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Polcag Spaces: I. Group-Like Structures

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Abstract In this paper, Polcag space is defined to investigate the relation between group-like structure and topological one. Also, we examined the properties of the group-like structure induced by a Polcag space and obtained some useful results.

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1. Introduction

It is an interesting idea to examine the relationship between algebraic structures and topological ones. To investigate this relation, we define a new mathematical structure: Polcag space. Moreover, we show that group-like structures can be induced by Polcag spaces, and give some useful results. In our next work, we will study topological structures by induced Polcag spaces, and we examine how the relationship between the induced topological structure and the induced algebraic one changes when changing some conditions.

We give the following definitions of group-like structures [1–6].

Definition 1.1. Let X be a set.

(1) X with a partial binary operation $\circ : X \times X \to X$ is said to be *a partial magma*.
(2) X with a binary operation $\circ : X \times X \to X$ is said to be *a magma*.
(3) A partial magma $(X, \circ)$ is called *a partial semigroup* if the followings hold:
(Partial associativity) For all $x, y, z \in X$ holding $x \circ y \in X$ and $y \circ z \in X$,
  (a) $x \circ (y \circ z) \in X$ iff $(x \circ y) \circ z \in X$ and
  (b) if $x \circ (y \circ z) \in X$ then $x \circ (y \circ z) = (x \circ y) \circ z$.
(4) A magma $(X, \circ)$ is said to be *a semigroup* if,
(Union) for all $x, y, z \in X$, $x \circ (y \circ z) = (x \circ y) \circ z$.
(5) A magma $(X, \circ)$ is said to be *a quasigroup* if the following condition holds:
(Divisibility) For all $x, y \in X$, there exists a pair of elements$u, v \in X$ holds the following conditions:

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(a) $x \circ u = y$
(b) $v \circ x = y$

(6) A semigroup $(X, \circ)$ is said to be *a monoid* if the following condition holds:
(Identity) For an element$e \in X$ and all $x \in X$, $x \circ e = e \circ x = x$.

(7) A quasigroup $(X, \circ)$ is called *a loop* if the following condition holds:
(Identity) For an element$e \in X$ and all $x \in X$, $x \circ e = e \circ x = x$.

(8) A monoid $(X, \circ)$ is said to be *a group* if the following condition holds:
(Inverse element) For all $x \in X$ and some element $y \in X$, $x \circ y = y \circ x = e$ where $e \in X$ is the identity element.
Equally, a loop $(X, \circ)$ is called *a group* if the following condition holds:
(Union) For all $x, y, z \in X$, $x \circ (y \circ z) = (x \circ y) \circ z$.

(9) The group $(X, \circ)$ is called *a commutative group* if:
(Commutativity) for all $x, y \in X$, $x \circ y = y \circ x$.

2. Polcag Space

Definition 2.1. Let$X$ be a set and $\mathcal{P}(X)$ is the power set of X. We define $\Phi = \{\varphi_x\}_{x \in X}$ and $\Psi = \{\psi_x\}_{x \in X}$ as two collections of functions from X to $\mathcal{P}(X)$ indexed by X. Then, the triplet (X, Φ, Ψ) is called *a raw structure.*

Definition 2.2. Given a raw structure (X, Φ, Ψ) and let $x, y \in X$. If $\varphi_y(x) \neq \emptyset$ ($\psi_y(x) \neq \emptyset$) then x is said to be *a φ_y-compatible* (*ψ_y-compatible*).

Definition 2.3. Given a raw structure (X, Φ, Ψ) and let $x, y, z \in X$.

(1) x is called *a φ_z conjugate of* y if $x \in \varphi_z(y)$.
(2) x is called *a ψ_z conjugate of* y if $x \in \psi_z(y)$.
(3) x is called *a z-conjugate of* y if x is both a φ_z conjugate and ψ_z conjugate of y.

Definition 2.4. A raw structure (X, Φ, Ψ) is called *a stable structure* if, for all $x, y, z \in X$,

$$x \text{ is a } \psi_z \text{ conjugate of } y \Leftrightarrow y \text{ is a } \varphi_z \text{ conjugate of } x.$$

Definition 2.5. A stable structure (X, Φ, Ψ) is said to satisfy *precedency property* if, for all distinct pair of elements x_1, x_2 and all $y \in X$,

$$\varphi_{x_1}(y) \cap \varphi_{x_2}(y) = \emptyset = \psi_{x_1}(y) \cap \psi_{x_2}(y).$$

Definition 2.6. A stable structure (X, Φ, Ψ) is called *a Polcag space* and denoted by $\mathbf{P} = (X, \Phi, \Psi)$ if (X, Φ, Ψ) satisfies the precedency property.

