x_n - u_k\|^2 + \tfrac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k)\rangle - \tfrac{2}{k}\|x_n - u_k\|^2\end{aligned}$$

$$= (1-\frac{2}{k})\|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k)\rangle. \tag{3.5}$$

So by (3.4) and (3.5), we have

$$(1-\frac{1}{k})^2\|x_n - u_k\|^2 \geq (1-\frac{1}{k})^2\|x_n - Wu_k\|^2 \geq (1-\frac{2}{k})\|x_n - u_k\|^2 + \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k)\rangle$$

and hence

$$\frac{1}{k^2}\|x_n - u_k\|^2 \geq \frac{2}{k}\langle f(u_k) - u_k, j(x_n - u_k)\rangle.$$

This implies that

$$\frac{1}{2k}\mu_n\|x_n - u_k\|^2 \geq \mu_n\langle f(u_k) - u_k, j(x_n - u_k)\rangle.$$

Since $u_k \to p \in F(W)$ *as* $k \to \infty$, we get

$$\mu_n\langle f(p) - p, j(x_n - p)\rangle \leq 0. \tag{3.6}$$

This completes the proof. □

Theorem 3.2. *Let E be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose C be a nonempty closed convex subset of E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that the following conditions are satisfied:*

i) $\lim_{n\to\infty}\beta_n = 0, \Sigma_{n=1}^{\infty}\beta_n = \infty$ *and* $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$*;*

ii) $\Sigma_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$*;*

iii) $\lim_{n\to\infty}\alpha_n = 0$ *or* $\alpha_n \in [0, a)$, $\exists a \in (0, 1)$.

Then $\{x_n\}$ is defined by (1.5) converges strongly to a point in $\cap_{n=1}^{\infty} F(T_n)$.

Proof. By the proved of Lemma 3.1, we have $\{x_n\}$ is bounded. So are $\{y_n\}$, $\{f(x_n)\}$ and $\{W_n x_n\}$. From (1.5), we have

$$\|x_{n+1} - y_n\| = \beta_n\|f(x_n) - y_n\| \to 0 \quad as \quad n \to \infty. \tag{3.7}$$

Next, we show that

$$\|x_{n+1} - x_n\| \to 0 \quad as \quad n \to \infty. \tag{3.8}$$

From $\{x_n\}, \{y_n\}, \{f(x_n)\}$ and $\{W_n x_n\}$ are bounded, we let

$$M = \sup\{\|y_n - f(x_n)\| + \|W_{n+1}x_n - x_n\| + \|W_n x_n - W_{n+1}x_n\|\}.$$

Moreover, we note that

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &= \|\beta_{n+1}f(x_{n+1}) + (1-\beta_{n+1})y_{n+1} - (\beta_n f(x_n) + (1-\beta_n)y_n)\| \\ &= \|\beta_{n+1}f(x_{n+1}) + (1-\beta_{n+1})y_{n+1} - (1-\beta_{n+1})y_n + (1-\beta_{n+1})y_n \\ &\quad -\beta_n f(x_n) - (1-\beta_n)y_n - \beta_{n+1}f(x_n) + \beta_{n+1}f(x_n)\| \\ &= \|(1-\beta_{n+1})(y_{n+1} - y_n) + (\beta_n - \beta_{n+1})y_n + \beta_{n+1}(f(x_{n+1}) - f(x_n)) \\ &\quad +(\beta_{n+1} - \beta_n)f(x_n)\| \\ &= \|(1-\beta_{n+1})(y_{n+1} - y_n) + (\beta_n - \beta_{n+1})(y_n - f(x_n)) \\ &\quad +\beta_{n+1}(f(x_{n+1}) - f(x_n))\| \\ &\le (1-\beta_{n+1})\|y_{n+1} - y_n\| + |\beta_n - \beta_{n+1}|\|y_n - f(x_n)\| + \beta_{n+1}\alpha\|x_{n+1} - x_n\| \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|\alpha_{n+1}x_{n+1} + (1-\alpha_{n+1})W_{n+1}x_{n+1} - (\alpha_n x_n + (1-\alpha_n)W_n x_n)\| \\ &= \alpha_{n+1}x_{n+1} + (1-\alpha_{n+1})W_{n+1}x_{n+1} - (1-\alpha_{n+1})W_{n+1}x_n + (1-\alpha_{n+1})W_{n+1}x_n \\ &\quad -\alpha_n x_n - (1-\alpha_n)W_n x_n - (1-\alpha_n)W_{n+1}x_n + (1-\alpha_n)W_{n+1}x_n - \alpha_{n+1}x_n \\ &\quad +\alpha_{n+1}x_n\| \\ &= \|(1-\alpha_{n+1})(W_{n+1}x_{n+1} - W_{n+1}x_n) + (\alpha_{n+1} - \alpha_n)W_{n+1}x_n \\ &\quad +(\alpha_n - \alpha_{n+1})(W_n x_n - W_{n+1}x_n) + \alpha_{n+1}(x_{n+1} - x_n) + (\alpha_{n+1} - \alpha_n)x_n\| \\ &= \|(1-\alpha_{n+1})(W_{n+1}x_{n+1} - W_{n+1}x_n) + (\alpha_{n+1} - \alpha_n)(W_{n+1}x_n - x_n) \\ &\quad +(\alpha_n - \alpha_{n+1})(W_n x_n - W_{n+1}x_n) + \alpha_{n+1}(x_{n+1} - x_n)\| \\ &\le (1-\alpha_{n+1})\|W_{n+1}x_{n+1} - W_{n+1}x_n\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}x_n - x_n\| \\ &\quad +|\alpha_n - \alpha_{n+1}|\|W_n x_n - W_{n+1}x_n\| + \alpha_{n+1}\|x_{n+1} - x_n\| \\ &\le \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}x_n - x_n\| + |\alpha_n - \alpha_{n+1}|\|W_n x_n - W_{n+1}x_n\| \end{aligned} \tag{3.10}$$

