Thai Journal of Mathematics
Volume 19 Number 1 (2021)
Pages 77–85

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Fixed Point Theorems on $b-$Metric Spaces via $C_F - b-$Simulation Functions

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Abstract The purpose of this paper is to present some fixed point results for $C_F - b-$simulation functions in complete $b-$metric spaces. One example is given to support the result.

MSC: 47H10; 54H25

Keywords: fixed point; $b-$metric space; $C-$class function; Property C_F; $C_F - b-$simulation function

Submission date: 24.11.2017 / Acceptance date: 19.12.2019

1. Introduction

The existence of fixed point theorem in Banach space was first investigated by Banach or the well known as the Banach contraction principle [1] in 1922.

Next, many authors extended and improved many fixed point results in connection with existing ones.

In 1989, Bakhtin [2] (see also Czerwik [3]) introduced the concept of a $b-$metric space (a special kind of metric space) and proved some fixed point theorems for some contraction mappings in $b-$metric spaces which generalize Banach's contraction principle in metric space.

In 2015, Khojasteh et at. [4] introduced the notion of a simulation function in connection with the generalization of Banach's contraction principle.

Recently, Roldán-LÓpez-de-Hierroet et al. [5] modified the notion of a simulation function and showed the existence and uniqueness of coincidence points of two nonlinear mappings, using the concept of a simulation function.

Very recently Demma et at. [6] introduced the notion of $b-$simulation in the setting of $b-$metric spaces and they established the existence and uniqueness of fixed points in $b-$metric spaces.

In this paper, we introduce the notion of $C_F - b-$simulation function and prove some fixed point theorems in complete $b-$metric spaces. Furthermore, we also give one example

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Published by The Mathematical Association of Thailand.

to illustrate the main results. As consequences of this study, we deduce several related results in fixed point theory in $b-$metric space.

2. Preliminaries

We begin with giving some notation and preliminaries that we shall need to state our results.

In the sequel, the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of all real numbers and the set of all natural numbers, respectively.

Definition 2.1. (see [7]). Let X be a nonempty set and let $d : X \times X \longrightarrow [0, \infty)$ be a function satisfying the conditions :

(m1) $d(x, y) = 0$ if and only if $x = y$;
(m2) $d(x, y) = d(y, x)$;
(m3) $d(x, z) \leq d(x, y) + d(y, z)$,

for all $x, y, z \in X$. Then d is called metric on X, and the pair (X, d) is called metric space.

Definition 2.2. (see [2]). Let X be a nonempty set and $K \geq 1$ be a given real number. A function $d : X \times X \longrightarrow [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied :

(b1) $d(x, y) = 0$ if and only if $x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq K[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space.

We can see from the definition of $b-$ metric that every metric space is $b-$ metric for $K = 1$, but the converse is not true.

Example 2.3. (see [8]). Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x-y|^2$. Then d is a b-metric on $\mathbb{R}$ with $K = 2$, but it is not a metric on $\mathbb{R}$.

Example 2.4. (see [9]). Let $X = l_p, (0 < p < 1)$, when

$$l_p = \{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\}.$$

Define $d : X \times X \to [0, \infty)$ by $d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$, where $x = (x_n) \in X$ and $y = (y_n) \in X$. Then d is a $b-$metric with $K = 2^{\frac{1}{p}}$.

Definition 2.5. (see [3]). Let $\{x_n\}$ be a sequence in $b-$metric space (X, d).Then $\{x_n\}$ is called a $b-$Cauchy sequence, if for all $\epsilon > 0$ there exist a positive integer N such that for $m, n \geq N$ we have $d(x_m, x_n) < \epsilon$.

Definition 2.6. (see [3]). A sequence $\{x_n\}$ is called $b-$convergent in $b-$metric space (X, d), if for all $\epsilon > 0$ and for $n \geq N$ we have $d(x_n, x) < \epsilon$, where x is called the limit point of the sequence $\{x_n\}$.

Definition 2.7. (see [3]). A $b-$metric space (X, d) is said to be complete if every Cauchy sequence in X converge to a point of X.

Definition 2.8. (see [10]). A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called $C-$class function if it is continuous and satisfies the following conditions:

(C1) $F(s, t) \leq s$;

(C2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

We denote $C-$class function as $\mathcal{C}$.