3. Main Results

Theorem 3.1. *Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space. The operation $\circ : X \times X \to X$ defined by*

$$\circ(x, y) := \begin{cases} z & y \text{ is a } \varphi_z \text{ conjugate of } x \\ \text{undefined} & \text{otherwise} \end{cases}$$

for all $x, y, z \in X$ is a partial binary operation on X.

Proof. Let $x_1 \circ y_1 = z_1$, $x_2 \circ y_2 = z_2$ and $z_1 \neq z_2$. By the definition of the operation $\circ$on X, $y_1$is a φ_{z_1}conjugate of x_1 and y_2 a φ_{z_2}conjugate of x_2 i.e., $y_1 \in \varphi_{z_1}(x_1)$ and $y_2 \in \varphi_{z_2}(x_2)$.

Let $x_1 = x_2$. In this case, by the precedency property one can obtain that $\varphi_{z_1}(x_1) \cap \varphi_{z_2}(x_2) = \varphi_{z_1}(x_1) \cap \varphi_{z_2}(x_1) = \emptyset$. On the other hand, it is known that $y_1 \in \varphi_{z_1}(x_1)$ and $y_2 \in \varphi_{z_2}(x_2)$. So, $y_1 \neq y_2$.

Let $y_1 = y_2$. Since $\mathbf{P}$ is stable, from the Definition 2.4, x_1 is a ψ_{z_1}conjugate of y_1 and x_2 is a ψ_{z_2}conjugate of y_2 i.e., $x_1 \in \psi_{z_1}(y_1)$ and $x_2 \in \psi_{z_2}(y_2)$. Therefore, by the precendecy property, $\psi_{z_1}(y_1) \cap \psi_{z_2}(y_2) = \psi_{z_1}(y_1) \cap \psi_{z_2}(y_1) = \emptyset$. On the other hand, it is known that $x_1 \in \psi_{z_1}(y_1)$ and $x_2 \in \psi_{z_2}(y_2)$. So, $x_1 \neq x_2$.

Thus, if $z_1 \neq z_2$ then $(x_1, y_1) \neq (x_2, y_2)$ which implies that $\circ$ is well-defined. ∎

Definition 3.2. A Polcag space $\mathbf{P} = (X, \Phi, \Psi)$ is called *a complete Polcag space* if, for all $x, y \in X$, there exists some $z \in X$ such that x is a φ_z conjugate of y.

Theorem 3.3. *Let a complete Polcag space $\mathbf{P} = (X, \Phi, \Psi)$ is given. In this case, the partial binary operation $\circ$ defined in Theorem 3.1 is a binary operation.*

Proof. Let $x, y \in X$. Since $\mathbf{P}$ is complete, there exists some $z \in X$ such that x is a φ_z conjugate of y from Definition 3.2. So, $x \in \varphi_z(y)$ according to Definition 2.3(1). Therefore, $y \circ x = z \in X$ from the definition of $\circ$, i.e., the operation $\circ$ is closed on X. ∎

Definition 3.4. Let $\mathbf{P} = (X, \Phi, \Psi)$ be a complete Polcag space. The set X with the partial binary operation $\circ$ defined in Theorem 3.1 is called *the partial group-like structure induced by* $\mathbf{P}$ and denoted by $(X, \circ)_{\mathbf{P}}$. If $\circ$ is a binary operation then $(X, \circ)_{\mathbf{P}}$ is called *a group-like structure induced by* $\mathbf{P}$.

Corollary 3.5. *Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space. The (partial) group-like structure $(X, \circ)_{\mathbf{P}}$ is a (partial) magma.*

Proof. It is clear from the definition of (partial) magma. ∎

Definition 3.6. A Polcag space $\mathbf{P} = (X, \Phi, \Psi)$ is called *relevant* if x is a ψ_w conjugate of s if and only if z is a φ_w conjugate of t for all $s, t, w \in X$ and all $x, y, z \in X$ such that y is a φ_t conjugate of x and a $\psi_s$conjugate of z.

Theorem 3.7. *If a Polcag space $\mathbf{P} = (X, \Phi, \Psi)$ is relevant then the partial magma $(X, \circ)_{\mathbf{P}}$ is a partial semigroup.*

Proof. Let $x \circ y = t$ and $y \circ z = s$. From the definition of the operation $\circ$, y is a φ_t conjugate of x and z a φ_s conjugate of y, i.e., $y \in \varphi_t(x)$ and $z \in \varphi_s(y)$.

