Thai Journal of Mathematics
Volume 19 Number 1 (2021)
Pages 93–112

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Coincidence Point Results for Generalized (ψ, θ, ϕ) − Contraction on Partially Ordered Metric Spaces

Bhavana Deshpande[1], Vishnu Narayan Mishra[2,*], Amrish Handa[3] and Lakshmi Narayan Mishra[4]

[1] *Department of Mathematics, Govt. P.G. Arts and Science College, Ratlam (M. P.), India*
e-mail : bhavnadeshpande@yahoo.com

[2] *Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur-484 887, India*
e-mail : vishnunarayanmishra@gmail.com

[3] *P. G. Arts and Science College, Ratlam (M. P.), India*
e-mail : amrishhanda83@gmail.com

[4] *Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India*
e-mail : lakshminarayanmishra04@gmail.com, lakshminarayan.mishra@vit.ac.in

Abstract We establish coincidence point theorem for g-non-decreasing mappings under generalized (ψ, θ, φ)-contraction on partially ordered metric spaces. With the help of the obtain result, we indicate the formation of a coupled coincidence point theorem of generalized compatible pair of mappings $F, G : X^2 \to X$. We apply our result to obtain the solution of integral equation and also give an example to show the degree of validity of our hypothesis. Our results modify, improve, sharpen, enrich and generalize various known results.

MSC: 47H10; 54H25

Keywords: coincidence point; coupled coincidence point; partially ordered metric space; generalized compatibility; mixed monotone mapping; commuting mapping

Submission date: 20.10.2017 / Acceptance date: 06.10.2019

1. Introduction and Preliminaries

In the sequel, X is a non-empty set. Given $n \in U^*(F)2115$ where $n \geq 2$, let X^n be the nth Cartesian product $X \times X \times ... \times X$ (n times). Let $g : X \to X$ be a mapping. For simplicity, we denote $g(x)$ by gx where $x \in X$.

Coupled fixed point theorems and coupled coincidence point theorems have appeared prominently in recent literature. Although the concept of coupled fixed points was introduced by Guo and Lakshmikantham [1].

---

*Corresponding author.

Published by The Mathematical Association of Thailand.

Definition 1.1. [1]. Let $F : X^2 \to X$ be a given mapping. An element $(x,\ y) \in X^2$ is called a coupled fixed point of F if

$$F(x,\, y) = x \ \text{ and } F(y,\, x) = y.$$

Later on, Gnana-Bhaskar and Lakshmikantham [2] presented some coupled fixed point theorems on partially ordered metric spaces, by defining the notion of mixed monotone property. Starting with the work of Gnana-Bhaskar and Lakshmikantham [2], where they established a coupled contraction principle, this line of research has developed rapidly in partially ordered metric spaces. References [3–18] are some examples of these works.

Definition 1.2. [2]. Let $(X,\ \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ be a given mapping. We say that F has the mixed monotone property if for all $x,\ y \in X$, we have

$$x_1,\, x_2 \in X,\, x_1 \preceq x_2 \Longrightarrow F(x_1,\, y) \preceq F(x_2,\, y),$$

and

$$y_1,\, y_2 \in X,\, y_1 \preceq y_2 \Longrightarrow F(x,\, y_1) \succeq F(x,\, y_2).$$

Thenafter, Lakshmikantham and Ciric [14] obtained coupled fixed/coincidence point results by extending the notion of mixed monotone property to mixed g -monotone property.

Definition 1.3. [14]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of the mappings F and g if

$$F(x,\, y) = gx \ \text{ and } F(y,\, x) = gy.$$

Definition 1.4. [14]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x,\ y) \in X^2$ is called a common coupled fixed point of the mappings F and g if

$$x = F(x,\, y) = gx \ \text{ and } y = F(y,\, x) = gy.$$

Definition 1.5. [14]. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if

$$gF(x,\, y) = F(gx,\, gy), \ \text{ for all } (x,\, y) \in X^2.$$

Definition 1.6. [14]. Let $(X,\ \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the mixed $g-$ monotone property if for all $x,\ y \in X,$ we have

$$x_1,\, x_2 \in X,\, gx_1 \preceq gx_2 \Longrightarrow F(x_1,\, y) \preceq F(x_2,\, y),$$

and

$$y_1,\, y_2 \in X,\, gy_1 \preceq gy_2 \Longrightarrow F(x,\, y_1) \succeq F(x,\, y_2).$$

If g is the identity mapping on X, then X, satisfies the mixed monotone property.

Subsequently, Choudhury and Kundu [6] improve the results of Lakshmikantham and Ciric [14], by introducing the concept of compatibility in coupled coincidence point context. Following it, various authors established coupled fixed/coincidence point results on partially ordered metric spaces.

Definition 1.7. [6]. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(gF(x_n,\, y_n),\, F(gx_n,\, gy_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n,\, x_n),\, F(gy_n,\, gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} F(x_n,\, y_n) &= \lim_{n\to\infty} gx_n = x,\\ \lim_{n\to\infty} F(y_n,\, x_n) &= \lim_{n\to\infty} gy_n = y, \text{ for some } x,\, y \in X.\end{aligned}$$

These studies applied to initial value problems defined by differential or integral equations. Hussain et al. [12] obtained some coupled coincidence point results with the help of newly defined concept of generalized compatibility.

Definition 1.8. [12]. Suppose that $F, G : X^2 \to X$ are two mappings. F is said to be G -increasing with respect to $\preceq$ if for all $x,\ y,\ u,\ v \in X$, with $G(x,\ y) \preceq G(u,\ v)$ we have $F(x,\ y) \preceq F(u,\ v)$.

Definition 1.9. [12]. Let $F,\ G : X^2 \to X$ be two mappings. We say that the pair $\{F, G\}$ is commuting if

$$F(G(x,\, y),\, G(y,\, x)) = G(F(x,\, y),\, F(y,\, x)), \text{ for all } x,\, y \in X.$$