Definition 2.9. (see [11]). A mapping $F : [0, \infty)^2 \to \mathbb{R}$ has a property C_F, if there exists a nonnegative real number C_F such that

(Cf1) $F(s, t) > C_F$ implies $s > t$;

(Cf2) $F(t, t) \leq C_F$, for all $t \in [0, \infty)$.

Let $\mathcal{C}_\mathcal{F}$ be the family of all $C-$class functions that have property C_F.

Example 2.10. The following function $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$ that have property C_F, for all $s, t \in [0, \infty)$

(1) $F(s, t) = s - t, \quad C_F = a, \quad a \in [0, \infty)$.

(2) $F(s, t) = s - \frac{(2+t)t}{1+t}, \quad C_F = 0$.

Definition 2.11. (see [4]). Let $\zeta : [0, \infty)^2 \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions

(ζ1) $\zeta(0, 0) = 0$;

(ζ2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Example 2.12. (see [4]). Let $\zeta_i : [0, \infty)^2 \to \mathbb{R}, \quad i = 1, 2, 3$ be defined by

(i) $\zeta_1(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, \infty)$, where $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$.

(ii) $\zeta_2(t, s) = s - \frac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty)^2 \to [0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.

(iii) $\zeta_3(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, \infty)$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

Then ζ_i for $i = 1, 2, 3$ are simulation functions.

Definition 2.13. (see [11]). A C_F-simulation is a mapping $\zeta_F : [0, \infty)^2 \to \mathbb{R}$ satisfies the following conditions

(ζ_F1) $\zeta_F(t, s) < F(s, t)$ for all $t, s > 0$ and $F \in \mathcal{C}$;

(ζ_F2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta_F(t_n, s_n) < C_F$.

Definition 2.14. (see [6]). Let (X, d) be a $b-$metric space with constant $K \geq 1$. A $b-$simulation function is a function $\xi : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(ξ1) $\xi(t, s) < s - t$, for all $t, s > 0$;

(ξ2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq K \lim_{n\to\infty} t_n < \infty,$$

then

$$\limsup_{n\to\infty} \xi(Kt_n, s_n) < 0.$$

Theorem 2.15. (see [6]). *Let (X, d) be a complete $b-$metric space with constant $K \geq 1$ and let $f : X \to X$ be a mapping. Suppose that there exists a $b-$simulation function ξ such that*

$$\xi(Kd(fx, fy), d(x, y)) \geq 0, \quad \text{for all } x, y \in X.$$

Then f has a unique fixed point.

3. Main Results

In this section, we define the $C_F - b-$simulation function and prove the existence of a fixed point for such mapping in complete $b-$metric spaces.

Definition 3.1. Let (X, d) be a $b-$metric space with a constant $K \geq 1$. A $C_F - b-$simulation function is a function $\xi_{C_F} : [0, \infty)^2 \to \mathbb{R}$ satisfying the following conditions:

(ξ_{C_F}1) $\xi_{C_F}(t, s) < F(s, t)$ for all $t, s > 0$, where $F : [0, \infty)^2 \to \mathbb{R}$ is element of $\mathcal{C}_{\mathcal{F}}$;

(ξ_{C_F}2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq K \lim_{n\to\infty} t_n < \infty,$$

then

$$\limsup_{n\to\infty} \xi_{C_F}(Kt_n, s_n) < C_F,$$

where C_F is a nonnegative real number.

Example 3.2. Let $\lambda \in [0, 1)$ and define $\xi_1 : [0, \infty)^2 \to \mathbb{R}$ by $\xi_1(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$ then ξ_1 is $C_F - b-$simulation function where $F(s, t) = s - t$ and $C_F = 0$.

Example 3.3. If $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous functions such that $\varphi(t) = 0$ if and only if $t = 0$ and define $\xi_2 : [0, \infty)^2 \to \mathbb{R}$ by $\xi_2(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, \infty)$, then ξ_2 is $C_F - b-$simulation function where $F(s, t) = s - t$ and $C_F = 0$.

Example 3.4. If $\varphi : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, and define $\xi_3 : [0, \infty)^2 \to \mathbb{R}$ by $\xi_3(t, s) = s\varphi(s) - t$ for all $t, s \in [0, \infty)$, then ξ_3 is $C_F - b-$simulation function where $F(s, t) = s - t$ and $C_F = 0$.