(a)$\Rightarrow$**:** Let $x \circ (y \circ z) \in X$. Then, there exists some $w \in X$ such that $x \circ (y \circ z) = w$. So, we have $x \circ s = w$, and from the definition of the operation $\circ$, s is a $\varphi_w$conjugate of x. Since $\mathbf{P}$ is stable, x is a $\psi_w$conjugate of s. z is a φ_w conjugate of t since $\mathbf{P}$ is relevant. Thus, from the definition of the operation $\circ$, we have $t \circ z = w$. Substituting $t = x \circ y$ in $t \circ z = w$, we have $(xoy) \circ z = w$, i.e., $(xoy) \circ z \in X$.
$\Leftarrow$: Let $(xoy) \circ z \in X$. Then, there exists some $v \in X$ such that $(xoy) \circ z = v$. So, we have $t \circ z = v$, and from the definition of the operation $\circ$, z is a φ_v conjugate of t. x is a ψ_v conjugate of s since $\mathbf{P}$ is relevant. Thus, from the definition of

the operation $\circ$, we have $x \circ s = v$. Substituting $s = y \circ z$ in $x \circ s = v$, we have $x \circ (y \circ z) = v$, i.e., $x \circ (y \circ z) \in X$.

(b): Let $x \circ (y \circ z) \in X$. Thus, we have obtained the equality $x \circ (y \circ z) = w = (x \circ y) \circ z$ from (a). So, the proof is completed.

∎

Corollary 3.8. *If a complete Polcag space* $\mathbf{P} = (X, \Phi, \Psi)$ *is relevant then the magma* $(X, \circ)_{\mathbf{P}}$ *is a semigroup.*

Proof. Let $x \circ (y \circ z) = w$. From the proof of Theorem 3.7, $(x \circ y) \circ z = w$ and so $x \circ (y \circ z) = w = (x \circ y) \circ z$. ∎

Definition 3.9. Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space and $x, y \in X$. x is called *y-correlated* if φ_y and $\psi_y$conjugates of x are nonempty sets.

Definition 3.10. Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space. $\mathbf{P}$ is called *correlated* if x is y-correlated for all $x, y \in X$.

Theorem 3.11. *If a complete Polcag space* $\mathbf{P} = (X, \Phi, \Psi)$ *is correlated then the magma* $(X, \circ)_{\mathbf{P}}$ *is a quasigroup.*

Proof. Let $x, y \in X$. Since $\mathbf{P}$ is correlated, x is y-correlated according to Definition 3.10. Also, φ_y and ψ_y conjugates of x are nonempty sets according to Definition 3.9, i.e., $\varphi_y(x) \neq \emptyset \neq \psi_y(x)$. Chosen $u, v \in X$ such that $u \in \varphi_y(x)$ and $v \in \psi_y(x)$. Then, u is a φ_y conjugate of x and v is a ψ_y conjugate of x. From the definition of the operation $\circ$ in Theorem 3.1, $x \circ u = y$ and $v \circ x = y$. Since the pair of elements x, y is chosen arbitrarily, the magma $(X, \circ)_{\mathbf{P}}$ satisfies the divisibility property. Thus, $(X, \circ)_{\mathbf{P}}$ is a quasigroup. ∎

Definition 3.12. Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space and $x, y \in X$. x is called *a phi-selfconjugate of y* if x is a φ_y conjugate of y; x is called *a psi-selfconjugate of y* if x is a ψ_y conjugate of y. x is called *a selfconjugate of y* if x is a phi-selfconjugate of y and psi-selfconjugate of y.

Definition 3.13. Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space. $a \in X$ is called *a common-conjugate element of* $\mathbf{P}$ and $\mathbf{P}$ is called *common-correlated* if a is a selfconjugate of every x in X.

Theorem 3.14. *If a complete, relevant Polcag space* $\mathbf{P} = (X, \Phi, \Psi)$ *is common-correlated then the semigroup* $(X, \circ)_{\mathbf{P}}$ *is a monoid.*

Proof. Let $x \in X$. Since $\mathbf{P}$ is common-correlated, from Definition 3.13, $\mathbf{P}$ has a common-conjugate element; say this element e. Then, e is a selfconjugate of x. From the Definition 3.12, e is both a phi-selfconjugate and a psi-selfconjugate of x, i.e., $e \in \varphi_x(x)$ and $e \in \psi_x(x)$. From the definition of the operation $\circ$ in Theorem 3.1, we have $x \circ e = x$ and $e \circ x = x$. Since the element x is chosen arbitrarily, $e \in X$ is the identity element of the semigroup $(X, \circ)_{\mathbf{P}}$. Thus, $(X, \circ)_{\mathbf{P}}$ is a monoid. ∎

Theorem 3.15. *If a complete, correlated Polcag space* $\mathbf{P} = (X, \Phi, \Psi)$ *is common-correlated then the quasigroup* $(X, \circ)_{\mathbf{P}}$ *is a loop.*