Definition 1.10. [12]. Suppose that $F,\ G : X^2 \to X$ are two mappings. An element $(x,\ y) \in X^2$ is called a coupled coincidence point of mappings F and G if

$$F(x,\, y) = G(x,\, y) \text{ and } F(y,\, x) = G(y,\, x).$$

Definition 1.11. [12]. Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ and $g : X \to X$ are two mappings. We say that F is g -increasing with respect to $\preceq$ if for any $x,\ y \in X$,

$$gx_1 \preceq gx_2 \text{ implies } F(x_1,\, y) \preceq F(x_2,\, y),$$

and

$$gy_1 \preceq gy_2 \text{ implies } F(x,\, y_1) \preceq F(x,\, y_2).$$

Definition 1.12. [12]. Let $(X,\ \preceq)$ be a partially ordered set, $F : X^2 \to X$ be a mapping. We say that F is increasing with respect to $\preceq$ if for any $x,\ y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1,\ y) \preceq F(x_2,\ y),$$

and

$$y_1 \preceq y_2 \text{ implies } F(x,\ y_1) \preceq F(x,\ y_2).$$

Definition 1.13. [12]. Let $F,\ G : X^2 \to X$ are two mappings. We say that the pair $\{F,\ G\}$ is generalized compatible if

$$\begin{aligned}\lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0,\\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0,\end{aligned}$$

whenever (x_n) and (y_n) are sequences in X such that

$$\begin{aligned}\lim_{n\to\infty} G(x_n,\ y_n) &= \lim_{n\to\infty} F(x_n,\ y_n) = x,\\ \lim_{n\to\infty} G(y_n,\ x_n) &= \lim_{n\to\infty} F(y_n,\ x_n) = y,\ \text{for some } x,\ y\in X.\end{aligned}$$

Obviously, a commuting pair is a generalized compatible but not conversely in general.

Erhan et al. [9], declared that the results established in Hussain et al. [12] can be deduce from the coincidence point results in the existing literature.

In [9], Erhan et al. recalled the following basic definitions:

Definition 1.14. [19, 20]. A coincidence point of two mappings $T,\ g: X\to X$ is a point $x\in X$ such that $Tx = gx$.

Definition 1.15. [9]. A partially ordered metric space $(X,\ d,\ \preceq)$ is a metric space $(X,\ d)$ provided with a partial order $\preceq$.

Definition 1.16. [2, 12]. A partially ordered metric space $(X,\ d,\ \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $\{x_n\}\subseteq X$ such that $\{x_n\}\to x$ and $x_n\preceq x_{n+1}$ (respectively, $x_n\succeq x_{n+1}$) for all $n\geq 0$, we have that $x_n\preceq x$ (respectively, $x_n\succeq x$) for all $n\geq 0$. $(X,\ d,\ \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.

Definition 1.17. [9]. Let$(X,\ \preceq)$ be a partially ordered set and let $T,\ g: X\to X$ be two mappings. We say that T is $(g,\ \preceq)$ -non-decreasing if $Tx\preceq Ty$ for all $x,\ y\in X$ such that $gx\preceq gy$. If g is the identity mapping on X, we say that T is $\preceq$ -non-decreasing.

Remark 1.18. [9]. If T is $(g,\ \preceq)$ -non-decreasing and $gx = gy$, then $Tx = Ty$. It follows that

$$gx = gy \Rightarrow \left\{\begin{array}{c} gx\preceq gy,\\ gy\preceq gx\end{array}\right\} \Rightarrow \left\{\begin{array}{c} Tx\preceq Ty,\\ Ty\preceq Tx\end{array}\right\} \Rightarrow Tx = Ty.$$

Definition 1.19. [9]. Let $(X,\ \preceq)$ be a partially ordered set and endow the product space X^2 with the following partial order:

$$(u,\ v)\sqsubseteq (x,\ y) \Leftrightarrow x\succeq u \text{ and } y\preceq v,\ \text{for all } (u,\ v),\ (x,\ y)\in X^2.$$

Definition 1.20. [6, 16, 21, 22]. Let $(X,\ d,\ \preceq)$ be a partially ordered metric space. Two mappings $T,\ g: X\to X$ are said to be O-compatible if

$$\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0,$$

provided that $\{x_n\}$ is a sequence in X such that $\{gx_n\}$ is $\preceq$ -monotone, that is, it is either non-increasing or non-decreasing with respect to $\preceq$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n \in X.$$

Recently Samet et al. [23] announced that most of the coupled fixed point theorems for single-valued mappings on partially ordered metric spaces can be derived from well-known fixed point theorems. Some of our basic references are [12, 22–26].

In this paper, we establish coincidence point theorem for g -non-decreasing mappings under generalized $(\psi,\ \theta,\ \varphi)$ -contraction on partially ordered metric spaces. With the help of the obtain result, we indicate the formation of a coupled coincidence point theorem of generalized compatible pair of mappings $F,\ G : X^2 \to X$. We apply our result to obtain the solution of integral equation and also give an example to show the degree of validity of our hypothesis. We modify, improve, sharpen, enrich and generalize the results of Alotaibi and Alsulami [4], Alsulami [5], Gnana-Bhaskar and Lakshmikantham [2], Harjani et al. [10], Harjani and Sadarangani [11], Lakshmikantham and Ciric [14], Luong and Thuan [15], Nieto and Rodriguez-Lopez [27], Ran and Reurings [28], Razani and Parvaneh [17] and many other famous results in the literature.

2. Main Results

Lemma 2.1. *Let $(X,\ d)$ be a metric space. Suppose $Y = X^2$ and define $\delta : Y \times Y \to [0, +\infty)$ by*

$$\delta((x,\ y),\ (u,\ v)) = \max\{d(x,\ u),\ d(y,\ v)\},\ \textit{for all } (x,\ y),\ (u,\ v) \in Y.$$

Then δ is metric on Y and $(X,\ d)$ is complete if and only if $(Y,\ \delta)$ is complete.

Definition 2.2. An altering distance function is a function $\psi : [0,\ +\infty) \to [0,\ +\infty)$ which satisfies the following conditions:

(i_ψ) ψ is continuous and non-decreasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

Theorem 2.3. *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and let $T,\ g : X \to X$ be two mappings such that the following properties are fulfilled:*

(i) $T(X) \subseteq g(X)$,

(ii) T is $(g,\ \preceq)$ -non-decreasing,

(iii) there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$,

(iv) there exist an altering distance function ψ, an upper semi-continuous function $\theta : [0,\ +\infty) \to [0,\ +\infty)$ and a lower semi-continuous function $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ such that

$$\psi\left(d(Tx,\ Ty)\right) \le \theta\left(d(gx,\ gy)\right) - \varphi\left(d(gx,\ gy)\right),$$

for all $x,\ y \in X$ with $gx \preceq gy$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Also assume that, at least, one of the following conditions holds.

(a) $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is O-compatible,

(b) $(X,\ d)$ is complete, T and g are continuous and commuting,

(c) $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular,

(d) $(X,\ d)$ is complete, $g(X)$ is closed and $(X,\ d,\ \preceq)$ is non-decreasing-regular,

(e) $(X,\ d)$ is complete, g is continuous, the pair $(T,\ g)$ is O-compatible and $(X,\ d,\ \preceq)$ is non-decreasing-regular.

Then T and g have, at least, a coincidence point.