Theorem 3.5. *Let (X, d) be a complete $b-$metric space with a constant $K \geq 1$ and let $f : X \to X$ be a mapping. Suppose that there exists a $C_F - b-$simulation function ξ_{C_F} such that*

$$\xi_{C_F}(Kd(fx, fy), d(x, y)) \geq C_F, \text{for all } x, y \in X. \tag{3.1}$$

Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be a sequence in X defined by $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$. If $x_{m-1} = x_m$ for some $m \in \mathbb{N}$, then $x_{m-1} = x_m = fx_{m-1}$, that is x_{m-1} is a fixed point of f. Therefore, suppose that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$.
We have devided the proof in 4 steps.

Step 1. We shall now prove that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Using (3.1) and ($\xi_{C_F}1$), for all $n \geq 0$,

$$\begin{aligned} C_F &\leq \xi_{C_F}(Kd(fx_{n-1}, fx_n), d(x_{n-1}, x_n)) \\ &= \xi_{C_F}(Kd(x_n, x_{n+1}), d(x_{n-1}, x_n)) \\ &< F(d(x_{n-1}, x_n), Kd(x_n, x_{n+1})). \end{aligned} \tag{3.2}$$

Form (C_f1) of definition 2.9 it follows that $d(x_{n-1}, x_n) > Kd(x_n, x_{n+1})$, for all $n \geq 0$. Then $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence of nonnegative real numbers. Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = r$.
Assume that $r > 0$. Applying the condition ($\xi_{C_F}2$), with $t_n = d(x_n, x_{n+1})$ and $s_n = d(x_{n-1}, x_n)$, it follows that

$$\limsup_{n\to\infty} \xi_{C_F}(Kd(x_n, x_{n+1}), d(x_{n-1}, x_n)) < C_F,$$

which contradics (3.2) because $\xi_{C_F}(Kd(x_n, x_{n+1}), d(x_{n-1}, x_n)) \geq C_F$. Thus $\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0$.

Step 2. We claim that the sequence $\{x_n\}$ is a bounded sequence. Assume that $\{x_n\}$ is not a bounded sequence. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_{k+1}}, x_{n_k}) > 1$$

and $d(x_m, x_{n_k}) \leq 1$, for $n_k \leq m \leq n_{k+1} - 1$. By (b3) of definition 2.2, we get

$$\begin{aligned} 1 &< d(x_{n_{k+1}}, x_{n_k}) \\ &\leq Kd(x_{n_{k+1}}, x_{n_{k+1}-1}) + Kd(x_{n_{k+1}-1}, x_{n_k}) \\ &\leq Kd(x_{n_{k+1}}, x_{n_{k+1}-1}) + K. \end{aligned} \tag{3.3}$$

Letting $k \to \infty$ in (3.3) and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we obtain

$$1 \leq \liminf_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq K \tag{3.4}$$

From (3.1) and property ($\xi_{C_F}1$), we have

$$\begin{aligned} C_F &\leq \xi_{C_F}(Kd(fx_{n_{k+1}-1}, fx_{n_k-1}), d(x_{n_{k+1}-1}, x_{n_k-1})) \\ &= \xi_{C_F}(Kd(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})) \\ &\leq F(d(x_{n_{k+1}-1}, x_{n_k-1}), Kd(x_{n_{k+1}}, x_{n_k})). \end{aligned} \tag{3.5}$$

Using (Cf1) of the definition 2.9 and (b3) of definition 2.2, it follows that

$$\begin{aligned} Kd(x_{n_{k+1}}, x_{n_k}) &< d(x_{n_{k+1}-1}, x_{n_k-1}) \\ &\leq Kd(x_{n_{k+1}-1}, x_{n_k}) + Kd(x_{n_k}, x_{n_k-1}) \\ &\leq K + Kd(x_{n_k}, x_{n_k-1}). \end{aligned} \tag{3.6}$$

Letting $k \to \infty$ in the above inequality and using (3.4), we deduce that there exist

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1 \ \text{ and } \ \lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = K.$$

Therefore by 3.5 and $(\xi_{C_F}2)$, with $t_k = d(x_{n_{k+1}}, x_{n_k})$ and $s_k = d(x_{n_{k+1}-1}, x_{n_k-1})$, we have

$$C_F \leq \limsup_{k\to\infty} \xi_{C_F}(Kd(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})) < C_F,$$

which is a contradiction. Hence the sequence $\{x_n\}$ is bounded.