for all $n \in \mathbb{N}$. Substituting (3.10) in (3.9), we have

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &= (1-\beta_{n+1})[\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}x_n - x_n\| \\ &\quad +|\alpha_n - \alpha_{n+1}|\|W_n x_n - W_{n+1}x_n\|] + |\beta_n - \beta_{n+1}|\|y_n - f(x_n)\| \\ &\quad +\beta_{n+1}\alpha\|x_{n+1} - x_n\| \\ &= (1-\beta_{n+1})\|x_{n+1} - x_n\| + (1-\beta_{n+1})|\alpha_{n+1} - \alpha_n|\|W_{n+1}x_n - x_n\| \\ &\quad +(1-\beta_{n+1})|\alpha_n - \alpha_{n+1}|\|W_n x_n - W_{n+1}x_n\| + |\beta_n - \beta_{n+1}|\|y_n - f(x_n)\| \\ &\quad +\beta_{n+1}\alpha\|x_{n+1} - x_n\| \\ &\le (1-\beta_{n+1})\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}x_n - x_n\| \\ &\quad +|\alpha_n - \alpha_{n+1}|\|W_n x_n - W_{n+1}x_n\| + |\beta_n - \beta_{n+1}|\|y_n - f(x_n)\| \\ &\quad +\beta_{n+1}\alpha\|x_{n+1} - x_n\| \\ &= (1-\beta_{n+1}(1-\alpha))\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|W_{n+1}x_n - x_n\| \\ &\quad +|\alpha_n - \alpha_{n+1}|\|W_n x_n - W_{n+1}x_n\| + |\beta_n - \beta_{n+1}|\|y_n - f(x_n)\| \\ &\le (1-\beta_{n+1}(1-\alpha))\|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n|M + |\beta_n - \beta_{n+1}|M. \end{aligned}$$

Put $b_n = 2|\alpha_{n+1} - \alpha_n|M + |\beta_n - \beta_{n+1}|M$. From (i) and (ii), we have

$$\Sigma_{n=1}^{\infty}|b_n| = 2\Sigma_{n=1}^{\infty}(|\alpha_{n+1} - \alpha_n|M) + \Sigma_{n=1}^{\infty}|\beta_n - \beta_{n+1}|M < \infty.$$

Therefore, it follows from Lemma 2.2, that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Next, we show that

$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle \le 0,$

where $p \in \cap_{n=1}^{\infty}F(T_n)$. Since $\lim_{n\to\infty}\beta_n = 0$, it follows from Lemma 3.1, we have

$$\mu_n\langle f(p)-p, j(x_n-p)\rangle \le 0, \tag{3.11}$$

where $p \in \cap_{n=1}^{\infty}F(T_n)$. From $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, thus

$$\lim_{n\to\infty}|\langle f(p)-p, j(x_{n+1}-p)\rangle - \langle f(p)-p, j(x_n-p)\rangle| = 0, \tag{3.12}$$

where $p \in \cap_{n=1}^{\infty}F(T_n)$. From (3.11), (3.14) and Lemma 2.5, we have