Proof. We divide the proof into four steps:

Step 1. We claim that there exists a sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is $\preceq$ -non-decreasing and $gx_{n+1} = Tx_n,$ for all $n \geq 0$. Starting from $x_0 \in X$ given in (iii) and taking into account that $Tx_0 \in T(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $Tx_0 = gx_1$. Then $gx_0 \preceq Tx_0 = gx_1$. Since T is $(g, \preceq)$ -non-decreasing, $Tx_0 \preceq Tx_1$. Now $Tx_1 \in T(X) \subseteq g(X)$, so there exists $x_2 \in X$ such that $Tx_1 = gx_2$. Then $gx_1 = Tx_0 \preceq Tx_1 = gx_2$. Since T is $(g, \preceq)$ -non-decreasing, $Tx_1 \preceq Tx_2$. Repeating this argument, there exists a sequence $\{x_n\}_{n\geq 0}$ such that $\{gx_n\}$ is $\preceq$ -non-decreasing, $gx_{n+1} = Tx_n \preceq Tx_{n+1} = gx_{n+2}$ and

$$gx_{n+1} = Tx_n \text{ for all } n \geq 0. \tag{2.1}$$

Step 2. We claim that $\{d(gx_n, gx_{n+1})\} \to 0$. Now, by contractive condition (iv) and (i_ψ), we have

$$\begin{aligned} \psi\left(d(gx_{n+1}, gx_{n+2})\right) &= \psi\left(d(Tx_n, Tx_{n+1})\right) \\ &\leq \theta\left(d(gx_n, gx_{n+1})\right) - \varphi\left(d(gx_n, gx_{n+1})\right), \end{aligned} \tag{2.2}$$

but we have $\psi\left(d(gx_n, gx_{n+1})\right) - \theta\left(d(gx_n, gx_{n+1})\right) + \varphi\left(d(gx_n, gx_{n+1})\right) > 0$. Then

$$\frac{\psi\left(d(gx_{n+1}, gx_{n+2})\right)}{\psi\left(d(gx_n, gx_{n+1})\right)} \leq \frac{\theta\left(d(gx_n, gx_{n+1})\right) - \varphi\left(d(gx_n, gx_{n+1})\right)}{\psi\left(d(gx_n, gx_{n+1})\right)} < 1.$$

Thus

$$\psi\left(d(gx_{n+1}, gx_{n+2})\right) < \psi\left(d(gx_n, gx_{n+1})\right). \tag{2.3}$$

Since ψ is non-decreasing, therefore

$$d(gx_{n+1}, gx_{n+2}) < d(gx_n, gx_{n+1}). \tag{2.4}$$

This shows that the sequence $\{\delta_n\}_{n=0}^{\infty}$ defined by $\delta_n = d(gx_n, gx_{n+1})$, is a decreasing sequence of positive numbers. Then there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = \delta. \tag{2.5}$$

We shall prove that $\delta = 0$. Suppose to the contrary that $\delta > 0$. Taking $n \to \infty$ in (2.2), by using the property of ψ, θ, φ and (2.5), we obtain

$$\psi(\delta) \leq \theta(\delta) - \varphi(\delta),$$

which implies that

$$\psi(\delta) - \theta(\delta) + \varphi(\delta) \leq 0,$$

which is a contradiction. Thus, by (2.5), we get

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{2.6}$$

Step 3. We claim that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, and

$$d(gx_{n(k)}, gx_{m(k)}) \geq \varepsilon, \text{ for } n(k) > m(k) > k.$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(gx_{n(k)-1}, gx_{m(k)}) < \varepsilon.$$

Now, by triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq d(gx_{n(k)},\ gx_{m(k)}) \\ &\leq d(gx_{n(k)},\ gx_{n(k)-1}) + d(gx_{n(k)-1},\ gx_{m(k)}) \\ &\leq d(gx_{n(k)},\ gx_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using (2.6), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)}) = \varepsilon. \tag{2.7}$$

By the triangle inequality, we have

$$\begin{aligned} & d(gx_{n(k)+1},\ gx_{m(k)+1}) \\ \leq\ & d(gx_{n(k)+1},\ gx_{n(k)}) + d(gx_{n(k)},\ gx_{m(k)}) + d(gx_{m(k)},\ gx_{m(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (2.6) and (2.7), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1},\ gx_{m(k)+1}) = \varepsilon. \tag{2.8}$$

As $n(k) > m(k)$, $gx_{n(k)} \succeq gx_{m(k)}$, by using contractive condition (iv), we have

$$\begin{aligned} & \psi\left(d(gx_{n(k)+1},\ gx_{m(k)+1})\right) \\ =\ & \psi\left(d(Tx_{n(k)},\ Tx_{m(k)})\right) \\ \leq\ & \theta\left(d(gx_{n(k)},\ gx_{m(k)})\right) - \varphi\left(d(gx_{n(k)},\ gx_{m(k)})\right). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (2.7), (2.8), we have

$$\psi(\varepsilon) \leq \theta(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction due to $\varepsilon > 0$. This shows that $\{gx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X.

Step 4. We claim that T and g have a coincidence point distinguishing between cases $(a) - (e)$. Suppose now that (a) holds, that is, $(X,\ d)$ is complete, T and g are continuous and the pair $(T,\ g)$ is O-compatible. Since $(X,\ d)$ is complete, therefore there exists $z \in X$ such that $\{gx_n\} \to z$. Now $Tx_n = gx_{n+1}$ for all $n \geq 0$, we also have that $\{Tx_n\} \to z$. As T and g are continuous, then $\{Tgx_n\} \to Tz$ and $\{ggx_n\} \to gz$. Taking into account that the pair $(T,\ g)$ is O-compatible,we deduce that $\lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$. In such a case, we conclude that $d(gz,\ Tz) = \lim_{n\to\infty} d(ggx_{n+1},\ Tgx_n) = \lim_{n\to\infty} d(gTx_n,\ Tgx_n) = 0$, that is, z is a coincidence point of T and g. Suppose now that (b) holds, that is, $(X,\ d)$ is complete, T and g are continuous and commuting. It is obvious because (b) implies (a). Suppose now that (c) holds, that is, $(g(X),\ d)$ is complete and $(X,\ d,\ \preceq)$ is non-decreasing-regular. As $\{gx_n\}$ is a Cauchy sequence in the complete space $(g(X),\ d)$, so there exist $y \in g(X)$ such that $\{gx_n\} \to y$. Let $z \in X$ be any point such that $y = gz$. In this case $\{gx_n\} \to gz$. Indeed, as $(X,\ d,\ \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to gz, we deduce that $gx_n \preceq gz$ for all $n \geq 0$. Applying the contractive condition (iv),

$$\psi\left(d(gx_{n+1},\ Tz)\right) = \psi\left(d(Tx_n,\ Tz)\right) \leq \theta\left(d(gx_n,\ gz)\right) - \varphi\left(d(gx_n,\ gz)\right).$$