Step 3. Now, we prove that the sequence $\{x_n\}$ is Cauchy in X.
Let $A_n = \sup\{d(x_i, x_j) : i, j \geq n\}, \quad n \in \mathbb{N}$.
Since the sequence $\{x_n\}$ is bounded, $A_n < \infty$ for all $n \in \mathbb{N}$ and since the sequence $\{A_n\}$ is a positive decreasing, there exists $A \geq 0$ such that

$$\lim_{n\to\infty} A_n = A. \tag{3.7}$$

Suppose that $A > 0$. Then by the definition of A_n, for every $k \in \mathbb{N}$ there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$A_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq A_k. \tag{3.8}$$

Letting $k \to \infty$ in (3.8), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = A. \tag{3.9}$$

By (3.1) and property $(\xi_{C_F}1)$, we have

$$\begin{aligned} C_F &\leq \xi_{C_F}(Kd(fx_{m_k-1}, fx_{n_k-1}), d(x_{m_k-1}, x_{n_k-1})) \\ &= \xi_{C_F}(Kd(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1})) \\ &< F(d(x_{m_k-1}, x_{n_k-1}), Kd(x_{m_k}, x_{n_k})). \end{aligned}$$

Using (Cf1) of the definition 2.9 and definition of A_n, we get

$$Kd(x_{m_k}, x_{n_k}) < d(x_{m_k-1}, x_{n_k-1}) \leq A_{k-1}. \tag{3.10}$$

Letting $k \to \infty$ in (3.10), using (3.7) and (3.9), we have

$$KA \leq \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq A. \tag{3.11}$$

From (3.11) we see that, if $K > 1$ then $A = 0$. If $K = 1$ then by the property $(\xi_{C_F}2)$ with $t_k = d(x_{m_k}, x_{n_k})$ and $s_k = d(x_{m_k-1}, x_{n_k-1})$, we get

$$C_F \leq \limsup_{k\to\infty} \xi_{C_F}(Kd(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1})) < C_F,$$

which is a contradiction. Thus $A = 0$, that is $\lim_{n\to\infty} A_n = 0$, for all $K \geq 1$. This prove that $\{x_n\}$ is a Cauchy sequence.

Step 4. We claim that f has a unique fixed point.
Since X is a complete $b-$metric space and $\{x_n\}$ is a Cauchy sequence in X, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{3.12}$$

We shall prove that z is a fixed point of f. Using (3.1) and property ($\xi_{C_F}1$), we obtain

$$\begin{aligned} C_F &\leq \xi_{C_F}(Kd(fx_n, fz), d(x_n, z)) \\ &< F(d(x_n, z), Kd(fx_n, fz)). \end{aligned}$$

From (Cf1) of definition 2.9, it follows that

$$d(x_n, z) > Kd(fx_n, fz), \quad \text{forall} \quad n \in \mathbb{N},$$

and consequently

$$\begin{aligned} d(z, fz) &\leq Kd(z, x_{n+1}) + Kd(x_{n+1}, fz) \\ &= Kd(z, x_{n+1}) + Kd(fx_n, fz) \\ &\leq Kd(z, x_{n+1}) + d(x_n, z). \end{aligned} \tag{3.13}$$

Letting $n \to \infty$ in (3.13), we get $d(z, fz) = 0$, that is z is a fixed point of f. Finally, we prove that z is the unique fixed point of f in X. Suppose that there exists $w \in X$ such that $fw = w$ and $w \neq z$.
Using (3.1) and property ($\xi_{C_F}1$), we obtain

$$\begin{aligned} C_F &\leq \xi_{C_F}(Kd(fw, fz), d(w, z)) \\ &< F(d(w, z), Kd(fw, fz)) \\ &= F(d(w, z), Kd(w, z)), \end{aligned}$$

using (Cf1) of definition 2.9, we get $d(w, z) > Kd(w, z)$.
Since $d(w, z) \neq 0$, $K < 1$, which is a contradiction. Therefore $w = z$. This complete the proof. ■

Corollary 3.6. ([12], Theorem 3.3). *Let (X, d) be a complete $b-$metric space with $K \geq 1$ and let $f : X \to X$ be a mapping. Suppose that there exists $\lambda \in (0, 1)$ such that*

$$Kd(fx, fy) \leq \lambda d(x, y), \quad \text{for all} \quad x, y \in X.$$

Then f has a unique fixed point.