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle \le 0, \tag{3.13}$$

for $p \in \cap_{n=1}^{\infty}F(T_n)$. Finally, we show that $x_n \to p$ strongly and this concludes the proof. Indeed, using Lemma 2.1, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\beta_n f(x_n) + (1-\beta_n)y_n - p\|^2\\
&= \|\beta_n(f(x_n)-p) + (1-\beta_n)(y_n-p)\|^2\\
&\le (1-\beta_n)^2\|y_n-p\|^2 + 2\beta_n\langle f(x_n)-p, j(x_{n+1}-p)\rangle\\
&\le (1-\beta_n)^2\|x_n-p\|^2 + 2\beta_n\langle f(x_n)-f(p), j(x_{n+1}-p)\rangle\\
&\quad +2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\le (1-\beta_n)^2\|x_n-p\|^2+2\beta_n\alpha\|x_n-p\|\|x_{n+1}-p\|+2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\le (1-\beta_n)^2\|x_n-p\|^2 + \beta_n\alpha(\|x_n-p\|^2 + \|x_{n+1}-p\|^2)\\
&\quad +2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\le (1-2\beta_n+\beta_n^2+\beta_n\alpha)\|x_n-p\|^2 + \beta_n\alpha\|x_{n+1}-p\|^2\\
&\quad +2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle.
\end{aligned}$$

It follows that

$$(1-\beta_n\alpha)\|x_{n+1}-p\|^2 \le (1-\beta_n(2-\alpha)+\beta_n^2)\|x_n-p\|^2 + 2\beta_n\langle f(p)-p, j(x_{n+1}-p)\rangle,$$

that is

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \tfrac{(1-\beta_n(2-\alpha))}{1-\beta_n\alpha}\|x_n-p\|^2 + \tfrac{\beta_n^2}{1-\beta_n\alpha}\|x_n-p\|^2 + \tfrac{2\beta_n}{1-\beta_n\alpha}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&= [1-\tfrac{2(1-\alpha)\beta_n}{1-\beta_n\alpha}]\|x_n-p\|^2 + \tfrac{2(1-\alpha)\beta_n}{1-\beta_n\alpha}[\tfrac{1}{1-\alpha}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad +\tfrac{\beta_n}{2(1-\alpha)}M_1]
\end{aligned}$$

for all $n \in \mathbb{N}$, where $M_1 \ge \|x_n-p\|^2 \ge 0$, $n \ge 1$. Now, we apply Lemma 2.2 and use (3.13), we have $\lim_{n\to\infty}\|x_n-p\|^2 = 0$. Consequently, we deduce that $\{x_n\}$ converges strongly to fixed point $p \in \cap_{n=1}^{\infty}F(T_n)$. This completes the proof. □

Theorem 3.3. *Let E be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty}F(T_n)$ is nonempty and $f \in \Pi_C$ with $\alpha \in (0,1)$. Suppose that the following condition are satisfying*

i) $\lim_{n\to\infty}\beta_n = 0$, *and* $\Sigma_{n=1}^{\infty}\beta_n = \infty$;

ii) $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$.

Then $\{x_n\}$ is defined by (1.5) converges strongly to a point in $\cap_{n=1}^{\infty}F(T_n)$.

Proof. By using the same arguments and techniques as those of Lemma 3.1, we note that $\{x_n\}$ is bounded, and so are the $\{W_nx_n\}, \{y_n\}$ and $\{f(x_n)\}$. Setting $\gamma_n = (1-\beta_n)\alpha_n, \forall n \geq 1$, it follows from $\lim_{n\to\infty}\beta_n = 0$ and (ii) that