Taking $n \to \infty$ in the above inequality, we get $d(gz,\ Tz) = 0$, that is, z is a coincidence point of T and g. Suppose now that (d) holds, that is, $(X,\ d)$ is complete, $g(X)$ is closed and $(X,\ d,\ \preceq)$ is non-decreasing-regular. It follows from the fact that a closed subset of a complete metric space is also complete. Then, $(g(X),\ d)$ is complete and $(X,$

$d, \preceq)$ is non-decreasing-regular. Thus (c) is applicable.Suppose now that (e) holds, that is, (X, d) is complete, g is continuous, the pair (T, g) is O-compatible and $(X, d, \preceq)$ is non-decreasing-regular. As (X, d) is complete, so there exists $z \in X$ such that $\{gx_n\} \to z$. Since $Tx_n = gx_{n+1}$ for all $n \geq 0$, we also have that $\{Tx_n\} \to z$. As g is continuous, then $\{ggx_n\} \to gz$. Furthermore, since the pair (T, g) is O-compatible, we have $\lim_{n\to\infty} d(ggx_{n+1}, Tgx_n) = \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. As $\{ggx_n\} \to gz$ the previous property means that $\{Tgx_n\} \to gz$. Indeed, as $(X, d, \preceq)$ is non-decreasing-regular and $\{gx_n\}$ is $\preceq$-non-decreasing and converging to z, we deduce that $gx_n \preceq z$ for all $n \geq 0$. Applying the contractive condition (iv), we get

$$\psi\left(d(ggx_{n+1}, Tz)\right) = \psi\left(d(Tgx_n, Tz)\right) \leq \theta\left(d(ggx_n, gz)\right) - \varphi\left(d(ggx_n, gz)\right).$$

Taking $n \to \infty$ in the above inequality, we get $d(gz, Tz) = 0$, that is, z is a coincidence point of T and g. ■

Now, we find the two dimensional version of Theorem 2.3. For this, we shall consider the partially ordered metric space $(X^2, \delta, \sqsubseteq)$, where δ was defined in Lemma 2.1 and $\sqsubseteq$ was introduced in Definition 1.19. We define the mappings $T_F, T_G : X^2 \to X^2$, for all $(x, y) \in X^2$, by

$$T_F(x, y) = (F(x, y), F(y, x)) \text{ and } T_G(x, y) = (G(x, y), G(y, x)).$$

Under these conditions, the following properties hold:

Lemma 2.4. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F, G : X^2 \to X$ be two mappings. Then*

(1) (X, d) is complete if and only if (X^2, δ) is complete.

(2) If $(X, d, \preceq)$ is regular, then $(X^2, \delta, \sqsubseteq)$ is also regular.

(3) If F is d-continuous, then T_F is δ-continuous.

(4) If F is G-increasing with respect to $\preceq$, then T_F is $(T_G, \sqsubseteq)$-non-decreasing.

(5) If there exist two elements $x_0, y_0 \in X$ with $G(x_0, y_0) \preceq F(x_0, y_0)$ and $G(y_0, x_0) \succeq F(y_0, x_0)$, then there exists a point $(x_0, y_0) \in X^2$ such that $T_G(x_0, y_0) \sqsubseteq T_F(x_0, y_0)$.

(6) For any $x, y \in X$, there exist $u, v \in X$ such that $F(x, y) = G(u, v)$ and $F(y, x) = G(v, u)$, then $T_F(X^2) \subseteq T_G(X^2)$.

(7) There exist an altering distance function ψ, an upper semi-continuous function $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$\begin{aligned} & \psi\left(d(F(x, y), F(u, v))\right) \\ \leq\ & \theta\left(\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}\right) \\ & -\varphi\left(\max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}\right), \end{aligned} \tag{2.9}$$

for all $x, y, u, v \in X$, with $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$, then

$$\begin{aligned} & \psi\left(\delta(T_F(x, y), T_F(u, v))\right) \\ \leq\ & \theta\left(\delta(T_G(x, y), T_G(u, v))\right) - \varphi\left(\delta(T_G(x, y), T_G(u, v))\right), \end{aligned}$$

for all $(x, y), (u, v) \in X^2$, where $T_G(x, y) \sqsubseteq T_G(u, v)$.

(8) If the pair $\{F, G\}$ is generalized compatible, then the mappings T_F and T_G are O-compatible in $(X^2, \delta, \sqsubseteq)$.

(9) A point $(x, y) \in X^2$ is a coupled coincidence point of F and G if and only if it is a coincidence point of T_F and T_G.

Proof. Item (1) follows from Lemma 2.1 and items (2), (3), (5), (6) and (9) are obvious.

(4) Assume that F is G -increasing with respect to $\preceq$ and let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Then $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$. Since F is G -increasing with respect to $\preceq$, we deduce that $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. Therefore $T_F(x, y) \sqsubseteq T_F(u, v)$ and this means that T_F is $(T_G, \sqsubseteq)$ -non-decreasing.

(7) Let $(x, y), (u, v) \in X^2$ be such that $T_G(x, y) \sqsubseteq T_G(u, v)$. Therefore $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$ and by using (2.9), we have

$$\begin{aligned} & \psi\left(d(F(x, y), F(u, v))\right) \\ \leq\ & \theta\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right) \\ & -\varphi\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right). \end{aligned}$$

Furthermore $G(y, x) \succeq G(v, u)$ and $G(x, y) \preceq G(u, v)$, the contractive condition (2.9) also guarantees that

$$\begin{aligned} & \psi\left(d(F(y, x), F(v, u))\right) \\ \leq\ & \theta\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right) \\ & -\varphi\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right). \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\left\{\psi\left(d(F(x, y), F(u, v))\right), \psi\left(d(F(y, x), F(v, u))\right)\right\} \\ \leq\ & \theta\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right) \\ & -\varphi\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right). \end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned} & \psi\left(\max\left\{d(F(x, y), F(u, v)), d(F(y, x), F(v, u))\right\}\right) \\ \leq\ & \theta\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right) \\ & -\varphi\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right). \end{aligned} \tag{2.10}$$