Proof. Let $x \in X$. Since $\mathbf{P}$ is common-correlated, from Definition 3.13, $\mathbf{P}$ has a common-conjugate element; say this element e. Then, e is a selfconjugate of x. From the Definition 3.12, e is both a phi-selfconjugate and a psi-selfconjugate of x, i.e., $e \in \varphi_x(x)$ and $e \in \psi_x(x)$. From the definition of the operation $\circ$ in Theorem 3.1, we have $x \circ e = x$ and $e \circ x = x$. Since the element x is chosen arbitrarily, $e \in X$ is the identity element of the quasigroup $(X, \circ)_{\mathbf{P}}$. Thus, $(X, \circ)_{\mathbf{P}}$ is a loop. ∎

Definition 3.16. Let $\mathbf{P} = (X, \Phi, \Psi)$ be a Polcag space and e a common-conjugate element of $\mathbf{P}$. $\mathbf{P}$ is said to be *inverse-correlated* if there exists an e-conjugate of every element in X.

Theorem 3.17. *Let a complete, relevant and common-correlated Polcag space* $\mathbf{P} = (X, \Phi, \Psi)$ *is inverse-correlated then the monoid* $(X, \circ)_{\mathbf{P}}$ *is a group.*

Proof. Let $x \in X$. Since $\mathbf{P}$ is common-correlated, from Definition 3.13, $\mathbf{P}$ has a common-conjugate element; say this element e. From Theorem 3.14, e is the identity element of $(X, \circ)_{\mathbf{P}}$. Since $\mathbf{P}$ is inverse-correlated, from the Definition 3.16, x has a e-conjugate; say this element y. From the Definition 2.3(3), x is both a φ_e conjugate of y and ψ_e conjugate of y. Again, from Definition 2.3(1-2), we have $x \in \varphi_e(y)$ and $x \in \psi_e(y)$. From the definition of the operation $\circ$ in the Theorem 3.1, we have $y \circ x = e$ and $x \circ y = e$. Since the element x is chosen arbitrarily, the monoid $(X, \circ)_{\mathbf{P}}$ satisfies the inverse element property. So, $(X, \circ)_{\mathbf{P}}$ is a group. ∎

Theorem 3.18. *If a complete, correlated and common-correlated Polcag space* $\mathbf{P} = (X, \Phi, \Psi)$ *is relevant then the loop* $(X, \circ)_{\mathbf{P}}$ *is a group.*

Proof. Let $x, y, z \in X$. Since $\mathbf{P}$ is complete, $x \circ y \in X$ and $y \circ z \in X$. Set $t = x \circ y$ and $s = y \circ z$. From the definition of the operation $\circ$ in Theorem 3.1, y is a φ_t conjugate of x and z is a φ_s conjugate of y, i.e., $y \in \varphi_t(x)$ and $z \in \varphi_s(y)$. Again, since $\mathbf{P}$ is complete, $x \circ s \in X$ and $t \circ z \in X$. Set $w = x \circ s$. Hence, from the definition of the operation $\circ$, s is a φ_w conjugate of x. Since $\mathbf{P}$ is stable, x is a ψ_w conjugate of s. z is a φ_w conjugate of t since $\mathbf{P}$ is relevant. From the definition of the operation $\circ$, $t \circ z = w$. Substituting $t = x \circ y$ in $t \circ z = w$, we have $(xoy) \circ z = w$. Also, since $w = x \circ s = x \circ (y \circ z)$, we have $(xoy) \circ z = x \circ (y \circ z)$. Since the elements x, y, z are chosen arbitrarily, the loop $(X, \circ)_{\mathbf{P}}$ satisfies the associativity property. Thus, $(X, \circ)_{\mathbf{P}}$ is a group. ∎

Definition 3.19. A Polcag space $\mathbf{P} = (X, \Phi, \Psi)$ is called *symmetric-correlated* if, for every pair x, y in X, there exists some $z \in X$ such that each of x, y is z-conjugate of the other.

Theorem 3.20. *Let* $\mathbf{P} = (X, \Phi, \Psi)$ *be a complete, relevant and common-correlated Polcag space and also, correlated or inverse correlated. If* $\mathbf{P}$ *is symmetric-correlated then the group* $(X, \circ)_{\mathbf{P}}$ *is commutative.*

Proof. Let $x, y \in X$. Since $\mathbf{P}$ is symmetric-correlated, there exists some $z \in X$ such that x is a z-conjugate of y from Definition 3.19. From Definition 2.3(3), x is both a φ_z conjugate of y and a ψ_z conjugate of y. Again, from Definition 2.3(1-2), $x \in \varphi_z(y)$ and $x \in \psi_z(y)$. From the definition of the operation $\circ$ in the Theorem 3.1, we have $x \circ y = z = y \circ x$. Since the elements x, y are chosen arbitrarily, the group $(X, \circ)_{\mathbf{P}}$ satisfies the commutativity property. So, the group $(X, \circ)_{\mathbf{P}}$ is commutative. ∎

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