$$0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1. \tag{3.14}$$

Define

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n. \tag{3.15}$$

We observe that

$$\begin{aligned}
z_{n+1} - z_n &= \tfrac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \tfrac{x_{n+1}-\gamma_n x_n}{1-\gamma_n} \\
&= \tfrac{\beta_{n+1}f(x_{n+1})+(1-\beta_{n+1})y_{n+1}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \tfrac{\beta_n f(x_n)+(1-\beta_n)y_n-\gamma_n x_n}{1-\gamma_n} \\
&= (\tfrac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \tfrac{\beta_n f(x_n)}{1-\gamma_n}) - \tfrac{(1-\beta_n)[\alpha_n x_n+(1-\alpha_n)W_nx_n]-\gamma_n x_n}{1-\gamma_n} \\
&\quad + \tfrac{(1-\beta_{n+1})[\alpha_{n+1}x_{n+1}+(1-\alpha_{n+1})W_{n+1}x_{n+1}]-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&= (\tfrac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \tfrac{\beta_n f(x_n)}{1-\gamma_n}) - \tfrac{\gamma_n x_n}{1-\gamma_n} - \tfrac{(1-\beta_n)(1-\alpha_n)W_nx_n}{1-\gamma_n} + \tfrac{\gamma_n x_n}{1-\gamma_n} + \tfrac{\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&\quad + \tfrac{(1-\beta_{n+1})(1-\alpha_{n+1})W_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \tfrac{\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&= (\tfrac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \tfrac{\beta_n f(x_n)}{1-\gamma_n}) - \tfrac{(1-\beta_n)(1-\alpha_n)W_nx_n}{1-\gamma_n} + \tfrac{(1-\beta_{n+1})(1-\alpha_{n+1})W_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&= (\tfrac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \tfrac{\beta_n f(x_n)}{1-\gamma_n}) - \tfrac{(1-\gamma_n)W_nx_n}{1-\gamma_n} + \tfrac{W_nx_n}{1-\gamma_n} - \tfrac{(1-\beta_n)W_nx_n}{1-\gamma_n} + \tfrac{(1-\gamma_{n+1})W_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&\quad - \tfrac{W_{n+1}x_{n+1}}{1-\gamma_{n+1}} + \tfrac{(1-\beta_{n+1})W_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\
&= (\tfrac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \tfrac{\beta_n f(x_n)}{1-\gamma_n}) + (W_{n+1}x_{n+1} - W_nx_n) + \tfrac{\beta_n W_nx_n}{1-\gamma_n} - \tfrac{\beta_{n+1}W_{n+1}x_{n+1}}{1-\gamma_{n+1}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \tfrac{\beta_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\|) + \tfrac{\beta_n}{1-\gamma_n}(\|f(x_n)\| + \|W_nx_n\|) \\
&\quad + \|W_{n+1}x_{n+1} - W_nx_n\| \\
&\leq \tfrac{\beta_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\|) + \tfrac{\beta_n}{1-\gamma_n}(\|f(x_n)\| + \|W_nx_n\|) \\
&\quad + \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\| \\
&\leq \tfrac{\beta_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\|) + \tfrac{\beta_n}{1-\gamma_n}(\|f(x_n)\| + \|W_nx_n\|) \\
&\quad + \|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_nx_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \tfrac{\beta_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\| + \|W_{n+1}x_{n+1}\|) \\
&\quad + \frac{\beta_n}{1-\gamma_n}(\|f(x_n)\| + \|W_nx_n\|) + \|W_{n+1}x_n - W_nx_n\|. \qquad (3.16)
\end{aligned}$$

From (1.4), since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &= \|\lambda_1T_1U_{n+1,2}x_n - \lambda_1T_1U_{n,2}x_n\| \\
&\leq \lambda_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \|\lambda_2T_2U_{n+1,3}x_n - \lambda_2T_2U_{n,3}x_n\| \\
&\leq \lambda_1\lambda_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\
&\leq \cdots \\
&\leq \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&\leq M\prod_{i=1}^{n}\lambda_i, \qquad (3.17)
\end{aligned}$$

where $M \geq 0$ is constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M$ for all $n \geq 0$.

Substituting (3.17) into (3.16), we have

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \leq \frac{\beta_{n+1}}{1-\gamma_{n+1}}(\|f(x_{n+1})\|+\|W_{n+1}x_{n+1}\|)+\frac{\beta_n}{1-\gamma_n}(\|f(x_n)\| +\|W_nx_n\|) + M\prod_{i=1}^{n}\lambda_i,$$

which implies that (noting that (i) and $0 < \lambda_i \leq b < 1, \forall i \geq 1$)

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence by Lemma 2.6, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Consequently

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = (1 - \gamma_n) \lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Argument of the proved in Theorem 3.2, we have

$$\limsup_{n\to\infty}\langle f(p) - p, j(x_n - p)\rangle \leq 0,$$

for $p \in \cap_{n=1}^{\infty}F(T_n)$. By using the same arguments and techniques as those Theorem 3.2, we have $\{x_n\}$ converges strongly to a point $p \in \cap_{n=1}^{\infty}F(T_n)$. This completes the proof. □

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