Thus, by using (2.10), we get

$$\begin{aligned} & \psi\left(\delta(T_F(x, y), T_F(u, v))\right) \\ =\ & \psi\left(\delta\left((F(x, y), F(y, x)), (F(u, v), F(v, u))\right)\right) \\ =\ & \psi\left(\max\left\{d(F(x, y), F(u, v)), d(F(y, x), F(v, u))\right\}\right) \\ \leq\ & \theta\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right) \\ & -\varphi\left(\max\left\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\right\}\right) \\ \leq\ & \theta\left(\delta(T_G(x, y), T_G(u, v))\right) - \varphi\left(\delta(T_G(x, y), T_G(u, v))\right). \end{aligned}$$

(8) Let $\{(x_n, y_n)\} \subseteq X^2$ be any sequence such that $T_F(x_n, y_n) \xrightarrow{\delta} (x, y)$ and $T_G(x_n, y_n) \xrightarrow{\delta} (x, y)$. Therefore,

$$\begin{aligned} & (F(x_n, y_n), F(y_n, x_n)) \xrightarrow{\delta} (x, y) \\ \Rightarrow\ & F(x_n, y_n) \xrightarrow{d} x \text{ and } F(y_n, x_n) \xrightarrow{d} y, \end{aligned}$$

and

$$\begin{aligned} & (G(x_n,\ y_n),\ G(y_n,\ x_n)) \xrightarrow{\delta} (x,\ y) \\ \Rightarrow \quad & G(x_n,\ y_n) \xrightarrow{d} x \text{ and } G(y_n,\ x_n) \xrightarrow{d} y. \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= \lim_{n\to\infty} G(x_n,\ y_n) = x \in X, \\ \lim_{n\to\infty} F(y_n,\ x_n) &= \lim_{n\to\infty} G(y_n,\ x_n) = y \in X. \end{aligned}$$

Since the pair $\{F,\ G\}$ is generalized compatible, we have that

$$\begin{aligned} \lim_{n\to\infty} d(F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ G(F(x_n,\ y_n),\ F(y_n,\ x_n))) &= 0, \\ \lim_{n\to\infty} d(F(G(y_n,\ x_n),\ G(x_n,\ y_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))) &= 0. \end{aligned}$$

In particular,

$$\begin{aligned} & \lim_{n\to\infty} \delta(T_G T_F(x_n,\ y_n),\ T_F T_G(x_n,\ y_n)) \\ = \ & \lim_{n\to\infty} \delta(T_G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ T_F(G(x_n,\ y_n),\ G(y_n,\ x_n))) \\ = \ & \lim_{n\to\infty} \delta\begin{pmatrix} (G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ G(F(y_n,\ x_n),\ F(x_n,\ y_n))), \\ (F(G(x_n,\ y_n),\ G(y_n,\ x_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{pmatrix} \\ = \ & \lim_{n\to\infty} \max\left\{ \begin{array}{l} d(G(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ F(G(x_n,\ y_n),\ G(y_n,\ x_n))), \\ d(G(F(y_n,\ x_n),\ F(x_n,\ y_n)),\ F(G(y_n,\ x_n),\ G(x_n,\ y_n))) \end{array} \right\} \\ = \ & 0. \end{aligned}$$

Hence, the mappings T_F and T_G are O-compatible in $(X^2,\ \delta,\ \sqsubseteq)$. ■

Theorem 2.5. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\ G : X^2 \to X$ be two generalized compatible mappings satisfying (2.9) such that F is G -increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0,\ y_0) \preceq F(x_0,\ y_0) \text{ and } G(y_0,\ x_0) \succeq F(y_0,\ x_0).$$

Suppose that for any $x,\ y \in X$, there exist $u,\ v \in X$ such that

$$F(x,\ y) = G(u,\ v) \text{ and } F(y,\ x) = G(v,\ u). \tag{2.11}$$

Also suppose that either

(a) F is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Then F and G have a coupled coincidence point.

Proof. It is only need to apply Theorem 2.3 to the mappings $T = T_F$ and $g = T_G$ in the partially ordered metric space $(X^2,\ \delta,\ \sqsubseteq)$ with the help of Lemma 2.4. ■

Corollary 2.6. *Let $(X,\ \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F,\ G : X^2 \to X$ be two commuting mappings satisfying (2.9) and (2.11), F is G -increasing with respect to $\preceq$, G is continuous and there exist two elements $x_0,\ y_0 \in X$ with*

$$G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0)$$

Also suppose that either
(*a*) F *is continuous or*
(*b*) $(X,\ d,\ \preceq)$ *is regular.*
Then F *and* G *have a coupled coincidence point.*

Now we form the results without mixed g -monotone property of F.

Corollary 2.7. *Let* $(X,\ \preceq)$ *be a partially ordered set such that there exists a complete metric* d *on* X. *Assume* $F : X^2 \to X$ *and* $g : X \to X$ *be two mappings such that* F *is* g *-increasing with respect to* $\preceq$ *and there exist an altering distance function* ψ, *an upper semi-continuous function* $\theta : [0,\ +\infty) \to [0,\ +\infty)$ *and a lower semi-continuous function* $\varphi\ : [0,\ +\infty) \to [0,\ +\infty)$ *such that*

$$\begin{aligned}\psi\left(d(F(x,\ y),\ F(u,\ v))\right) \quad &\leq \quad \theta\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right) \\ &\qquad -\varphi\left(\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\}\right),\end{aligned} \tag{2.12}$$

for all $x,\ y,\ u,\ v \in X$, *with* $gx \preceq gu$ *and* $gy \succeq gv$, *where* $\theta(0) = \varphi(0) = 0$ *and* $\psi(t) - \theta(t) + \varphi(t) > 0$ *for all* $t > 0$. *Suppose that* $F(X^2) \subseteq g(X)$, g *is continuous and the pair* $\{F,\ g\}$ *is compatible.Also suppose that either*
(*a*) F *is continuous or*
(*b*) $(X,\ d,\ \preceq)$ *is regular.*
If there exist two elements $x_0,\ y_0 \in X$ *with*

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F *and* g *have a coupled coincidence point.*

Corollary 2.8. *Let* $(X,\ \preceq)$ *be a partially ordered set such that there exists a complete metric* d *on* X. *Assume* $F : X^2 \to X$ *and* $g : X \to X$ *be two mappings such that* F *is* g *-increasing with respect to* $\preceq$ *and satisfying* (2.12). *Suppose that* $F(X^2) \subseteq g(X)$, g *is continuous and the pair* $\{F,\ g\}$ *is commuting. Also suppose that either*
(*a*) F *is continuous or*
(*b*) $(X,\ d,\ \preceq)$ *is regular.If there exist two elements* $x_0,\ y_0 \in X$ *with*

$$gx_0 \preceq F(x_0,\ y_0) \text{ and } gy_0 \succeq F(y_0,\ x_0).$$

Then F *and* g *have a coupled coincidence point.*

Now, we deduce the result without mixed monotone property of F.