Proof. It follows from Theorem 3.5 using the $C_F - b-$simulation function

$$\xi_{C_F}(t, s) = \lambda s - t,$$

for all $t, s \geq 0$ and $F(s, t) = s - t$ and $C_F = 0$. ■

Corollary 3.7. (Rhoades Type [13]). *Let (X, d) be a complete $b-$metric space with $K \geq 1$ and let $f : X \to X$ be a mapping. Suppose that there exists a lower semi-continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ such that*

$$Kd(fx, fy) \leq d(x, y) - \varphi(d(x, y)) \quad \text{for all} \quad x, y \in X.$$

Then f has a unique fixed point.

Proof. The result follows from Theorem 3.5 by taking as the $C_F - b-$simulation function

$$\xi_{C_F}(t, s) = s - \varphi(s) - t,$$

for all $t, s \geq 0$ and $F(s, t) = s - t$ and $C_F = 0$. ■

Corollary 3.8. (Reich Type [14]). *Let (X, d) be a complete $b-$metric space with $K \geq 1$ and let $f : X \to X$ be a mapping. Suppose that there exists a function $\varphi : [0, \infty) \to [0, 1)$ with $\limsup_{t \to r^+} \varphi(t) < 1$ for all $r > 0$ such that*

$$Kd(fx, fy) \leq \varphi(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. It follows from Theorem 3.5 using the $C_F - b-$simulation function

$$\xi_{C_F}(t, s) = s\varphi(s) - t,$$

for all $t, s \geq 0$ and $F(s, t) = s - t$ and $C_F = 0$. ∎

Corollary 3.9. (Boyd-Wong type [15]). *Let (X, d) be a complete $b-$metric space with $K \geq 1$ and let $f : X \to X$ be a mapping. Suppose that there exists an upper semi-continuous function $\eta : [0, \infty) \to [0, \infty)$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ such that*

$$Kd(fx, fy) \leq \eta(d(x, y)) \text{ for all } x, y \in X.$$

Then f has a unique fixed point.

Proof. It follows from Theorem 3.5 using the $C_F - b-$simulation function

$$\xi_{C_F}(t, s) = \eta(s) - t,$$

for all $t, s \geq 0$ and $F(s, t) = s - t$ and $C_F = 0$. ∎

Example 3.10. Let $X = [0, 1]$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = (x - y)^2$. Then (X, d) is a complete $b-$metric space with $K = 2$. Define $f : X \to X$ by $fx = \frac{ax}{1+x}$ for all $x \in X$ and $a \in (0, \frac{1}{\sqrt{2}}]$. Let $\xi_{C_F} : [0, \infty)^2 \to \mathbb{R}$ define by $\xi_{C_F}(t, s) = \frac{s}{s+1} - t$, let $F(s, t) = s-t$ and $C_F = 0$, we have ξ_{C_F} is $C_F - b-$simulation function. Indeed, we obtain

$$\begin{aligned}
\xi_{C_F}(2d(fx, fy), d(x, y)) &= \frac{d(x, y)}{d(x, y) + 1} - 2d(fx, fy) \\
&= \frac{(x - y)^2}{(x - y)^2 + 1} - 2(\frac{ax}{1 + x} - \frac{ay}{1 + y})^2 \\
&= \frac{(x - y)^2}{(x - y)^2 + 1} - \frac{2a^2(x - y)^2}{((1 + x)(1 + y))^2} \\
&\geq \frac{(x - y)^2}{(x - y)^2 + 1} - \frac{2a^2(x - y)^2}{1 + (x - y)^2} \\
&= \frac{(x - y)^2 - 2a^2(x - y)^2}{(x - y)^2 + 1} \\
&\geq 0 = C_F, \quad \text{for all } x, y \in X.
\end{aligned}$$

Thus all the conditions of Theorem 3.5 are satisfied. Hence f has a unique fixed point $x = 0$.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

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