Corollary 2.9. *Let* $(X,\ \preceq)$ *be a partially ordered set such that there exists a complete metric* d *on* X. *Assume* $F : X^2 \to X$ *be an increasing mapping with respect to* $\preceq$ *and*

$$\begin{aligned}&\psi\left(d(F(x,\ y),\ F(u,\ v))\right) \\ \leq\quad &\theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right) - \varphi\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right),\end{aligned} \tag{2.13}$$

for all $x,\ y,\ u,\ v \in X$, *with* $x \preceq u$ *and* $y \succeq v$, *where* $\theta(0) = \varphi(0) = 0$ *and* $\psi(t) - \theta(t) + \varphi(t) > 0$ *for all* $t > 0$. *Also suppose that either*
(*a*) F *is continuous or*
(*b*) $(X,\ d,\ \preceq)$ *is regular.*
If there exist two elements $x_0,\ y_0 \in X$ *with*

$$x_0 \preceq F(x_0,\ y_0) \text{ and } y_0 \succeq F(y_0,\ x_0).$$

Then F *has a coupled fixed point.*

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Theorem 2.5, we get the following result:

Corollary 2.10. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ be two generalized compatible mappings such that F is G-increasing with respect to $\preceq$, G is continuous and satisfying*

$$d(F(x, y), F(u, v)) \leq k \max\{d(G(x, y), G(u, v)), d(G(y, x), G(v, u))\}, \tag{2.14}$$

for all $x, y, u, v \in X$, with $G(x, y) \preceq G(u, v)$ and $G(y, x) \succeq G(v, u)$, where $k < 1$ and there exist two elements $x_0, y_0 \in X$ with

$$G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).$$

Suppose that for any $x, y \in X$, there exist $u, v \in X$ satisfying (2.11). Also suppose that either
(a) F is continuous or
(b) $(X, d, \preceq)$ is regular.
Then F and G have a coupled coincidence point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Corollary 2.6, we get the following result:

Corollary 2.11. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F, G : X^2 \to X$ be two commuting mappings such that F is G-increasing with respect to $\preceq$, G is continuous and satisfying (2.11) and (2.14) and there exist two elements $x_0, y_0 \in X$ with*

$$G(x_0, y_0) \preceq F(x_0, y_0) \text{ and } G(y_0, x_0) \succeq F(y_0, x_0).$$

Also suppose that either
(a) F is continuous or
(b) $(X, d, \preceq)$ is regular.
Then F and G have a coupled coincidence point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Corollary 2.7, we get the following result:

Corollary 2.12. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g-increasing with respect to $\preceq$ and satisfying*

$$d(F(x, y), F(u, v)) \leq k \max\{d(gx, gu), d(gy, gv)\}, \tag{2.15}$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$, where $k < 1$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F, g\}$ is compatible. Also suppose that either
(a) F is continuous or
(b) $(X, d, \preceq)$ is regular.If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Corollary 2.8, we get the following result:

Corollary 2.13. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ and $g : X \to X$ be two mappings such that F is g -increasing with respect to $\preceq$ and satisfying (2.15). Suppose that $F(X^2) \subseteq g(X)$, g is continuous and the pair $\{F, g\}$ is commuting. Also suppose that either*
(a) F is continuous or
(b) $(X, d, \preceq)$ is regular.
If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0).$$

Then F and g have a coupled coincidence point.

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Corollary 2.9, we get the following result:

Corollary 2.14. *Let $(X, \preceq)$ be a partially ordered set such that there exists a complete metric d on X. Assume $F : X^2 \to X$ be an increasing mapping with respect to $\preceq$ and*

$$d(F(x,y), F(u,v)) \leq k \max\{d(x, u), d(y, v)\}, \tag{2.16}$$

for all $x, y, u, v \in X$, with $x \preceq u$ and $y \succeq v$, where $k < 1$. Also suppose that either
(a) F is continuous or
(b) $(X, d, \preceq)$ is regular.If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \text{ and } y_0 \succeq F(y_0, x_0).$$

Then F has a coupled fixed point.

Example 2.15. Let $X = U^*(F)211d$ be a metric space with the metric $d : X^2 \to [0, +\infty)$ defined by $d(x, y) = |x - y|$, for all $x, y \in X$, with the natural ordering of real numbers $\leq$. Let $F, G : X^2 \to X$ be defined as

$$F(x, y) = \begin{cases} \frac{x^2 - y^2}{3}, \text{ if } x \geq y, \\ 0, \text{ if } x < y, \end{cases}$$

and

$$G(x, y) = \begin{cases} x^2 - y^2, \text{ if } x \geq y, \\ 0, \text{ if } x < y. \end{cases}$$

Firstly, we shall show that the contractive condition of Theorem 2.5 should satisfy by the mappings F and G. Let $\psi(t) = \theta(t) = t$ and $\varphi(t) = \frac{2t}{3}$ for $t \geq 0$. Now, for all x,

$y,\ u,\ v \in X$ such that $G(x,\ y) \preceq G(u,\ v)$ and $G(y,\ x) \succeq G(v,\ u)$, we have

$$\begin{aligned}
& \psi\left(d(F(x,\ y),\ F(u,\ v))\right) \\
= \ & d(F(x,\ y),\ F(u,\ v)) \\
= \ & \left|\frac{x^2-y^2}{3} - \frac{u^2-v^2}{3}\right| \\
= \ & \frac{1}{3}\left|(x^2-y^2)-(u^2-v^2)\right| \\
= \ & \frac{1}{3}\left|G(x,\ y)-G(u,\ v)\right| \\
= \ & \frac{1}{3}d(G(x,\ y),\ G(u,\ v)) \\
\leq \ & \frac{1}{3}\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\} \\
\leq \ & \theta\left(\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\}\right) \\
& -\varphi\left(\max\left\{d(G(x,\ y),\ G(u,\ v)),\ d(G(y,\ x),\ G(v,\ u))\right\}\right).
\end{aligned}$$

Thus the contractive condition of Theorem 2.5 is satisfied for all $x,\ y,\ u,\ v \in X$. Furthermore, like in [12], all the other conditions of Theorem 2.5 are satisfied and $z = (0,\ 0)$ is a coupled coincidence point of F and G.

3. Application to Integral Equations

We study the existence of the solution to a Fredholm nonlinear integral equation, as an application of the results established in previous section. We shall consider the following integral equation

$$x(p) = \int_a^b (K_1(p,\ q)+K_2(p,\ q))\left[f(q,\ x(q))+g(q,\ x(q))\right]dq + h(p) \tag{3.1}$$

for all $p \in I = [a,\ b]$.

Let Θ denote the set of all functions $\theta : [0,\ +\infty) \to [0,\ +\infty)$ having the following properties:

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p) \leq p$.

Definition 3.1. [15]. A pair $(\alpha,\ \beta) \in X^2$ with $X = C(I,\ U^*(F)211d)$, where $C(I,\ U^*(F)211d)$ denote the set of all continuous functions from I to $U^*(F)211d$, is called a coupled lower-upper solution of (3.1) if, for all $p \in I$,

$$\begin{aligned}
\alpha(p) \ \leq \ & \int_a^b K_1(p,\ q)\left[f(q,\ \alpha(q))+g(q,\ \beta(q))\right]dq \\
& + \int_a^b K_2(p,\ q)\left[f(q,\ \beta(q))+g(q,\ \alpha(q))\right]dq + h(p),
\end{aligned}$$

and

$$\beta(p) \geq \int_a^b K_1(p,\ q)\left[f(q,\ \beta(q))+g(q,\ \alpha(q))\right]dq$$
$$+\int_a^b K_2(p,\ q)\left[f(q,\ \alpha(q))+g(q,\ \beta(q))\right]dq+h(p).$$

Theorem 3.2. *Consider the integral equation (3.1) with* $K_1,\ K_2 \in C(I\times I,\ U^*(F)211d)$, $f,\ g \in C(I\times U^*(F)211d,\ U^*(F)211d)$ *and* $h\in C(I,\ U^*(F)211d)$. *We assume that the functions* $K_1,\ K_2,\ f,\ g$ *fulfill the following conditions:*
(i) $K_1(p,\ q)\geq 0$ *and* $K_2(p,\ q)\leq 0$ *for all* $p,\ q\in I$.
(ii) There exist positive numbers $\lambda,\ \mu$ *and* $\theta\in\Theta$ *such that for all* $x,\ y\in U^*(F)211d$ *with* $x\succeq y$, *the following conditions hold:*

$$0 \leq f(q,\ x)-f(q,\ y)\leq \lambda\theta(x-y), \tag{3.2}$$
$$-\mu\theta(x-y) \leq g(q,\ x)-g(q,\ y)\leq 0. \tag{3.3}$$

(iii)

$$\max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b[K_1(p,\ q)-K_2(p,\ q)]dq\leq\frac{1}{6}. \tag{3.4}$$

Suppose that there exists a coupled lower-upper solution $(\alpha,\ \beta)$ *of (3.1) and Condition (3.1) is satisfied. Then the integral equation (3.1) has a solution in* $C(I,\ U^*(F)211d)$.

Proof. Consider $X=C(I,\ U^*(F)211d)$ with the following partial order

$$x\preceq y \Longleftrightarrow x(p)\leq y(p),\ \forall p\in I,$$

for all $x,\ y\in C(I,\ U^*(F)211d)$. Evidently X is a complete metric space with respect to the sup metric

$$d(x,\ y)=\sup_{p\in I}|x(p)-y(p)|\,.$$

Define on X^2 the following partial order: for $(x,\ y),\ (u,\ v)\in X^2$,

$$(x,\ y)\preceq(u,\ v)\Longleftrightarrow x(p)\leq u(p) \text{ and } y(p)\geq v(p),\ \forall p\in I.$$

Define now the mapping $F: X^2\to X$ by

$$F(x,\ y)(p) = \int_a^b K_1(p,\ q)\left[f(q,\ x(q))+g(q,\ y(q))\right]dq$$
$$+\int_a^b K_2(p,\ q)\left[f(q,\ y(q))+g(q,\ x(q))\right]dq+h(p),$$

for all $p \in I$. We can easily prove, like in [12], that F is increasing. Let $\psi(t) = \theta(t) = t$ and $\varphi(t) = \frac{2t}{3}$ for $t \geq 0$. Then, for all $x,\ y,\ u,\ v \in X$ with $x \succeq u$ and $y \preceq v$, we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ = & \int_a^b K_1(p,\ q)[(f(q,\ x(q)) - f(q,\ u(q))) - (g(q,\ v(q)) - g(q,\ y(q)))]dq \\ & - \int_a^b K_2(p,\ q)[(f(q,\ v(q)) - f(q,\ y(q))) - (g(q,\ x(q)) - g(q,\ u(q)))]dq \end{aligned}$$

Thus, by using (3.2) and (3.3), we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\theta\left(x(q) - u(q)\right) + \mu\theta\left(v(q) - y(q)\right)\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\lambda\theta\left(v(q) - y(q)\right) + \mu\theta\left(x(q) - u(q)\right)\right]dq. \end{aligned} \tag{3.5}$$

Since the function θ is non-decreasing and $x \succeq u,\ \ y \preceq v,$ we have

$$\begin{aligned} \theta\left(x(q) - u(q)\right) & \leq \theta\left(\sup_{q \in I}|x(q) - u(q)|\right) = \theta(d(x,\ u)), \\ \theta\left(v(q) - y(q)\right) & \leq \theta\left(\sup_{q \in I}|v(q) - y(q)|\right) = \theta(d(y,\ v)). \end{aligned}$$

Hence by (3.5), in view of the fact that $K_2(p,\ \ q) \leq 0,$ we obtain

$$\begin{aligned} & |F(x,\ y)(p) - F(u,\ v)(p)| \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\theta(d(x,\ u)) + \mu\theta(d(y,\ v))\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\lambda\theta(d(y,\ v)) + \mu\theta(d(x,\ u))\right]dq, \\ \leq & \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(x,\ u)) + \max\{\lambda,\ \mu\}\theta(d(y,\ v))\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\theta(d(y,\ v)) + \max\{\lambda,\ \mu\}\theta(d(x,\ u))\right]dq, \end{aligned}$$

as all the quantities on the right hand side of (3.5) are non-negative. Now, taking the supremum with respect to p, by using (3.4), we get

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \max\{\lambda,\ \mu\} \sup_{p\in I} \int_a^b (K_1(p,\ q) - K_2(p,\ q))\, dq. [\theta(d(x,\ u)) + \theta(d(y,\ v))] \\ \leq\ & \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6}. \end{aligned}$$

Thus

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6}. \tag{3.6}$$

Now, since θ is non-decreasing, we have

$$\begin{aligned} \theta(d(x,\ u)) &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \\ \theta(d(y,\ v)) &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \end{aligned}$$

which implies, by (ii_θ), that

$$\begin{aligned} \frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{2} &\leq \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right) \\ &\leq \max\left\{d(x,\ u),\ d(y,\ v)\right\}. \end{aligned}$$

Hence

$$\frac{\theta(d(x,\ u)) + \theta(d(y,\ v))}{6} \leq \frac{1}{3} \max\left\{d(x,\ u),\ d(y,\ v)\right\}. \tag{3.7}$$

Thus by (3.6) and (3.7), we have

$$d(F(x,\ y),\ F(u,\ v)) \leq \frac{1}{3} \max\left\{d(x,\ u),\ d(y,\ v)\right\}. \tag{3.8}$$

Now, by (3.8), we have

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \frac{1}{3} \max\left\{d(x,\ u),\ d(y,\ v)\right\} \\ \leq\ & \theta\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right) - \varphi\left(\max\left\{d(x,\ u),\ d(y,\ v)\right\}\right), \end{aligned}$$

which is the contractive condition (2.13) of Corollary 2.9. Let $(\alpha,\ \beta) \in X \times X$ be a coupled upper-lower solution of (3.1), then we have $\alpha(p) \leq F(\alpha,\ \beta)(p)$ and $\beta(p) \geq F(\ \beta,\ \alpha)(p)$, for all $p \in I$. Thus all the hypothesis of Corollary 2.9 are satisfied. Consequently, F has a coupled fixed point $(x,\ y) \in X \times X$ which is the solution of integral equation (3.1) in $X = C(I,\ U^*(F)211d)$. ■

Remark 3.3. With the help of the technique used in [22, 23, 29–32], we obtain tripled, quadruple and in general, multidimensional version of coincidence point theorems established in Theorem 2.3.

Acknowledgements

The authors express their sincere thanks to referees for their valuable suggestions and comments.

References

[1] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. 11 (5) (1987) 623–632.

[2] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (7) (2006) 1379–1393.

[3] M. Abbas, B. Ali, W. Sintunavarat, P. Kumam, Tripled fixed point and tripled coincidence point theorems in intuitionistic fuzzy normed spaces, Fixed Point Theory Appl. 2012 (2012) Article no. 187.

[4] A. Alotaibi, S.M. Alsulami, Coupled coincidence points for monotone operators in partially ordered metric spaces, Fixed Point Theory Appl. 2011 (2011) Article no. 44.

[5] S.M. Alsulami, Some coupled coincidence point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions, Fixed Point Theory Appl. 2013 (2013) Article no. 194.

[6] B.S. Choudhury, A. Kundu, A coupled coincidence point results in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73 (2010) 2524–2531.

[7] B. Deshpande, A. Handa, Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations, Afr. Mat. 26 (3–4) (2015) 317–343.

[8] B. Deshpande, A. Handa, Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces, Adv. Fuzzy Syst. 2014 (2014) Article ID 348069.

[9] I.M. Erhan, E. Karapinar, A. Roldan, N. Shahzad, Remarks on coupled coincidence point results for a generalized compatible pair with applications, Fixed Point Theory Appl. 2014 (2014) Article no. 207.

[10] J. Harjani, B. Lopez, K. Sadarangani, Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal. 74 (2011) 1749–1760.

[11] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal. 72 (2010) 1188–1197.

[12] N. Hussain, M. Abbas, A. Azam, J. Ahmad, Coupled coincidence point results for a generalized compatible pair with applications, Fixed Point Theory Appl. 2014 (2014) Article no. 62.

[13] E. Karapinar, P. Kumam, I.M. Erhan, Coupled fixed points on partially ordered G-metric spaces, Fixed Point Theory Appl. 2012 (2012) Article no. 174.

[14] V. Lakshmikantham, L. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (12) (2009) 4341–4349.

[15] N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74 (2011) 983–992.

[16] N.V. Luong, N.X. Thuan, Coupled points in ordered generalized metric spaces and application to integro-differential equations, Comput. Math. Appl. 62 (11) (2011) 4238–4248.

[17] A. Razani, V. Parvaneh, Coupled coincidence point results for (ψ, α, β)-weak contractions in partially ordered metric spaces, J. Appl. Math. 2012 (2012) Article ID 496103.

[18] W. Sintunavarat, P. Kumam, Y.J. Cho, Coupled fixed point theorems for nonlinear contractions without mixed monotone property, Fixed Point Theory Appl. 2012 (2012) Article no. 170.

[19] R.P. Agarwal, R.K. Bisht, N. Shahzad, A comparison of various noncommuting conditions in metric fixed point theory and their applications, Fixed Point Theory Appl. 2014 (2014) Article no. 38.

[20] K. Goebel, A coincidence theorem, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. 16 (1968) 733–735.

[21] N.M. Hung, E. Karapinar, N.V. Luong, Coupled coincidence point theorem for O-compatible mappings via implicit relation, Abstr. Appl. Anal. 2012 (2012) Article ID 796964.

[22] S.A. Al-Mezel, H. Alsulami, E. Karapinar and A. Roldan, Discussion on multidimensional coincidence points via recent publications, Abstr. Appl. Anal. 2014 (2014) Article ID 287492.

[23] B. Samet, E. Karapinar, H. Aydi, V.C. Rajic, Discussion on some coupled fixed point theorems, Fixed Point Theory Appl. 2013 (2013) Article no. 50.

[24] M. Imdad, S. Chauhan, P. Kumam, Fixed point theorems for two hybrid pairs of non-self mappings under joint common limit range property in metric spaces, J. Nonlinear Convex Anal. 16 (2) (2015) 243–254.

[25] C. Mongkolkeha, W. Sintunavarat, P. Kumam, Fixed point theorems for contraction mappings in modular metric spaces, Fixed Point Theory Appl. 2011 (2011) Article no. 93.

[26] W. Sintunavarat, P. Kumam, Fixed point theorems for a generalized intuitionistic fuzzy contraction in intuitionistic fuzzy metric spaces, Thai J. Math. 10 (1) (2012) 123–135.

[27] J.J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations 22 (2005) 223–239.

[28] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443.

[29] E. Karapinar, A. Roldan, A note on n-Tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces, J. Inequal. Appl. 2013 (2013) Article no. 567.

[30] E. Karapinar, A. Roldan, C. Roldan, J. Martinez-Moreno, A note on N-fixed point theorems for nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2013 (2013) Article no. 310.

[31] E. Karapinar, A. Roldan, N. Shahzad, W. Sintunavarat, Discussion on coupled and tripled coincidence point theorems for $\phi-$contractive mappings without the mixed g-monotone property, Fixed Point Theory Appl. 2014 (2014) Article no. 92.

[32] A. Roldan, J. Martinez-Moreno, C. Roldan, E. Karapinar, Some remarks on multidimensional fixed point theorems, Fixed Point Theory Appl. 2013 (2013) Article no. 158.